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Inequalities of Hermite-Hadamard type for AH-convex functions

Sever S. Dragomir

Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. Some inequalities of Hermite-Hadamard type for AH-convex functions defined on convex subsets in real or complex linear spaces are given. The case for functions of one real variable is explored in depth. Applications for special means are provided as well.

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1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [41]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [41]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [1]-[19], [22]-[24], [25]-[34] and [35]-[44].

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x,y \in X,\ x \neq y$. Define the segment

$$[x,y] := \{(1-t)x+ty,\ t \in [0,1]\}.$$

We consider the function $f:[x,y] \to \mathbb{R}$ and the associated function

$$g(x,y):[0,1] \to \mathbb{R},\ g(x,y)(t) := f[(1-t)x+ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [20, p. 2], [21, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

Let X be a linear space and C a convex subset in X. A function $f : C \to \mathbb{R}\setminus\{0\}$ is called *AH-convex (concave)* on the convex set C if the following inequality holds

$$f\left((1-\lambda)x + \lambda y\right) \leq (\geq) \frac{1}{(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)}} = \frac{f(x)f(y)}{(1-\lambda)f(y) + \lambda f(x)} \tag{AH}$$

for any $x, y \in C$ and $\lambda \in [0, 1]$.

An important case which provides many examples is that one in which the function is assumed to be positive for any $x \in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)} \leq (\geq) \frac{1}{f\left((1-\lambda)x + \lambda y\right)}$$

for any $x, y \in C$ and $\lambda \in [0, 1]$.

Therefore we can state the following fact:

Criterion 1.1. *Let X be a linear space and C a convex subset in X. The function $f : C \to (0, \infty)$ is AH-convex (concave) on C if and only if $\frac{1}{f}$ is concave (convex) on C in the usual sense.*

If we apply the Hermite-Hadamard inequality (1.2) for the function $\frac{1}{f}$ then we state the following result:

Proposition 1.2. *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0, \infty)$ is AH-convex (concave) on C, then*

$$\frac{f(x)+f(y)}{2f(x)f(y)} \leq (\geq) \int_0^1 \frac{d\lambda}{f\left((1-\lambda)x + \lambda y\right)} \leq (\geq) \frac{1}{f\left(\frac{x+y}{2}\right)} \tag{1.3}$$

for any $x, y \in C$.

Motivated by the above results, in this paper we establish some new Hermite-Hadamard type inequalities for AH-convex (concave) functions, first in the general setting of linear spaces and then in the particular case of functions of a real variable. Some examples for special means are provided as well.

2. Some Hermite-Hadamard type inequalities

The following result holds:

Theorem 2.1. *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0, \infty)$ is AH-convex (concave) on C, then for any $x, y \in C$ we have*

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \le (\ge) \frac{G^2(f(x), f(y))}{L(f(x), f(y))}, \tag{2.1}$$

where the Logarithmic mean of positive numbers a, b is defined as

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \\ \\ a & \text{if } a = b, \end{cases}$$

and the geometric mean is $G = \sqrt{ab}$.

Proof. Let $x, y \in C$ with $x \neq y$. If $f : C \to (0, \infty)$ is AH-convex (concave) on C, then $\frac{1}{f}$ is concave (convex) on C. This implies that the function

$$\varphi_{x,y} : [0,1] \to (0,\infty), \varphi_{x,y}(t) = \frac{1}{f((1-\lambda)x + \lambda y)}$$

is concave (convex) on $[0,1]$ and therefore continuous on $(0,1)$ with $\varphi_{x,y}(0) = \frac{1}{f(x)}$ and $\varphi_{x,y}(1) = \frac{1}{f(y)}$. The function $[0,1] \ni t \mapsto f((1-t)x + ty)$ is continuous on $(0,1)$ and since $f(x), f(y) > 0$ are finite, then the Lebesgue integral $\int_0^1 f((1-t)x + ty)\, dt$ exists and by (AH) we have

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \le (\ge) f(x) f(y) \int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)}. \tag{2.2}$$

If $f(y) = f(x)$, then

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)} = \frac{1}{f(y)}.$$

If $f(y) \neq f(x)$, then by changing the variable $u = \lambda(f(x) - f(y)) + f(y)$ we have

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)} = \frac{\ln f(x) - \ln f(y)}{f(x) - f(y)} = \frac{1}{L(f(x), f(y))}.$$

By the use of (2.2) we get the desired result (2.1). □

Remark 2.2. Using the following well known inequalities

$$H(a,b) \le G(a,b) \le L(a,b)$$

we have

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \le \frac{G^2(f(x), f(y))}{L(f(x), f(y))} \le G(f(x), f(y)) \tag{2.3}$$

for any $x, y \in C$, provided that $f : C \to (0, \infty)$ is AH-convex.

If $f : C \to (0,\infty)$ is AH-concave, then

$$\begin{aligned}\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda &\geq \frac{G^2(f(x),f(y))}{L(f(x),f(y))} \\ &\geq \frac{G(f(x),f(y))}{L(f(x),f(y))}H(f(x),f(y))\end{aligned} \tag{2.4}$$

for any $x, y \in C$.

Theorem 2.3. *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0,\infty)$ is AH-convex (concave) on C, then for any $x, y \in C$ we have*

$$f\left(\frac{x+y}{2}\right) \leq (\geq) \frac{\int_0^1 f((1-\lambda)x+\lambda y) f(\lambda x+(1-\lambda)y)\,d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda}. \tag{2.5}$$

Proof. By the definition of AH-convexity (concavity) we have

$$f\left(\frac{u+v}{2}\right) \leq (\geq) \frac{2f(u)f(v)}{f(u)+f(v)} \tag{2.6}$$

for any $u, v \in C$.

Let $x, y \in C$ and $\lambda \in [0,1]$. If we take in (2.6) $u = (1-\lambda)x+\lambda y$ and $v = \lambda x+(1-\lambda)y$, then we get

$$f\left(\frac{x+y}{2}\right) \leq (\geq) \frac{2f((1-\lambda)x+\lambda y) f(\lambda x+(1-\lambda)y)}{f((1-\lambda)x+\lambda y)+f(\lambda x+(1-\lambda)y)},$$

which is equivalent to

$$\begin{aligned}&\frac{1}{2}f\left(\frac{x+y}{2}\right)[f((1-\lambda)x+\lambda y)+f(\lambda x+(1-\lambda)y)] \\ &\leq (\geq) f((1-\lambda)x+\lambda y) f(\lambda x+(1-\lambda)y).\end{aligned} \tag{2.7}$$

Integrating the inequality on $[0,1]$ over $\lambda \in [0,1]$ and taking into account that

$$\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda = \int_0^1 f(\lambda x+(1-\lambda)y)\,d\lambda$$

we deduce from (2.7) the desired result (2.5). ☐

Remark 2.4. By the Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\begin{aligned}&\int_0^1 f((1-\lambda)x+\lambda y) f(\lambda x+(1-\lambda)y)\,d\lambda \\ &\leq \left[\int_0^1 f^2((1-\lambda)x+\lambda y)\,d\lambda \int_0^1 f^2(\lambda x+(1-\lambda)y)\,d\lambda\right]^{1/2} \\ &= \int_0^1 f^2((1-\lambda)x+\lambda y)\,d\lambda\end{aligned} \tag{2.8}$$

for any $x, y \in C$.

If the function $f : C \to (0, \infty)$ is AH-convex on C, then we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{\int_0^1 f((1-\lambda)x+\lambda y) f(\lambda x+(1-\lambda)y) d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y) d\lambda} \tag{2.9}$$
$$\leq \frac{\int_0^1 f^2((1-\lambda)x+\lambda y) d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y) d\lambda}.$$

If the function $\psi_{x,y}(t) = f((1-t)x+ty)$, for some given $x, y \in C$ with $x \neq y$, is monotonic nondecreasing on $[0, 1]$, then $\chi_{x,y}(t) = f(tx+(1-t)y)$ is monotonic nonincreasing on $[0, 1]$ and by Čebyšev's inequality for monotonic opposite functions we have

$$\int_0^1 f((1-\lambda)x+\lambda y) f(\lambda x+(1-\lambda)y) d\lambda$$
$$\leq \int_0^1 f((1-\lambda)x+\lambda y) d\lambda \int_0^1 f(\lambda x+(1-\lambda)y) d\lambda$$
$$= \left(\int_0^1 f((1-\lambda)x+\lambda y) d\lambda\right)^2.$$

So, for some given $x, y \in C$ with $x \neq y$, $\psi_{x,y}(t) = f((1-t)x+ty)$ is monotonic nondecreasing (nonincreasing) on $[0, 1]$ and if the function $f : C \to (0, \infty)$ is AH-convex on C, then we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{\int_0^1 f((1-\lambda)x+\lambda y) f(\lambda x+(1-\lambda)y) d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y) d\lambda} \tag{2.10}$$
$$\leq \int_0^1 f((1-\lambda)x+\lambda y) d\lambda.$$

If $(X, \|\cdot\|)$ is a normed space, then the function $g : X \to [0, \infty)$, $g(x) = \|x\|^p$, $p \geq 1$ is convex and then the function $f : C \subset X \to (0, \infty)$, $f(x) = \frac{1}{\|x\|^p}$ is AH-*concave* on any convex subset of X which does not contain $\{0\}$.

Utilising (2.1) we have

$$\int_0^1 \frac{d\lambda}{\|(1-\lambda)x+\lambda y\|^p} \geq \frac{1}{L(\|x\|^p, \|y\|^p)}, \tag{2.11}$$

for any linearly independent $x, y \in X$ and $p \geq 1$.

Making use of (2.5) we also have

$$\int_0^1 \frac{d\lambda}{\|(1-\lambda)x+\lambda y\|^p} \tag{2.12}$$
$$\geq \left\|\frac{x+y}{2}\right\|^p \int_0^1 \frac{d\lambda}{\|(1-\lambda)x+\lambda y\|^p \|\lambda x+(1-\lambda)y\|^p}$$

for any linearly independent $x, y \in X$ and $p \geq 1$.

3. More results for scalar case

If the function f is defined on an interval I and $a, b \in I$ with $a < b$, then

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda = \frac{1}{b-a}\int_a^b f(t)\, dt$$

and the inequalities (1.3), (2.1) and (2.5) can be written as

$$\frac{f(a)+f(b)}{2f(a)f(b)} \le (\ge) \frac{1}{b-a}\int_a^b \frac{1}{f(t)} dt \le (\ge) \frac{1}{f\left(\frac{a+b}{2}\right)}, \tag{3.1}$$

$$\frac{1}{b-a}\int_a^b f(t)\, dt \le (\ge) \frac{G^2(f(a), f(b))}{L(f(a), f(b))}, \tag{3.2}$$

and

$$f\left(\frac{a+b}{2}\right) \le (\ge) \frac{\int_a^b f(t) f(a+b-t)\, dt}{\int_a^b f(t)\, dt}, \tag{3.3}$$

respectively, where $f : I \to (0, \infty)$ is assumed to be AH-convex (concave) on I.

The following proposition holds:

Proposition 3.1. *Let $f : I \to (0, \infty)$ be AH-convex (concave) on I. Let $x, y \in \mathring{I}$, the interior of I, then there exists $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$ such that*

$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{\varphi(y)}{f(y)}(y - x) \tag{3.4}$$

holds.

Proof. Let $x, y \in \mathring{I}$. Since the function $\frac{1}{f}$ is concave (convex) then the lateral derivatives $f'_-(y), f'_+(y)$ exists for $y \in \mathring{I}$ and $\left(\frac{1}{f}\right)'_{-(+)}(y) = -\frac{f'_{-(+)}(y)}{f^2(y)}$.

Since $\frac{1}{f}$ is concave (convex) then we have the *gradient inequality*

$$\frac{1}{f(y)} - \frac{1}{f(x)} \ge (\le) \lambda(y)(y - x) = -\lambda(y)(x - y)$$

with $\lambda(y) \in \left[-\frac{f'_+(y)}{f^2(y)}, -\frac{f'_-(y)}{f^2(y)}\right]$, which is equivalent to

$$\frac{1}{f(y)} - \frac{1}{f(x)} \ge (\le) \frac{\varphi(y)}{f^2(y)}(x - y) \tag{3.5}$$

with $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$.

The inequality (3.5) can be also written as

$$1 - \frac{f(y)}{f(x)} \ge (\le) \frac{\varphi(y)}{f(y)}(x - y)$$

or as

$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{\varphi(y)}{f(y)}(y - x)$$

and the inequality (3.4) is proved. □

Corollary 3.2. *Let $f : I \to (0, \infty)$ be AH-convex (concave) on I. If f is differentiable on $\mathring{I}$ then for any $x, y \in \mathring{I}$, we have*

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{f'(y)}{f(y)} (y - x). \tag{3.6}$$

The following result also holds:

Theorem 3.3. *Let $f : I \to (0, \infty)$ be AH-convex (concave) on I. If $a, b \in I$ with $a < b$, then we have the inequality*

$$\frac{1}{b-a} \int_a^b f^2(t)\, dt \leq (\geq) \left[\frac{b-s}{b-a} f(b) + \frac{s-a}{b-a} f(a) \right] f(s) \tag{3.7}$$

for any $s \in [a, b]$.

In particular, we have

$$\frac{1}{b-a} \int_a^b f^2(t)\, dt \leq (\geq) f\left(\frac{a+b}{2}\right) \frac{f(a) + f(b)}{2} \tag{3.8}$$

and

$$\frac{1}{b-a} \int_a^b f^2(t)\, dt \leq (\geq) f(a) f(b). \tag{3.9}$$

Proof. If the function $f : I \to (0, \infty)$ is AH-convex (concave) on I, then the function f is differentiable almost everywhere on I and we have the inequality

$$\frac{f(t)}{f(s)} - 1 \leq (\geq) \frac{f'(t)}{f(t)} (t - s) \tag{3.10}$$

for every $s \in [a, b]$ and almost every $t \in [a, b]$.

Multiplying (3.10) by $f(t) > 0$ and integrating over $t \in [a, b]$ we have

$$\frac{1}{f(s)} \int_a^b f^2(t)\, dt - \int_a^b f(t)\, dt \leq (\geq) \int_a^b f'(t)(t - s)\, dt. \tag{3.11}$$

Integrating by parts we have

$$\int_a^b f'(t)(t - s)\, dt = f(b)(b - s) + f(a)(s - a) - \int_a^b f(t)\, dt$$

and by (3.11) we get the desired result (3.7).

We observe that (3.8) follows by (3.7) for $s = \frac{a+b}{2}$ while (3.9) follows by (3.7) for either $s = a$ or $s = b$. $\square$

Remark 3.4. By the Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\left(\frac{1}{b-a} \int_a^b f(t)\, dt \right)^2 \leq \frac{1}{b-a} \int_a^b f^2(t)\, dt$$

and if we assume that $f : I \to (0, \infty)$ is AH-convex on I, then we have

$$\frac{1}{b-a} \int_a^b f(t)\, dt \leq \left(\frac{1}{b-a} \int_a^b f^2(t)\, dt \right)^{1/2} \leq \sqrt{f\left(\frac{a+b}{2}\right) \frac{f(a) + f(b)}{2}} \tag{3.12}$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \left(\frac{1}{b-a}\int_a^b f^2(t)\,dt\right)^{1/2} \le \sqrt{f(a)\,f(b)}. \tag{3.13}$$

The following result also holds:

Theorem 3.5. *Let $f : I \to (0,\infty)$ be AH-convex (concave) on I. If $a, b \in I$ with $a < b$, then we have the inequality*

$$\begin{aligned}&\int_a^b \ln f(t)\,dt + \frac{1}{f(s)}\int_a^b f(t)\,dt \\ &\le (\ge)\, b-a+(b-s)\ln f(b)+(s-a)\ln f(a)\end{aligned} \tag{3.14}$$

for any $s \in [a,b]$.

In particular, we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \frac{1}{f\left(\frac{a+b}{2}\right)}\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le (\ge)\, 1+\ln\sqrt{f(b)\,f(a)}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \left[\frac{f(b)+f(a)}{2f(a)\,f(b)}\right]\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le (\ge)\, 1+\ln\sqrt{f(b)\,f(a)}.\end{aligned} \tag{3.16}$$

Proof. Integrating the inequality (3.10) over $t \in [a,b]$ we have

$$\frac{1}{f(s)}\int_a^b f(t)\,dt-(b-a) \le (\ge) \int_a^b \frac{f'(t)}{f(t)}(t-s)\,dt. \tag{3.17}$$

Observe that

$$\begin{aligned}\int_a^b \frac{f'(t)}{f(t)}(t-s)\,dt &= \int_a^b (\ln f(t))'(t-s)\,dt \\ &= (t-s)\ln f(t)\big|_a^b - \int_a^b \ln f(t)\,dt \\ &= (b-s)\ln f(b)+(s-a)\ln f(a)-\int_a^b \ln f(t)\,dt\end{aligned}$$

and by (3.17) we get

$$\begin{aligned}&\frac{1}{f(s)}\int_a^b f(t)\,dt-(b-a) \\ &\le (\ge)\,(b-s)\ln f(b)+(s-a)\ln f(a)-\int_a^b \ln f(t)\,dt,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_a^b \ln f(t)\,dt + \frac{1}{f(s)}\int_a^b f(t)\,dt \\ &\leq (\geq)\, b-a+(b-s)\ln f(b)+(s-a)\ln f(a)\end{aligned}$$

for any $s \in [a,b]$.

If we take in (3.14) $s=\frac{a+b}{2}$ then we get the desired result (3.15).

If we take in (3.14) $s=a$ and $s=b$ we get

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(a)}\int_a^b f(t)\,dt \leq (\geq)\, b-a+(b-a)\ln f(b)$$

and

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(b)}\int_a^b f(t)\,dt \leq (\geq)\, b-a+(b-a)\ln f(a),$$

which by addition produces

$$\begin{aligned}&2\int_a^b \ln f(t)\,dt + \frac{1}{f(a)}\int_a^b f(t)\,dt + \frac{1}{f(b)}\int_a^b f(t)\,dt \\ &\leq (\geq)\, 2(b-a)+(b-a)\ln f(b)+(b-a)\ln f(a)\end{aligned}$$

and then

$$\begin{aligned}&\int_a^b \ln f(t)\,dt + \left[\frac{f(b)+f(a)}{2f(a)f(b)}\right]\int_a^b f(t)\,dt \\ &\leq (\geq)\, b-a+(b-a)\ln\sqrt{f(b)f(a)},\end{aligned}$$

which is equivalent to (3.16). □

Remark 3.6. We observe that

$$(b-s)\ln f(b)+(s-a)\ln f(a)=0$$

iff

$$s=\frac{b\ln f(b)-a\ln f(a)}{\ln f(b)-\ln f(a)}=\frac{L(f(a),f(b))}{L\left([f(a)]^a,[f(b)]^b\right)}.$$

If

$$s=\frac{L(f(a),f(b))}{L\left([f(a)]^a,[f(b)]^b\right)}\in I$$

then from (3.14) we have

$$\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \frac{1}{f\left(\frac{L(f(a),f(b))}{L\left([f(a)]^a,[f(b)]^b\right)}\right)}\frac{1}{b-a}\int_a^b f(t)\,dt \leq (\geq)\, 1. \tag{3.18}$$

Let $(X, \|\cdot\|)$ be a normed space and $x, y \in X$ two linearly independent vectors on X. Since the function $g : [0,1] \to (0,\infty)$, $g(t) = \|(1-t)x + ty\|^p$, $p \geq 1$ is convex on $[0,1]$, then the function $f : [0,1] \to (0,\infty)$, $g(t) = \frac{1}{\|(1-t)x+ty\|^p}$ is AH-concave on $[0,1]$.

Making use of the inequalities (3.8) and (3.9) we get

$$\left\|\frac{x+y}{2}\right\|^p \int_0^1 \frac{1}{\|(1-t)x+ty\|^{2p}} dt \geq \frac{\|x\|^p + \|y\|^p}{2\|x\|^p \|y\|^p} \tag{3.19}$$

and

$$\int_0^1 \frac{1}{\|(1-t)x+ty\|^{2p}} dt \geq \frac{1}{\|x\|^p \|y\|^p}. \tag{3.20}$$

4. Applications for special means

Let us recall the following means:

a) The *arithmetic mean*

$$A(a,b) := \frac{a+b}{2}, \quad a,b > 0,$$

b) The *geometric mean*

$$G(a,b) := \sqrt{ab}; \quad a,b \geq 0,$$

c) The *harmonic mean*

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}; \quad a,b > 0,$$

d) The *identric mean*

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad b \neq a \\ a & \text{if} \quad b = a \end{cases}; \quad a,b > 0$$

e) The *logarithmic mean*

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if} \quad b \neq a \\ a & \text{if} \quad b = a \end{cases}; \quad a,b > 0$$

f) The $p-$*logarithmic mean*

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if} \quad b \neq a, \quad p \in \mathbb{R}\setminus\{-1,0\} \\ a & \text{if} \quad b = a \end{cases}; \quad a,b > 0.$$

It is well known that, if $L_{-1} := L$ and $L_0 := I$, then the function $\mathbb{R} \ni p \mapsto L_p$ is monotonically strictly increasing. In particular, we have

$$H(a,b) \leq G(a,b) \leq L(a,b) \leq I(a,b) \leq A(a,b).$$

Consider the function

$$f(t) = t^p = \frac{1}{t^{-p}}$$

if $-p > 1$ or $-p < 0$, i.e. $p \in (-\infty, -1) \cup (0, \infty)$ then the function $f(t) = t^p, t > 0$ is AH-concave. If $p \in (-1, 0)$ then the function $f(t) = t^p, t > 0$ is AH-convex.

Now, if we write the inequality (3.2) for the function $f(t) = t^p$ and $0 < a < b$ we get

$$\frac{1}{b-a}\int_a^b t^p dt \leq (\geq) \frac{G^2(a^p, b^p)}{L(a^p, b^p)}, \tag{4.1}$$

where $p \in (-1, 0)\, (p \in (-\infty, -1) \cup (0, \infty))$.

Now, observe that

$$\frac{1}{b-a}\int_a^b t^p dt = \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} = L_p^p(a,b),$$

$$\begin{aligned} L(a^p, b^p) &= \frac{b^p - a^p}{\ln b^p - \ln a^p} = \frac{b^p - a^p}{p(b-a)} \frac{b-a}{\ln b - \ln a} \\ &= L_{p-1}^{p-1}(a,b)\, L(a,b),\ p \in \mathbb{R} \backslash \{0, 1\} \end{aligned}$$

and

$$G^2(a^p, b^p) = G^{2p}(a,b).$$

Then by (4.1) we get

$$L_p^p(a,b)\, L_{p-1}^{p-1}(a,b)\, L(a,b) \leq (\geq)\, G^{2p}(a,b), \tag{4.2}$$

where $p \in (-1, 0)\, (p \in (-\infty, -1) \cup (0, \infty) \setminus \{1\})$.

If we write the inequality (3.8) for the function $f(t) = t^p$ and $0 < a < b$ we get

$$\frac{1}{b-a}\int_a^b t^{2p} dt \leq (\geq) \left(\frac{a+b}{2}\right)^p \frac{a^p + b^p}{2} \tag{4.3}$$

where $p \in (-1, 0)\, (p \in (-\infty, -1) \cup (0, \infty))$.

Since

$$\frac{1}{b-a}\int_a^b t^{2p} dt = L_{2p}^{2p}(a,b),\ p \in \mathbb{R} \backslash \left\{-\frac{1}{2}, 0\right\},$$

$$\left(\frac{a+b}{2}\right)^p = A^p(a,b),\ \frac{a^p + b^p}{2} = A(a^p, b^p),$$

then by (4.3) we have

$$L_{2p}^{2p}(a,b) \leq (\geq)\, A^p(a,b)\, A(a^p, b^p) \tag{4.4}$$

where $p \in (-1, 0) \setminus \left\{-\frac{1}{2}\right\} (p \in (-\infty, -1) \cup (0, \infty))$.

Now consider the function $f(t) = \ln t$, $t > 1$. The function

$$g(t) := \frac{1}{\ln t},\ t > 1$$

is convex on $(1,\infty)$. If we apply the inequality (3.2) for the AH-concave function $f(t)=\ln t$, $t>1$ on $[a,b]\subset(1,\infty)$, then we get

$$\ln I(a,b)\geq\frac{G^{2}(\ln a,\ln b)}{L(\ln a,\ln b)}. \tag{4.5}$$

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Sever S. Dragomir
Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia

School of Computational & Applied Mathematics
University of the Witwatersrand
Johannesburg 2050, South Africa
e-mail: `sever.dragomir@vu.edu.au`