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Existence theory for implicit fractional q-difference equations in Banach spaces

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Abstract. This paper deals with some existence results for a class of implicit fractional q-difference equations. The results are based on the fixed point theory in Banach spaces and the concept of measure of noncompactness. An illustrative example is given in the last section.

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1. Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences [27]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs [1, 2, 3, 20, 26, 30], the papers [21, 22, 29] and the references therein. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations and inclusions with Caputo fractional derivative; [2, 19]. Implicit fractional differential equations were analyzed by many authors; see, for instance [1, 2, 4, 12, 13, 14] and the references therein.

Fractional q-difference equations were initiated at the beginning of the 19th century [5, 15], and received significant attention in recent years. Some interesting details about initial and boundary value problems of q-difference and fractional q-difference equations can be found in [7, 8, 16, 17] and references therein.

Recently, in [3], the authors applied the measure of noncompactness to some classes of functional Riemann–Liouville or Caputo fractional differential equations in

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Banach spaces. Motivated by the above papers, we discuss the existence of solutions for the following implicit fractional q-difference equation

$$({}^C D_q^\alpha u)(t) = f(t, u(t), ({}^C D_q^\alpha u)(t)), \; t \in I := [0, T], \tag{1.1}$$

with the initial condition

$$u(0) = u_0, \tag{1.2}$$

where $q \in (0,1)$, $\alpha \in (0,1]$, $T > 0$, $f : I \times E \times E \to E$ is a given function, E is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^C D_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

This paper initiates the study of implicite fractional q-difference equations on Banach spaces.

2. Preliminaries

Consider the Banach space $C(I) := C(I, E)$ of continuous functions from I into E equipped with the usual supremum (uniform) norm

$$\|u\|_\infty := \sup_{t \in I} \|u(t)\|.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to E$ which are Bochner integrable with the norm

$$\|v\|_1 = \int_0^T \|v(t)\| dt.$$

Let us recall some definitions and properties of fractional q-calculus. For $a \in \mathbb{R}$, we set

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q-analogue of the power $(a - b)^n$ is

$$(a - b)^{(0)} = 1, \; (a - b)^{(n)} = \Pi_{k=0}^{n-1} (a - bq^k); \; a, b \in \mathbb{R}, \; n \in \mathbb{N}.$$

In general,

$$(a - b)^{(\alpha)} = a^\alpha \Pi_{k=0}^{\infty} \left(\frac{a - bq^k}{a - bq^{k+\alpha}} \right); \; a, b, \alpha \in \mathbb{R}.$$

Definition 2.1. [18] The q-gamma function is defined by

$$\Gamma_q(\xi) = \frac{(1 - q)^{(\xi - 1)}}{(1 - q)^{\xi - 1}}; \; \xi \in \mathbb{R} - \{0, -1, -2, ...\}$$

Notice that the q-gamma function satisfies $\Gamma_q(1 + \xi) = [\xi]_q \Gamma_q(\xi)$.

Definition 2.2. [18] The q-derivative of order $n \in \mathbb{N}$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$,

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t) - u(qt)}{(1 - q)t}; \; t \neq 0, \quad (D_q u)(0) = \lim_{t \to 0} (D_q u)(t),$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t); \; t \in I, \; n \in \{1, 2, \ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3. [18] The q-integral of a function $u : I_t \to E$ is defined by

$$(I_q u)(t) = \int_0^t u(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0).$$

Definition 2.4. [6] The Riemann–Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0,\infty)$ of a function $u : I \to E$ is defined by $(I_q^0 u)(t) = u(t)$, and

$$(I_q^\alpha u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s) d_q s;\ t \in I.$$

Lemma 2.5. [24] *For $\alpha \in \mathbb{R}_+ := [0,\infty)$ and $\lambda \in (-1,\infty)$ we have*

$$(I_q^\alpha (t-a)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(t-a)^{(\lambda+\alpha)};\ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

Definition 2.6. [25] The Riemann–Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$, and

$$(D_q^\alpha u)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u)(t);\ t \in I,$$

where $[\alpha]$ is the integer part of α.

Definition 2.7. [25] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $({}^C D_q^0 u)(t) = u(t)$, and

$$({}^C D_q^\alpha u)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t);\ t \in I.$$

Lemma 2.8. [25] *Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$(I_q^\alpha\ {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha\ {}^C D_q^\alpha u)(t) = u(t) - u(0).$$

From the above lemma and in order to define a solution for the problem (1.1)-(1.2), we conclude with the following lemma.

Lemma 2.9. *Let $f : I \times E \times E \to E$ such that $f(\cdot, u, v) \in C(I)$, for each $u, v \in E$. Then the problem (1.1)-(1.2) is equivalent to the problem of obtaining solutions of the integral equation*

$$g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)),$$

and if $g(\cdot) \in C(I)$ is the solution of this equation, then

$$u(t) = u_0 + (I_q^\alpha g)(t).$$

Definition 2.10. [9, 10, 11, 28] Let X be a Banach space and let Ω_X be the family of bounded subsets of X. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \to [0, \infty)$ defined by

$$\mu(M) = \inf\{\epsilon > 0 : M \subset \cup_{j=1}^m M_j, \mathrm{diam}(M_j) \leq \epsilon\},$$

where $M \in \Omega_X$.

The measure of noncompactness satisfies the following properties

(1) $\mu(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).
(2) $\mu(M) = \mu(\overline{M})$.
(3) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
(4) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.
(5) $\mu(cM) = |c|\mu(M),\ c \in \mathbb{R}$.
(6) $\mu(\mathrm{conv}\, M) = \mu(M)$.

For our purpose we will need the following fixed point theorem:

Theorem 2.11. *(Monch's fixed point theorem [23]). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \overline{V} \text{ is compact}, \tag{2.1}$$

holds for every subset V of D, then N has a fixed point.

3. Main results

In this section, we are concerned with existence results for the problem (1.1)-(1.2).

Definition 3.1. By a solution of problem (1.1)-(1.2), we mean a continuous function u that satisfies the equation (1.1) on I and the initial condition (1.2).

The following hypotheses will be used in the sequel.

(H_1) The function $f : I \times E \times E \to E$ is continuous.
(H_2) There exists a continuous function $p \in C(I, \mathbb{R}_+)$, such that

$$\|f(t, u, v)\| \leq p(t);\ \textit{for}\ t \in I,\ \textit{and}\ u, v \in E,$$

(H_3) For each bounded set $B \subset E$ and for each $t \in I$, we have

$$\mu(f(t, B, {}^C D_q^r B)) \leq p(t)\mu(B),$$

where ${}^C D_q^r B = \{{}^C D_q^r w : w \in B\}$, and μ is a measure of noncompactness on E.

Set

$$p^* = \sup_{t \in I} p(t), \text{ and } L := \sup_{t\in I} \int_0^T \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s.$$

Theorem 3.2. *Assume that the hypotheses $(H_1)-(H_3)$ hold. If*

$$\ell := Lp^* < 1, \tag{3.1}$$

then the problem (1.1)*-*(1.2) *has at least one solution defined on I.*

Proof. By using Lemma 2.9, we transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $N : C(I) \to C(I)$ defined by

$$(Nu)(t) = u_0 + (I_q^\alpha g)(t);\ t \in I, \tag{3.2}$$

where $g \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)),\ \text{or } g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)).$$

For any $u \in C(I)$ and each $t \in I$, we have

$$\begin{aligned} \|(Nu)(t)\| &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s)| d_q s \\ &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &\leq \|u_0\| + p^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\ &\leq \|u_0\| + Lp^* \\ &:= R. \end{aligned}$$

Thus

$$\|N(u)\|_\infty \leq R. \tag{3.3}$$

This proves that N transforms the ball $B_R := B(0,R) = \{w \in C : \|w\|_\infty \leq R\}$ into itself.

We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.11. The proof will be given in three steps.

Step 1. *$N : B_R \to B_R$ is continuous.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, we have

$$\|(Nu_n)(t) - (Nu)(t)\| \leq \int_0^t \tfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|(g_n(s) - g(s))\| d_q s,$$

where $g_n, g \in C(I)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, we get

$$g_n(t) \to g(t) \text{ as } n \to \infty, \text{ for each } t \in I.$$

Hence

$$\|N(u_n) - N(u)\|_\infty \leq L\|g_n - g\|_\infty \to 0 \ \ as \ \ n \to \infty.$$

Step 2. *$N(B_R)$ is bounded and equicontinuous.*
Since $N(B_R) \subset B_R$ and B_R is bounded, then $N(B_R)$ is bounded.
Next, let $t_1, t_2 \in I$, $t_1 < t_2$ and let $u \in B_R$. Thus, we have

$$\|(Nu)(t_2) - (Nu)(t_1)\| \leq \left\| \int_0^{t_2} \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q s - \int_0^{t_1} \frac{(t_1 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q s \right\|.$$

where $g \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)).$$

Hence, we get

$$\begin{aligned} \|(Nu)(t_2) - (Nu)(t_1)\| &\leq \int_{t_1}^{t_2} \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &+ \int_0^{t_1} \left| \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| d_q s \\ &\leq p^* \int_{t_1}^{t_2} \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &+ p^* \int_0^{t_1} \left| \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| d_q s. \end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.
Step 3. *The implication (2.1) holds.*
Now let V be a subset of B_R such that $V \subset \overline{N(V)} \cup \{0\}$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on I. By (H_3) and the properties of the measure μ, for each $t \in I$, we have

$$\begin{aligned} v(t) &\leq \mu((NV)(t) \cup \{0\}) \\ &\leq \mu((NV)(t)) \\ &\leq \int_0^t \frac{(tq - s)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) \mu(V(s)) d_q s \\ &\leq \int_0^t \frac{(tq - s)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) v(s) d_q s \\ &\leq Lp^* \|v\|_\infty. \end{aligned}$$

Thus

$$\|v\|_\infty \leq \ell \|v\|_\infty.$$

From (3.1), we get $\|v\|_\infty = 0$, that is, $v(t) = \mu(V(t)) = 0$, for each $t \in I$, and then $V(t)$ is relatively compact in E. In view of the Ascoli–Arzelà theorem, V is relatively compact in B_R. Applying now Theorem 2.11, we conclude that N has a fixed point which is a solution of the problem (1.1)-(1.2).

4. An example

Let

$$l^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_{l^1} = \sum_{n=1}^{\infty} |u_n|.$$

Consider the following problem of implicit fractional $\frac{1}{4}-$difference equations

$$\begin{cases} ({}^cD_{\frac{1}{4}}^{\frac{1}{2}} u_n)(t) = f_n(t, u(t), ({}^cD_{\frac{1}{4}}^{\frac{1}{2}} u)(t)); \ t \in [0,1], \\ u(0) = (0, 0, \dots, 0, \dots), \end{cases} \tag{4.1}$$

where

$$\begin{cases} f_n(t,u,v) = \dfrac{t^{\frac{-1}{4}} (2^{-n} + u_n(t)) \sin t}{64L(1 + \|u\|_{l^1} + \sqrt{t})(1 + \|u\|_{l^1} + \|v\|_{l^1})}, \ t \in (0,1], \\ \qquad f_n(0,u,v) = 0, \end{cases}.$$

with

$$f = (f_1, f_2, \dots, f_n, \dots), \ and \ u = (u_1, u_2, \dots, u_n, \dots).$$

For each $t \in (0,1]$, we have

$$\begin{aligned} \|f(t,u(t))\|_{l^1} &= \sum_{n=1}^{\infty} |f_n(s, u_n(s))| \\ &\le \frac{t^{\frac{-1}{4}} |\sin t|}{64L(1 + \|u\|_{l^1} + \sqrt{t})(1 + \|u\|_{l^1} + \|v\|_{l^1})} (1 + \|u\|_{l^1}) \\ &\le \frac{t^{\frac{-1}{4}} |\sin t|}{64L}. \end{aligned}$$

Thus, the hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{t^{\frac{-1}{4}} |\sin t|}{64L}; \ t \in (0,1], \\ \qquad p(0) = 0. \end{cases}$$

So, we have $p^* \le \frac{1}{64L}$, and then

$$Lp^* = \frac{1}{64} < 1.$$

Simple computations show that all conditions of Theorem 3.2 are satisfied. Hence, the problem (4.1) has at least one solution defined on $[0, 1]$.

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