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Fekete-Szegö problems for generalized Sakaguchi type functions associated with quasi-subordination

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Abstract. In the present paper, the authors introduce a generalized Sakaguchi type non-Bazilevic function class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ of analytic functions involving quasi-subordination and obtain bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for the functions belonging to the above and associated classes. Some important and useful special cases of the main results are also pointed out.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk:

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$$

having the normalized power series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}). \tag{1.1}$$

A function $f(z) \in \mathcal{A}$ is said to be univalent in $\mathbb{U}$ if $f(z)$ is one-to-one in $\mathbb{U}$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ (see [3]).
For two functions f and g in $\mathcal{A}$, we say that f is *subordinate* to g in $\mathbb{U}$, and write as

$$f \prec g \text{ in } \mathbb{U} \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{1.2}$$

If the function g is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For a brief survey on the concept of subordination, we refer to the works in [3, 10, 13, 27].

Further, a function $f(z)$ is said to be quasi-subordinate to $g(z)$ in the unit disk $\mathbb{U}$ if there exists the functions $\varphi(z)$ and $w(z)$ (with constant coefficient zero) which are analytic and bounded by one in the unit disk $\mathbb{U}$ such that

$$\frac{f(z)}{\varphi(z)} \prec g(z) \quad (z \in \mathbb{U}). \tag{1.3}$$

We denote the quasi-subordination by

$$f(z) \prec_q g(z) \quad (z \in \mathbb{U}). \tag{1.4}$$

Also, we note that quasi-subordination (1.4) is equivalent to

$$f(z) = \varphi(z) g(w(z)) \quad (z \in \mathbb{U}). \tag{1.5}$$

One may observe that when $\varphi(z) \equiv 1$ $(z \in \mathbb{U})$, the quasi-subordination $\prec_q$ becomes the usual subordination $\prec$. If we put $w(z) = z$ in (1.5), then the quasi-subordination (1.5) becomes the majorization. In this case, we have

$$f(z) \prec_q g(z) \Longrightarrow f(z) = \varphi(z) g(z) \Longrightarrow f(z) \ll g(z) \quad (z \in \mathbb{U}).$$

The concept of majorization is due to MacGreogor [12] and quasi-subordination is thus a generalization of the usual subordination as well as the majorization. The work on quasi-subordination is quite extensive which includes some recent expository investigations in [1, 7, 9, 14, 21, 22].

Recently, Frasin [5] introduced and studied a generalized Sakaguchi type classes $\mathcal{S}(\alpha, s, t)$ and $\mathcal{T}(\alpha, s, t)$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha, s, t)$ if it satisfies

$$\Re\left[\frac{(s-t) z f'(z)}{f(sz) - f(tz)}\right] > \alpha \tag{1.6}$$

for some α $(0 \leq \alpha < 1)$, $s,\ t \in \mathbb{C}$, $|s-t| \leq 1$, $s \neq t$ and $z \in \mathbb{U}$.

We also denote by $\mathcal{T}(\alpha, s, t)$, the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in \mathcal{S}(\alpha, s, t)$. For $s = 1$, the class $\mathcal{S}(\alpha, 1, t)$ becomes the subclass $\mathcal{S}^*(\alpha, t)$ studied by Owa et al. [17, 18]. If $t = -1$ in $\mathcal{S}(\alpha, 1, t)$, then the class $\mathcal{S}(\alpha, 1, -1) = \mathcal{S}_s(\alpha)$ was introduced by Sakaguchi [23] and is called Sakaguchi function of order α (see [2, 17]), whereas $\mathcal{S}_s(0) \equiv \mathcal{S}_s$ is the class of starlike functions with respect to symmetrical points in $\mathbb{U}$. Further, $\mathcal{S}(\alpha, 1, 0) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{T}(\alpha, 1, 0) \equiv \mathcal{C}(\alpha)$ are the familiar classes of starlike functions of order α $(0 \leq \alpha < 1)$ and convex function of order α $(0 \leq \alpha < 1)$, respectively.

Obradovic [16] introduced a class of functions $f \in \mathcal{A}$ which satisfies the inequality:

$$\Re\left[f'(z)\left(\frac{z}{f(z)}\right)^{1+\lambda}\right] > 0 \quad (0 < \lambda < 1; z \in \mathbb{U}), \tag{1.7}$$

and he calls such functions as functions of non-Bazilevič type.

By $\mathcal{P}$, we denote the class of functions ϕ analytic in $\mathbb{U}$ such that $\phi(0) = 1$ and $\Re(\phi(z)) > 0$.

Ma and Minda [11] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general subordination function. They introduced a class $S^*(\phi)$ defined by

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \mathbb{U}) \right\}, \tag{1.8}$$

where $\phi \in \mathcal{P}$ and $\phi(\mathbb{U})$ is symmetrical about the real axis and $\phi'(0) > 0$. A function $f \in S^*(\phi)$ is called a Ma and Minda starlike function with respect to ϕ.
Recently, Sharma and Raina [25] introduced and studied a generalized Sakaguchi type non-Bazilevic function class $\mathcal{G}_q^\lambda(\phi, b)$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{G}_q^\lambda(\phi, b)$ if it satisfies the condition that

$$\left[f'(z) \left(\frac{(1-b)z}{f(z) - f(bz)} \right)^\lambda - 1 \right] \prec_q (\phi(z) - 1) \quad (1 \neq b \in \mathbb{C}, |b| \leq 1, \lambda \geq 0; z \in \mathbb{U}). \tag{1.9}$$

Motivated by aforementioned works, we introduce here a new subclass of $\mathcal{A}$ which is defined as follows:
Definition 1.1. Let $\phi \in \mathcal{P}$ be univalent and $\phi(\mathbb{U})$ symmetrical about the real axis and $\phi'(0) > 0$. For $s,\ t \in \mathbb{C},\ s \neq t,\ |s - t| \leq 1,\ \lambda,\ \beta \geq 0$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ if it satisfies the condition that

$$\left[(1-\beta) f'(z) + \beta \frac{f(z)}{z} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] \left[\frac{(s-t)z}{f(sz) - f(tz)} \right]^\lambda - 1 \prec_q (\phi(z) - 1) \ (z \in \mathbb{U}), \tag{1.10}$$

where the powers are considered to be having only principal values.

By specializing the parameters $\lambda,\ \beta,\ s,$ and t in Definition 1.1 above, we obtain various subclasses which have been studied recently. To illustrate these subclasses, we observe the following:
(i) When $\beta = 0, s = 1$, then the class $\mathcal{M}_q^{\lambda,0}(\phi, 1, t) = \mathcal{G}_q^\lambda(\phi, t)$ which was studied recently by Sharma and Raina [25].
(ii) Next, when $\beta = t = 0$, $\lambda = s = 1$; $\beta = \lambda = t = 0$, $s = 1$ and $\lambda = \beta = s = 1$, $t = 0$; then the classes $\mathcal{M}_q^{1,0}(\phi, 1, 0)$, $\mathcal{M}_q^{0,0}(\phi, 1, 0)$ and $\mathcal{M}_q^{1,,1}(\phi, 1, 0)$ which, respectively, reduce to the classes $\mathcal{S}_q^*(\phi)$, $\mathcal{R}_q(\phi)$ and $\mathcal{C}_q(\phi)$ were studied earlier by Mohd and Darus [14].

From the Definition 1.1, it follows that $f \in \mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ if and only if there exists an analytic function $\varphi(z)$ with $|\varphi(z)| \leq 1$ $(z \in \mathbb{U})$ such that

$$\frac{\left[(1-\beta) f'(z) + \beta \frac{f(z)}{z} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] \left[\frac{(s-t)z}{f(sz)-f(tz)} \right]^\lambda - 1}{\varphi(z)} \prec (\phi(z) - 1) \ (z \in \mathbb{U}). \tag{1.11}$$

If we set $\varphi(z) \equiv 1$ $(z \in \mathbb{U})$ in (1.11), then the class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ is denoted by $\mathcal{M}^{\lambda,\beta}(\phi, s, t)$ satisfying the condition that

$$\left[(1-\beta) f'(z) + \beta \frac{f(z)}{z} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] \left[\frac{(s-t)z}{f(sz) - f(tz)} \right]^\lambda \prec \phi(z) \quad (z \in \mathbb{U}). \tag{1.12}$$

It may be noted that for $\beta = 0$, $s = \lambda = 1$ and for real t, the class $\mathcal{M}^{1,0}(\phi, 1, t) = \mathcal{S}^*(\phi, t)$ which was studied by Goyal and Goswami [6].

It is well-known (see [3]) that for $f \in \mathcal{S}$ given by (1.1), there holds a sharp inequality for the functional $|a_3 - a_2^2|$. Fekete-Szegö [4] obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ for $f \in \mathcal{S}$ when μ is real and thus the determination of the sharp upper bounds for such a nonlinear functional for any compact family $\mathcal{F}$ of functions in $\mathcal{S}$ is popularly known as the Fekete-Szegö problem for $\mathcal{F}$. Fekete-Szegö problems for several subclasses of $\mathcal{S}$ have been investigated by many authors including [19, 20, 24]; see also [26].

The aim of this paper is to obtain the coefficient estimates including a Fekete-Szegö inequality of functions belonging to the classes $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ and $\mathcal{M}^{\lambda,\beta}(\phi, s, t)$ and the class involving the majorization. Some consequences of the main results are also pointed out.

We need the following lemma in our investigations.

Lemma 1.2. ([8, p.10]) *Let the Schwarz function $w(z)$ be given by*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \mathbb{U}), \tag{1.13}$$

then

$$|w_1| \leq 1, \ |w_2 - \mu w_1^2| \leq 1 + (|\mu| - 1)|w_1|^2 \leq \max\{1, |\mu|\},$$

where $\mu \in \mathbb{C}$. The result is sharp for the function $w(z) = z$ or $w(z) = z^2$.

2. Main results

Let $f \in \mathcal{A}$ of the form (1.1), then for s, $t \in \mathbb{C}$, $|s - t| \leq 1, s \neq t$, we may write that

$$\frac{f(sz) - f(tz)}{s - t} = z + \sum_{n=2}^{\infty} \gamma_n a_n z^n, \tag{2.1}$$

where

$$\gamma_n = \frac{s^n - t^n}{s - t} = s^{n-1} + s^{n-2} t + \cdots + t^{n-1} \quad (n \in \mathbb{N}). \tag{2.2}$$

Therefore for $\lambda \geq 0$, we have

$$\left[\frac{(s-t)z}{f(sz) - f(tz)}\right]^\lambda = 1 - \lambda \gamma_2 a_2 z + \lambda \left[\frac{\lambda + 1}{2} \gamma_2^2 a_2^2 - \gamma_3 a_3\right] z^2 + \cdots . \tag{2.3}$$

Unless otherwise stated, throughout the sequel, we assume that

$$\lambda \gamma_n \neq (n-1)^2 \beta + n;$$

and that for real s, t:

$$\lambda \gamma_n < (n-1)^2 \beta + n, \ n = 2, 3, 4, \ldots .$$

Let the function $\phi \in \mathcal{P}$ be of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots \quad (B_1 \in \mathbb{R}, B_1 > 0), \tag{2.4}$$

and $\varphi(z)$ analytic in $\mathbb{U}$ be of the form

$$\varphi(z) = c_0 + c_1 z + c_2 z^2 + \cdots (c_0 \neq 0). \tag{2.5}$$

We now state and prove our first main result.

Theorem 2.1. *Let the function* $f \in \mathcal{A}$ *of the form* (1.1) *be in the class* $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$, *then*

$$|a_2| \le \frac{B_1}{|2+\beta-\lambda\gamma_2|}, \tag{2.6}$$

and for any $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \le \frac{B_1}{|3+4\beta-\lambda\gamma_3|} \max\left\{1, \left|\frac{B_2}{B_1} - B_1 R\right|\right\}, \tag{2.7}$$

where

$$R = \frac{(3+4\beta-\lambda\gamma_3)\mu}{(2+\beta-\lambda\gamma_2)^2} - \frac{\lambda\left(4+2\beta-(1+\lambda)\gamma_2\right)\gamma_2+4\beta}{2(2+\beta-\lambda\gamma_2)^2} \tag{2.8}$$

and γ_n $(n \in \mathbb{N})$ *is given by* (2.2). *The result is sharp.*

Proof. Let $f \in \mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$. In view of Definition 1.1, there exists then a Schwarz function $w(z)$ given by (1.13) and an analytic function $\varphi(z)$ given by (2.5) such that

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} - 1 = \varphi(z)(\phi(w(z))-1), \tag{2.9}$$

which can be expressed as

$$\begin{aligned}\varphi(z)(\phi(w(z))-1) &= (c_0 + c_1 z + c_2 z^2 + \cdots)\left(B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + \cdots\right)\\ &= c_0 B_1 w_1 z + \left\{c_0(B_1 w_2 + B_2 w_1^2) + c_1 B_1 w_1\right\} z^2 + \cdots. \end{aligned} \tag{2.10}$$

Using now the series expansions for $f'(z)$ and $f''(z)$ from (1.1), we obtain that

$$(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left[1+\frac{zf''(z)}{f'(z)}\right] = 1 + (2+\beta)a_2 z + \left((3+4\beta)a_3 - 2\beta a_2^2\right)z^2 + \cdots. \tag{2.11}$$

Thus, it follows from (2.3) and (2.11) that

$$\begin{aligned}&\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} - 1\\ &= (2+\beta-\lambda\gamma_2)a_2 z + \left[(3+4\beta-\lambda\gamma_3)a_3 - \lambda\left(2+\beta-\frac{1+\lambda}{2}\gamma_2\right)\gamma_2 a_2^2 - 2\beta a_2^2\right]z^2 + \cdots.\end{aligned} \tag{2.12}$$

Making use of (2.10) and (2.12) in (2.9) and equating the coefficients of z and z^2 in the resulting expression, we get

$$(2+\beta-\lambda\gamma_2)a_2 = c_0 B_1 w_1 \tag{2.13}$$

and

$$(3+4\beta-\lambda\gamma_3)a_3 - \lambda\left[2+\beta-\frac{1+\lambda}{2}\gamma_2\right]\gamma_2 a_2^2 - 2\beta a_2^2 = c_0(B_1 w_2 + B_2 w_1^2) + c_1 B_1 w_1. \tag{2.14}$$

Now (2.13) yields that

$$a_2 = \frac{c_0 B_1 w_1}{2+\beta-\lambda\gamma_2}. \tag{2.15}$$

From (2.14), we have

$$a_3 = \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[c_1 w_1 + c_0\left\{w_2 + \left(\frac{c_0\lambda\left(1+\frac{2+\beta-\gamma_2}{2+\beta-\lambda\gamma_2}\right)\gamma_2 B_1}{2(2+\beta-\lambda\gamma_2)}\frac{2\beta c_0 B_1}{(2+\beta-\lambda\gamma_2)^2} + \frac{B_2}{B_1}\right) w_1^2\right\}\right]. \quad (2.16)$$

Hence, for any complex number μ, we have

$$a_3 - \mu a_2^2 = \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[c_1 w_1 + c_0\left\{w_2 + \left(\frac{\lambda(4+2\beta-(1+\lambda)\gamma_2)\gamma_2 + 4\beta}{2(2+\beta-\lambda\gamma_2)^2} c_0 B_1 + \frac{B_2}{B_1}\right) w_1^2\right\}\right] - \mu\frac{c_0^2 B_1^2 w_1^2}{(2+\beta-\lambda\gamma_2)^2}$$

$$= \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[c_1 w_1 + \left(w_2 + \frac{B_2}{B_1}w_1^2\right)c_0 - B_1 c_0^2 w_1^2 R\right], \quad (2.17)$$

where R is given by (2.8).
Since $\varphi(z)$ given by (2.5) is analytic and bounded in the open unit disk $\mathbb{U}$, hence upon using [15, p. 172], we have for some y ($|y| \leq 1$):

$$|c_0| \leq 1 \text{ and } c_1 = (1-c_0^2)y. \quad (2.18)$$

Putting the value of c_1 from (2.18) into (2.17), we finally get

$$a_3 - \mu a_2^2 = \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[y w_1 + \left(w_2 + \frac{B_2}{B_1}w_1^2\right)c_0 - (B_1 w_1^2 R + w_1 y)c_0^2\right]. \quad (2.19)$$

If $c_0 = 0$, then (2.19) gives

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}. \quad (2.20)$$

On the other hand, if $c_0 \neq 0$, then we consider

$$T(c_0) = y w_1 + \left(w_2 + \frac{B_2}{B_1}w_1^2\right)c_0 - (B_1 w_1^2 R + w_1 y)c_0^2. \quad (2.21)$$

The expression (2.21) is a quadratic polynomial in c_0 and hence analytic in $|c_0| \leq 1$. The maximum value of $|T(c_0)|$ is attained at $c_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$, and hence, we have

$$\max|T(c_0)| = \max_{0\leq\theta<2\pi}|T(e^{i\theta})| = |T(1)|$$

$$= \left|w_2 - \left(B_1 R - \frac{B_2}{B_1}\right)w_1^2\right|.$$

Thus from (2.19), we get

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}\left|w_2 - \left(B_1 R - \frac{B_2}{B_1}\right)w_1^2\right|, \quad (2.22)$$

and in view of Lemma 1.2, we obtain that

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}\max\left\{1, \left|B_1 R - \frac{B_2}{B_1}\right|\right\}. \quad (2.23)$$

The desired assertion (2.7) follows now from (2.20) and (2.23).
The result is sharp for the function $f(z)$ given by

$$\left[(1-\beta)f'(z)+\beta\frac{f(z)}{z}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda}=\phi(z)$$

or

$$\left[(1-\beta)f'(z)+\beta\frac{f(z)}{z}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda}=\phi(z^2).$$

This completes the proof of Theorem 2.1. □

By setting $\beta=t=0,\ \lambda=s=1$ in the Theorem 2.1, we obtain the following sharp results for the subclass $\mathcal{S}_q^*(\phi)$.

Corollary 2.2. *Let $f\in\mathcal{A}$ of the form (1.1) be in the class $\mathcal{S}_q^*(\phi)$, then*

$$|a_2|\le B_1,$$

and for any $\mu\in\mathbb{C}$:

$$|a_3-\mu a_2^2|\le\frac{B_1}{2}\max\left\{1,\left|\frac{B_2}{B_1}+(1-2\mu)B_1\right|\right\}.$$

The result is sharp.

Next, putting $\beta=\lambda=s=1$ and $t=0$ in Theorem 2.1, we obtain the following sharp results for the class $\mathcal{C}_q(\phi)$.

Corollary 2.3. *Let $f\in\mathcal{A}$ of the form (1.1) belong to the class $\mathcal{C}_q(\phi)$, then*

$$|a_2|\le\frac{B_1}{2},$$

and for any $\mu\in\mathbb{C}$:

$$|a_3-\mu a_2^2|\le\frac{B_1}{6}\max\left\{1,\left|\frac{B_2}{B_1}+\left(1-\frac{3\mu}{2}\right)B_1\right|\right\}.$$

The result is sharp.

Further, by putting $\beta=\lambda=t=0$ and $s=1$ in Theorem 2.1, we get the following sharp results for the class $\mathcal{R}_q(\phi)$.

Corollary 2.4. *Let $f\in\mathcal{A}$ of the form (1.1) belong to the class $\mathcal{R}_q(\phi)$, then*

$$|a_2|\le\frac{B_1}{2},$$

and for any $\mu\in\mathbb{C}$:

$$|a_3-\mu a_2^2|\le\frac{B_1}{3}\max\left\{1,\left|\frac{B_2}{B_1}-\frac{3\mu}{4}B_1\right|\right\}.$$

The result is sharp.

Remark 2.5. The Fekete-Szegö type inequalities mentioned above for the classes $\mathcal{S}_q^*(\phi)$, $\mathcal{C}_q(\phi)$ and $\mathcal{R}_q(\phi)$ improve similar results obtained earlier in [14].

The next theorem gives the result for the class $\mathcal{M}^{\lambda,\beta}(\phi,s,t)$.

Theorem 2.6. *Let $f \in \mathcal{A}$ of the form (1.1) belong to the class $\mathcal{M}^{\lambda,\beta}(\phi, s, t)$, then*

$$|a_2| \leq \frac{B_1}{|2+\beta-\lambda\gamma_2|},$$

and for any $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|} \max\left\{1, \left|\frac{B_2}{B_1} - B_1 R\right|\right\},$$

where R is given by (2.8) and γ_n $(n \in \mathbb{N})$ is given by (2.2). The result is sharp.

Proof. The proof is similar to Theorem 2.1. Let $f \in \mathcal{M}^{\lambda,\beta}(\phi, s, t)$. If $\varphi(z) \equiv 1$, then (2.5) gives $c_0 = 1$ and $c_n = 0$ $(n \in \mathbb{N})$. Therefore, in view of (2.15), (2.17) and by an application of Lemma 1.2, we obtain the desired assertion. The result is sharp for the function $f(z)$ given by

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^\lambda = \phi(z)$$

or

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^\lambda = \phi(z^2).$$

□

The next theorem gives the result based on majorization.

Theorem 2.7. *Let $s, t \in \mathbb{C},\ s \neq t,\ |s-t| \leq 1$.*

If a function $f \in \mathcal{A}$ of the form (1.1) satisfies

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^\lambda - 1 \ll \phi(z) - 1 \quad (z \in \mathbb{U}), \tag{2.24}$$

then

$$|a_2| \leq \frac{B_1}{|2+\beta-\lambda\gamma_2|},$$

and for any $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|} \max\left\{1, \left|\frac{B_2}{B_1} - B_1 R\right|\right\},$$

where R is given by (2.8) and γ_n $(n \in \mathbb{N})$ is defined as (2.2). The result is sharp.

Proof. Assume that (2.24) holds true. Hence, by the definition of majorization there exists an analytic function $\varphi(z)$ given by (2.5) such that for $z \in \mathbb{U}$ we have

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^\lambda - 1 = \varphi(z)(\phi(z)-1). \tag{2.25}$$

Following similar steps as in the proof of Theorem 2.1 and by setting $w(z) \equiv 1$, so that $w_1 = 1$ and $w_n = 0,\ n \geq 2$, we obtain

$$a_2 = \frac{c_0 B_1}{2+\beta-\lambda\gamma_2},$$

so that

$$|a_2| \leq \frac{B_1}{|2+\beta-\lambda\gamma_2|}$$

and

$$a_3 - \mu a_2^2 = \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[c_1 + \frac{B_2}{B_1}c_0 - B_1 c_0^2 R\right]. \tag{2.26}$$

On putting the value of c_1 from (2.18) in (2.26), we get

$$a_3 - \mu a_2^2 = \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[y + \frac{B_2}{B_1}c_0 - (B_1 R + y)c_0^2\right]. \tag{2.27}$$

If $c_0 = 0$, then (2.27) yields

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}. \tag{2.28}$$

But if $c_0 \neq 0$, then we define the function

$$H(c_0) := y + \frac{B_2}{B_1}c_0 - (B_1 R + y)c_0^2. \tag{2.29}$$

The expression (2.29) is a polynomial in c_0 and hence analytic in $|c_0| \leq 1$. The maximum value of $|H(c_0)|$ occurs at $c_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$, and we have

$$\max_{0\leq\theta<2\pi} H(e^{i\theta}) = |H(1)|.$$

From (2.27), we get

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}\left|B_1 R - \frac{B_2}{B_1}\right|. \tag{2.30}$$

Thus, the assertion of Theorem 2.7 follows from (2.28) and (2.30). The result is sharp for the function given by

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} = \phi(z) \quad (z \in \mathbb{U}).$$

This completes the proof of Theorem 2.7. □

Next, we determine the bounds for the functional $|a_3 - \mu a_2^2|$ for real μ, s and t for the class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$.

Corollary 2.8. *Let the function $f \in \mathcal{A}$ given by (1.1) be in the class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$, then (for real values of μ, s, t):*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[B_1 Q + \frac{B_2}{B_1}\right] & \mu \leq \alpha_1, \\ \frac{B_1}{3+4\beta-\lambda\gamma_3} & \alpha_1 \leq \mu \leq \alpha_1 + 2\rho, \\ -\frac{B_1}{3+4\beta-\lambda\gamma_3}\left[B_1 Q + \frac{B_2}{B_1}\right] & \mu \geq \alpha_1 + 2\rho, \end{cases} \tag{2.31}$$

where

$$\alpha_1 = \frac{\lambda(4+2\beta-(1+\lambda)\gamma_2)\gamma_2 + 4\beta}{2(3+4\beta-\lambda\gamma_3)} - \frac{(2+\beta-\lambda\gamma_2)^2}{(3+4\beta-\lambda\gamma_3)}\left(\frac{1}{B_1} - \frac{B_2}{B_1^2}\right), \tag{2.32}$$

$$\rho = \frac{(2+\beta-\lambda\gamma_2)^2}{(3+4\beta-\lambda\gamma_3)B_1}, \tag{2.33}$$

$$Q = \frac{\lambda\{4+2\beta-(1+\lambda)\gamma_2\}\gamma_2+4\beta-2\mu(3+4\beta-\lambda\gamma_3)}{2(2+\beta-\lambda\gamma_2)^2}$$

and γ_n $(n \in \mathbb{N})$ *is given by* (2.2)*. Each of the estimates in* (2.31) *is sharp.*

Proof. For $s,\ t,\ \mu \in \mathbb{R}$, the above bounds can be obtained from (2.7), respectively, under the following cases:

$$B_1R - \frac{B_2}{B_1} \le -1,\ -1 \le B_1R - \frac{B_2}{B_1} \le 1 \text{ and } B_1R - \frac{B_2}{B_1} \ge 1,$$

where R is given by (2.8). We also note the following:
(i) When $\mu < \alpha_1$ or $\mu > \alpha_1 + 2\rho$, then the equality holds if and only if $w(z) = z$ or one of its rotations.
(ii) When $\alpha_1 < \mu < \alpha_1 + 2\rho$, then the inequality holds if and only if $w(z) = z^2$ or one of its rotation.
(iii) Equality holds for $\mu = \alpha_1$ if and only if $w(z) = \frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \le \epsilon \le 1)$ or one of its rotations, while for $\mu = \alpha_1 + 2\rho$, the equality holds if and only if $w(z) = -\frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \le \epsilon \le 1)$, or one of its rotations. □

The second part of assertion in (2.31) can be improved further.

Theorem 2.9. *Let* $f \in \mathcal{A}$ *of the form* (1.1) *belong to the class* $\mathcal{M}_q^{\lambda,\beta}(\phi,s,t)$*, then (for* $s,t,\mu \in \mathbb{R}$ $(\alpha_1 \le \mu \le \alpha_1 + 2\rho)$*)*

$$|a_3 - \mu a_2^2| + (\mu - \alpha_1)|a_2|^2 \le \frac{B_1}{3+4\beta-\lambda\gamma_3} \quad (\alpha_1 \le \mu \le \alpha_1 + \rho) \tag{2.34}$$

and

$$|a_3 - \mu a_2^2| + (\alpha_1 + 2\rho - \mu)|a_2|^2 \le \frac{B_1}{3+4\beta-\lambda\gamma_3} \quad (\alpha_1 + \rho, < \mu < \alpha_1 + 2\rho) \tag{2.35}$$

where α_1 *and* ρ *are given by* (2.32) *and* (2.33)*, respectively, and* γ_3 *is given by* (2.2)*.*

Proof. Let $f \in \mathcal{M}_q^{\lambda,\beta}(\phi,s,t)$. For $s,t,\mu \in \mathbb{R}$ and $\alpha_1 \le \mu \le \alpha_1 + \rho$, and in view of (2.15) and (2.22), we get

$$|a_3 - \mu a_2^2| + (\mu - \alpha_1)|a_2|^2 \le \frac{B_1}{3+4\beta-\lambda\gamma_3}$$

$$\cdot \left[|w_2| - \frac{B_1(3+4\beta-\lambda\gamma_3)}{(2+\beta-\lambda\gamma_2)^2}(\mu-\alpha_1-\rho)|w_1|^2 + \frac{B_1(3+4\beta-\lambda\gamma_3)}{(2+\beta-\lambda\gamma_2)^2}(\mu-\alpha_1)|w_1|^2 \right].$$

Hence, by virtue of Lemma 1.2, we have

$$|a_3 - \mu a_2^2| + (\mu - \alpha_1)|a_2|^2 \le \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[1 - |w_1|^2 + |w_1|^2\right],$$

which yields the assertion (2.34).
If $\alpha_1 + \rho < \mu < \alpha_1 + 2\rho$, then again from (2.15) and (2.22) and Lemma 1.2, we obtain

$$|a_3 - \mu a_2^2| + (\alpha_1 + 2\rho - \mu)|a_2|^2 \le \frac{B_1}{3+4\beta-\lambda\gamma_3}$$

$$\cdot \left[|w_2| + \frac{B_1(3+4\beta-\lambda\gamma_3)}{(2+\beta-\lambda\gamma_2)^2}(\mu-\alpha_1-\rho)|w_1|^2 + \frac{B_1(3+4\beta-\lambda\gamma_3)}{(2+\beta-\lambda\gamma_2)^2}(\alpha_1+2\rho-\mu)|w_1|^2\right]$$

$$\leq \frac{B_1}{3+4\beta-\lambda\gamma_3}[1-|w_1|^2+|w_1|^2],$$

which gives the estimate (2.35). □

We conclude this paper by remarking that the above theorems include several previously established results for particular values of the parameters λ, s, t and β. Thus, if we set $\beta = 0, s = 1$ in Theorems 2.1 and 2.6 above, we arrive at the Fekete-Szegö type inequalities for the classes $\mathcal{G}_q^\lambda(\phi, t)$ and $\mathcal{G}^\lambda(\phi, t)$, respectively, studied by Sharma and Raina [25]. Further, the majorization result and improvement of bounds given by Theorems 2.7 and 2.9 provide extensions of similar results due to Sharma and Raina [25].

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