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Global existence and stability of solution for a $p-$Kirchhoff type hyperbolic equation with damping and source terms

Amar Ouaoua, Aya Khaldi and Messaoud Maouni

Abstract. In this paper, we consider a nonlinear $p-$Kirchhoff type hyperbolic equation with damping and source terms

$$u_{tt} - M\left(\int_\Omega |\nabla u|^p \, dx\right)\Delta_p u + |u_t|^{m-2} u_t = |u|^{r-2} u.$$

Under suitable assumptions and positive initial energy, we prove the global existence of solution by using the potential energy and Nehari's functionals. Finally, the stability of equation is established based on Komornik's integral inequality.

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1. Introduction

In this article, we consider the following value problem

$$\begin{cases} u_{tt} - M\left(\int_\Omega |\nabla u|^p \, dx\right)\Delta_p u + |u_t|^{m-2} u_t = |u|^{r-2} u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$ and

$$M(s) = a + bs$$

with positive parameters a, b, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $p \geq 2$.

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In the past few years, much effort has been devoted to nonlocal problems because of their wide applications in both physics and biology. For exemple the following hyperbolic equation with a nonlocal coefficient are as follows:

$$\varepsilon u_{tt}^{\varepsilon}+u_{t}^{\varepsilon}-M\left(\int_{\Omega}\left|\nabla u^{\varepsilon}\right|^{p}dx\right)\Delta_{p}u^{\varepsilon}=f\left(x,\ t,\ u^{\epsilon}\right), \tag{1.2}$$

where $M\left(s\right)=a+bs,\ a,\,b>0$ and $p>1$, in a bounded domain $\Omega\subset\mathbb{R}^{n}$ is a potential model for damped small transversal vibrations of an elastic string with uniform density ε (see [6]). For $p=2$, such nonlocal equations were first proposed by Kirchhoff [7] in 1883 and therefore were usually referred to as Kirchhoff equations.

Equation (1.1) can be viewed as a generalization of a model introduced by Kirchhoff [15]. The following Kirchhoff type equation

$$u_{tt}-M\left(\left\|\nabla u\right\|_{2}^{2}\right)\Delta u+g\left(u_{t}\right)=f\left(u\right), \tag{1.3}$$

have been discussed by many authors. For $g\left(u_{t}\right)=u_{t}$, the global existence and blow up results can by found in ([13], [15]), for $g\left(u_{t}\right)=\left|u_{t}\right|^{m-2}u_{t}, p>2$, the main results of existence and blow up are in ([5], [11]). The absence of the damping term $\left|u_{t}\right|^{m-2}u_{t}$ in equation (1.1), when $M(s)=a+bs^{\gamma}$ $(\gamma>0)$ and $p=2$, the existence of the global solution was investigated by many authors (see [1]-[4], [9], [10], [15], [16]). The works of K. Ono [12]-[14] deal with equation (1.3) in two cases with $f(u)=\left|u\right|^{r-2}u$, $p>2$. In the first case, for $g(u_{t})=-u_{t}$ or u_{t}, he considered $M(s)=a+bs^{\gamma}$, where $a\geq 0,\ b\geq 0,\ a+b>0,\ \gamma>0$. He showed that the local solutions blow up at finite time with $E(0)>0$ by applying the concavity method. Moreover, he combined the so-called potential well method and concavity method to show blow-up properties with $E(0)>0$. While in the second case, for $g(u_{t})=\left|u_{t}\right|^{m-2}u_{t},\ m>2$, he treated $M(s)=a+bs^{\gamma}$, where $b>0,\ a=0$ and $\gamma\geq 1$. He proved that the local solution is not global when $p>max(2\gamma+2,m)$ and $E(0)<0$.

The paper is organized as follows. In section 2, we introduce some notations and Lemma needed in the next sections to prove the main result. In section 3, we use the energy and Nihari functionals to prove the global existence of the solutions. In section 4, we use the energy method to prove the result based on Komornik's integral inequality.

2. Preliminaries

We begin this section with some notations and definitions. Denote by $\left\|.\right\|_{p}$, the $L^{p}\left(\Omega\right)$ norm of a Lebesgue function $u\in L^{p}\left(\Omega\right)$ for $p\geq 1$. We use $W_{0}^{1,p}\left(\Omega\right)$ to denote the well-known Sobolev space such that both u and $\left|\nabla u\right|$ are in $W_{0}^{1,p}\left(\Omega\right)$ equipped with the norm $\left\|u\right\|_{W_{0}^{1,p}(\Omega)}=\left\|\nabla u\right\|_{p}$.

Lemma 2.1. *Let s be a number with $2\leq s\leq+\infty$ if $n\leq p$ and $2\leq s\leq\frac{pn}{n-p}$ if $n>p$. Then there is a constant c_{*} depending on Ω and s such that*

$$\left\|u\right\|_{s}\leq c_{*}\left\|\nabla u\right\|_{p},\quad\forall u\in W_{0}^{1,p}\left(\Omega\right).$$

Theorem 2.2. *Suppose that* $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ *and*

$$2p < r \leq p^*,$$

where

$$p^* = \begin{cases} \frac{np}{n-p}, & \text{if } n > p, \\ +\infty & \text{if } n \leq p. \end{cases}$$

Then problem (1.1) *has a unique weak solution such that*

$$\begin{aligned} u &\in L^\infty\left((0,T),\ W_0^{1,p}(\Omega)\right), \\ u_t &\in L^\infty\left((0,T),\ L^2(\Omega)\right) \cap L^m(\Omega \times (0,\ T)), \\ u_{tt} &\in L^2\left((0,T),\ W^{-1,p'}(\Omega)\right). \end{aligned}$$

3. Global existence

In this section, we state and prove our result, we define the potential energy functional and the Nehari's functional, by the following

$$E(t) = E(u(t)) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{a}{p}\|\nabla u(t)\|_p^p + \frac{b}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\|u(t)\|_r^r. \quad (3.1)$$

$$J(t) = J(u(t)) = \frac{a}{p}\|\nabla u(t)\|_p^p + \frac{b}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\|u(t)\|_r^r. \quad (3.2)$$

$$I(t) = I(u(t)) = a\|\nabla u(t)\|_p^p + b\|\nabla u(t)\|_p^{2p} - \|u(t)\|_r^r. \quad (3.3)$$

We can considering $a = b = 1$, and this does not change the general result of (1.1).

Lemma 3.1. *Under the assumptions of theorem* 2.2, *we have*

$$E'(t) = -\|u_t(t)\|_m^m \leq 0, \quad t \in [0,\ T]. \quad (3.4)$$

and

$$E(t) \leq E(0).$$

Proof. We multiply the first equation of (1.1) by u_t and integrating over the domain Ω, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}\int_\Omega |\nabla u(t)|^p\,dx + \frac{1}{2p}\left(\int_\Omega |\nabla u(t)|^p\,dx\right)^2 - \frac{1}{r}\|u(t)\|_r^r\right) = -\|u_t(t)\|_m^m,$$

then

$$E'(t) = -\|u_t(t)\|_m^m \leq 0.$$

Integrating (3.4) over $(0,\ t)$, we obtain $E(t) \leq E(0)$. □

Lemma 3.2. *Assume that the assumptions of theorem* 2.2 *hold,*

$$I(0) > 0,$$

and

$$\beta_1 + \beta_2 < 1, \tag{3.5}$$

where

$$\beta_1 := \alpha c_*^r \left(\frac{pr}{r-p} E(0)\right)^{\frac{r-p}{p}}, \quad \beta_2 := (1-\alpha)\, c_*^r \left(\frac{2pr}{r-2p} E(0)\right)^{\frac{r-2p}{2p}}$$

with $0 < \alpha < 1$, c_* *is the best embedding constant of* $W_0^{1,\,p}(\Omega) \hookrightarrow L^r(\Omega)$, *then* $I(t) > 0$, *for all* $t \in [0,\ T]$.

Proof. By continuity, there exists T_*, such that

$$I(t) \geq 0, \quad \text{for all } t \in [0,\ T_*]. \tag{3.6}$$

Now, we have for all $t \in [0,\ T_*]$:

$$\begin{aligned} J(t) &= J(u(t)) = \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\|u(t)\|_r^r \\ &\geq \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\left(\|\nabla u(t)\|_p^p + \|\nabla u(t)\|_p^{2p} - I(t)\right) \\ &\geq \frac{r-p}{pr}\|\nabla u(t)\|_p^p + \frac{r-2p}{2pr}\|\nabla u(t)\|_p^{2p} + \frac{1}{r} I(t) \end{aligned}$$

using (3.6), we obtain

$$\frac{r-p}{pr}\|\nabla u(t)\|_p^p + \frac{r-2p}{2pr}\|\nabla u(t)\|_p^{2p} \leq J(t), \qquad \text{for all } t \in [0,\ T_*]. \tag{3.7}$$

By the definition of E, we get

$$\|\nabla u(t)\|_p^p \leq \frac{pr}{r-p} E(t) \leq \frac{pr}{r-p} E(0) \tag{3.8}$$

and

$$\|\nabla u(t)\|_p^{2p} \leq \frac{2pr}{r-2p} E(t) \leq \frac{2pr}{r-2p} E(0) \tag{3.9}$$

On the other hand, we have

$$\|u(t)\|_r^r = \alpha\ \|u(t)\|_r^r + (1-\alpha)\|u(t)\|_r^r$$

By the embedding of $W_0^{1,\,p}(\Omega) \hookrightarrow L^r(\Omega)$, we obtain

$$\begin{aligned} \|u(t)\|_r^r &\leq \alpha\ c_*^r \|\nabla u(t)\|_p^r + (1-\alpha)\, c_*^r \|\nabla u(t)\|_p^r \\ &\leq \alpha\ c_*^r \|\nabla u(t)\|_p^{r-p} \times \|\nabla u(t)\|_p^p + (1-\alpha)\, c_*^r \|\nabla u(t)\|_p^{r-2p} \times \|\nabla u(t)\|_p^{2p} \end{aligned}$$

By (3.8) and (3.9), we get

$$\|u(t)\|_r^r \leq \beta_1 \|\nabla u(t)\|_p^p + \beta_2 \|\nabla u(t)\|_p^{2p}, \qquad \text{for all } t \in [0,\ T_*]. \tag{3.10}$$

Since $\beta_1 + \beta_2 < 1$, then

$$\|u(t)\|_r^r < \|\nabla u(t)\|_p^p + \|\nabla u(t)\|_p^{2p}, \qquad \text{for all } t \in [0,\ T_*].$$

This implies that

$$I(t) > 0, \qquad \text{for all } t \in [0,\ T_*].$$

By repeating the above procedure, we can extend T_* to T. ☐

Theorem 3.3. *Under the assumptions of lemma* 3.2, *the local solution of* (1.1) *is global.*

Proof. We have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\|u(t)\|_r^r \\ &\geq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{r-p}{pr}\|\nabla u(t)\|_p^p + \frac{r-2p}{2pr}\|\nabla u(t)\|_p^{2p}. \end{aligned}$$

So that

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p \leq C\ E(t). \tag{3.11}$$

By Lemma 3.1, we obtain

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p \leq C\ E(0). \tag{3.12}$$

This implies that the local solution is global in time. ☐

4. Stability of solution

In this section our main result is established based in Komornik's integral inequality [8]. For this, we need the following Lemma:

Lemma 4.1. *Suppose that the assumptions of Lemma* 3.2 *and* $m > p$, *hold, then there exists a positive constant* c *such that*

$$\int_\Omega |u(t)|^m\, dx \leq cE(t). \tag{4.1}$$

Proof. By using (3.8), we obtain

$$\begin{aligned} \int_\Omega |u(t)|^m\, dx = \|u(t)\|_m^m &\leq c_*^m \|\nabla u(t)\|_p^m \\ &\leq c_*^m \|\nabla u(t)\|_p^{m-p} \times \|\nabla u(t)\|_p^p \\ &\leq c_*^m \|\nabla u(t)\|_p^{m-p} \times \frac{rp}{r-p} E(t) \leq cE(t). \end{aligned}$$

☐

Now, we state our main result:

Theorem 4.2. *Let the assumptions of Lemma* 3.2, *then, there exists constants* $C, \zeta > 0$, *such that*

$$\begin{aligned} E(t) &\leq \frac{C}{(1+t)^{\frac{2}{m-2}}}, \quad \text{for all } t \geq 0 \quad \text{if } m > 2. \\ E(t) &\leq Ce^{-\zeta t}, \qquad\quad \text{for all } t \geq 0 \quad \text{if } m = 2. \end{aligned}$$

Proof. Multiplying first equation of (1.1) by $u(t)\,E^q(t)$ $(q>0)$, and integrating over $\Omega\times(S,\ T)$, we obtain

$$\int_S^T\int_\Omega E^q(t)\left[u(t)\,u_{tt}(t)-u(t)\left(M\left(\int_\Omega|\nabla u|^p\,dx\right)\Delta_p u+|u_t|^{m-2}u_t\right)\right]dxdt$$
$$=\int_S^T E^q(t)\int_\Omega|u(t)|^r\,dxdt$$

So that

$$\int_S^T\int_\Omega E^q(t)\Big[(u(t)\,u_t(t))_t-|u_t(t)|^2+|\nabla u(t)|^p+\|\nabla u(t)\|_p^p\,|\nabla u(t)|^p$$
$$+u(t)\,|u_t|^{m-2}u_t\Big]\,dxdt=\int_S^T E^q(t)\int_\Omega|u(t)|^r\,dxdt$$

We add and subtract the term

$$\int_S^T E^q(t)\int_\Omega\Big[\beta_1|\nabla u(t)|^p+\beta_2\|\nabla u(t)\|_p^p\,|\nabla u(t)|^p+(2+\beta_1+\beta_2)\,|u_t(t)|^2\Big]\,dxdt,$$

and use (3.10), to get

$$(1-\beta_1)\int_S^T E^q(t)\int_\Omega\Big[|\nabla u(t)|^p+|u_t(t)|^2\Big]\,dxdt$$
$$+(1-\beta_2)\int_S^T E^q(t)\int_\Omega\Big[\|\nabla u(t)\|_p^p\,|\nabla u(t)|^p+|u_t(t)|^2\Big]\,dxdt$$
$$+\int_S^T E^q(t)\int_\Omega\Big[(u(t)\,u_t(t))_t-(3-\beta_1-\beta_2)\,|u_t(t)|^2\Big]\,dxdt$$
$$+\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m-2}\,dxdt$$
$$=-\int_S^T E^q(t)\int_\Omega\Big[\beta_1|\nabla u(t)|^p+\beta_2\|\nabla u(t)\|_p^p\,|\nabla u(t)|^p-|u(t)|^r\Big]\,dxdt\le 0. \tag{4.2}$$

It is clear that

$$\gamma\int_S^T E^q(t)\int_\Omega\left[\frac{1}{p}|\nabla u(t)|^p+\frac{1}{2p}\|\nabla u(t)\|_p^p|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}-\frac{|u(t)|^r}{r}\right]dxdt$$

$$\begin{aligned}\le\ &(1-\beta_1)\int_S^T E^q(t)\int_\Omega\left[\frac{1}{p}|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}\right]dxdt\\&+(1-\beta_2)\int_S^T E^q(t)\int_\Omega\left[\frac{1}{2p}\|\nabla u(t)\|_p^p|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}\right]dxdt\end{aligned}\tag{4.3}$$

where $\gamma=Min\left((1-\beta_1),\ (1-\beta_2)\right)$. By (4.2) , (4.3) and definition of $E(t)$, we get

$$\begin{aligned}\gamma\int_S^T E^{q+1}(t)\,dt\ \le\ &-\int_S^T E^q(t)\int_\Omega\left(u(t)\,u_t(t)\right)_t dxdt\\&+(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega|u_t(t)|^2\,dxdt\\&-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m-2}\,dxdt.\end{aligned}\tag{4.4}$$

Using the definition of $E(t)$ and the following expression

$$\begin{aligned}\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)\ =\ &qE^{q-1}(t)\,\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx\\&+E^q(t)\int_\Omega\left(u(t)\,u_t(t)\right)_t dx.\end{aligned}$$

Inequality (4.4), becomes

$$\begin{aligned}\gamma\int_S^T E^{q+1}(t)\,dt\le q\int_S^T E^{q-1}(t)\,\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx\\-\int_S^T\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dt-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m-2}\,dxdt\\+(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega|u_t(t)|^2\,dxdt.\end{aligned}\tag{4.5}$$

In the sequel, we denote by c the various constants.
We estimate the terms in the right-hand side of (4.5) as follow:

By (3.4) and Young's inequality, we obtain

$$
\begin{aligned}
& q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx \\
\leq\ & q\int_S^T E^{q-1}(t)\left(-E'(t)\right)\int_\Omega\left[\frac{1}{p}\left|u(t)\right|^p+\frac{p-1}{p}\left|u_t(t)\right|^{\frac{p}{p-1}}\right]dxdt
\end{aligned}
\tag{4.6}
$$

Since, $1\leq\frac{p}{p-1}<2$, by the embedding of $L^2(\Omega)\hookrightarrow L^{\frac{p}{p-1}}(\Omega)$, we have

$$
\begin{aligned}
& q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx \\
\leq\ & q\int_S^T E^{q-1}(t)\left(-E'(t)\right)\int_\Omega\left[\frac{1}{p}\left|u(t)\right|^p+c\frac{p-1}{p}\left|u_t(t)\right|^2\right]dxdt
\end{aligned}
$$

Thus, by (3.11), we find

$$
\begin{aligned}
& q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx \\
\leq\ & c\int_S^T E^q(t)\left(-E'(t)\right)dt \\
\leq\ & cE^{q+1}(S)-cE^{q+1}(T) \\
\leq\ & cE^q(0)\,E(S)\leq cE(S).
\end{aligned}
\tag{4.7}
$$

For the second term, we have

$$
\begin{aligned}
& -\int_S^T\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dxdt \\
\leq\ & \left|E^q(t)\int_\Omega u(S)\,u_t(S)\,dx-E^q(t)\int_\Omega u(T)\,u_t(T)\,dx\right| \\
\leq\ & E^q(t)\left|\int_\Omega u(x,S)\,u_t(x,S)\,dx\right|+E^q(t)\left|\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \\
\leq\ & cE^{q+1}(S)+cE^{q+1}(T) \\
\leq\ & cE^q(0)\,E(S)\leq cE(S).
\end{aligned}
\tag{4.8}
$$

For the third term, we use the following Young inequality:

$$XY \leq \frac{\varepsilon}{\lambda_1} X^{\lambda_1} + \frac{1}{\lambda_2 \varepsilon^{\frac{\lambda_2}{\lambda_1}}} Y^{\lambda_2},\ X,\ Y \geq 0,\ \varepsilon > 0 \text{ and } \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1,$$

with $\lambda_1 = m$, $\lambda_2 = \frac{m}{m-1}$.
By (3.4) and Lemma 4.1, we have

$$\begin{aligned}
& -\int_S^T E^q(t) \int_\Omega u(t)\, u_t(t) \left|u_t(t)\right|^{m-2} dxdt \\
\leq\ & \int_S^T E^q(t) \left(\varepsilon c \int_\Omega \left|u(t)\right|^m dx + c_\varepsilon \int_\Omega \left|u_t(t)\right|^m dx \right) dt \\
\leq\ & \varepsilon c \int_S^T E^q(t) \int_\Omega \left|u(t)\right|^m dxdt + c_\varepsilon \int_S^T E^q(t) \left(-E'(t)\right) dt \\
\leq\ & \varepsilon c \int_S^T E^{q+1}(t)\, dt + c_\varepsilon E(S).
\end{aligned} \tag{4.9}$$

For the last term of (4.5), we have

$$\begin{aligned}
& (3 - \beta_1 - \beta_2) \int_S^T E^q(t) \int_\Omega \left|u_t(t)\right|^2 dxdt \\
\leq\ & c \int_S^T E^q(t) \left(\int_\Omega \left|u_t(t)\right|^m dx \right)^{\frac{2}{m}} dt \\
\leq\ & c \int_S^T E^q(t) \left(-E'(t)\right)^{\frac{2}{m}} dt.
\end{aligned} \tag{4.10}$$

By Young's inequality with $\lambda_1 = (q+1)/q$ and $\lambda_2 = q+1$, we have

$$\int_S^T E^q(t) \left(-E'(t)\right)^{\frac{2}{m}} dt \leq \varepsilon c \int_S^T E^{q+1}(t)\, dt + c_\varepsilon \int_S^T \left(-E'(t)\right)^{\frac{2(q+1)}{m}} dt.$$

We take $q = \frac{m}{2} - 1$, to find

$$\int_S^T E^q(t) \left(-E'(t)\right)^{\frac{2}{m}} dt \leq \varepsilon c \int_S^T E^{q+1}(t)\, dt + c_\varepsilon \int_S^T \left(-E'(t)\right) dt.$$

This implies

$$\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m}} dt \le \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \tag{4.11}$$

Substituting (4.11) into (4.10), we obtain

$$(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \le \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \tag{4.12}$$

By insert (4.7), (4.8), (4.9) and (4.12) in (4.5), we arrive at

$$\gamma\int_S^T E^{\frac{m}{2}}(t)\,dt \le \varepsilon c \int_S^T E^{\frac{m}{2}}(t)\,dt + c_\varepsilon E(S).$$

Choosing ε small enough for that

$$\int_S^T E^{\frac{m}{2}}(t)\,dt \le cE(S).$$

By taking T goes to ∞, we get

$$\int_S^\infty E^{\frac{m}{2}}(t)\,dt \le cE(S).$$

By Komornik's integral inequality yields the result. □

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References

[1] Autuori, G., Pucci, P., *Kirchhoff systems with dynamic boundary conditions*, Nonlinear Anal. Theory, Methods Appl., **73**(2010), no. 7, 1952-1965.

[2] Autuori, G., Pucci, P., Salvatori, MC., *Global nonexistence for nonlinear Kirchhoff systems*, Arch. Ration. Mech. Anal., **196**(2010), no. 2, 489-516.

[3] Benaissa, A., Messaoudi, S.A., *Blow-up of solutions for Kirchhoff equation of q-Laplacian type with nonlinear dissipation*, Colloq. Math., **94**(2002), no. 1, 103-109.

[4] Gao, Q., Wang, Y., *Blow-up of the solution for higher-order Kirchhoff-type equations with nonlinear dissipation*, Cent. Eur. J. Math., **9**(2011), no. 3, 686-698.

[5] Georgiev, V., Todorova, G., *Existence of a solution of the wave equation with nonlinear damping and source term*, J. Dfferential Equations., **109**(1994), no. 2, 295-308.

[6] Ghisi, M., Gobbino, M., *Hyperbolic-parabolic singular perturbation for middly degenerate Kirchhoff equations: time-decay estimates*, J. Differential Equations, **245**(2008), no. 10, 2979-3007.

[7] Kirchhoff, G., *Mechanik*, Teubner, 1883.

[8] Komornik, V., *Exact Controllability and Stabilization the Multiplier Method*, Paris: Masson – John Wiley, 1994.

[9] Li, F., *Global existence and blow-up of solutions for a higher-order Kirchhoff-type equation with nonlinear dissipation*, Appl. Math. Lett., **17**(2004), no. 12, 1409-1414.

[10] Messaoudi, S.A., Said Houari, B., *A blow-up result for a higher-order nonlinear Kirchhoff-type hyperbolic equation*, Appl. Math. Lett., **20**(2007), no. 8, 866-871.

[11] Messaoudi, S.A., Talahmeh, A., *Blowup in solutions of a quasilinear wave equation with variable-exponent nonlinearities*, Math. Methods Appl. Sci., **40**(2017), no. 18, 6976-6986.

[12] Ono, K., *Blowing up and global existence of solutions for some degenerate nonlinear wave equations with some dissipation*, Nonlinear Anal. Theory, Methods Appl., **30**(1997), no. 2, 4449-4457.

[13] Ono, K., *Global existence, decay, and blow-up of solutions for some mildly degenerate nonlinear Kirchhoff strings*, J. Differential Equations., **137**(1997), no. 2, 273-301.

[14] Ono, K., *On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation*, Math. Methods Appl. Sci., **20**(1997), no. 2, 151-177.

[15] Wu, S.T., Tsai, L.Y., *Blow-up of solutions for some nonlinear wave equations of Kirchhoff type with some dissipation*, Nonlinear Anal. Theory Methods Appl., **65**(2006), no. 2, 243-264.

[16] Zeng, R., Mu, C.L., Zhou, S.M., *A blow-up result for Kirchhoff-type equations with high energy*, Math. Methods Appl. Sci., **34**(2011), no. 4, 479-486.

Amar Ouaoua
Laboratory of Applied Mathematics and History and Didactics of Mathematics,
Faculty of Sciences,
University of 20 August 1955, Skikda, Algeria
e-mail: `a.ouaoua@univ-skikda.dz & ouaouaama21@gmail.com`

Aya Khaldi
Laboratory of Applied Mathematics and History and Didactics of Mathematics,
Faculty of Sciences,
University of 20 August 1955, Skikda, Algeria
e-mail: `ayakhaldi21@gmail.com`

Messaoud Maouni
Laboratory of Applied Mathematics and History and Didactics of Mathematics,
Faculty of Sciences,
University of 20 August 1955, Skikda, Algeria
e-mail: `m.maouni@univ-skikda.dz & maouni21@gmail.com`