Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 53–62

Certain sufficient conditions for parabolic starlike and uniformly close-to-convex functions

Richa Brar and Sukhwinder Singh Billing

Abstract. In the present paper, we study certain differential subordinations and obtain sufficient conditions for parabolic starlikeness and uniformly close-to-convexity of analytic functions.

Mathematics Subject Classification (2010): 30C80, 30C45.

Keywords: Analytic function, univalent function, parabolic starlike function, uniformly close-to-convex function, differential subordination.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{E} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore, Taylor's series expansion of $f \in \mathcal{A}$, is given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)),\ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called a dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant

is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}$ is said to be parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{E}. \tag{1.2}$$

The class of parabolic starlike functions is denoted by $\mathcal{S_P}$. A function $f \in \mathcal{A}$ is said to be uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - 1\right|, z \in \mathbb{E}, \tag{1.3}$$

for some $g \in \mathcal{S_P}$. Let UCC denote the class of all such functions. Note that the function $g(z) \equiv z \in \mathcal{S_P}$. Therefore, for $g(z) \equiv z$, condition (1.3) becomes:

$$\Re\left(f'(z)\right) > |f'(z) - 1|, z \in \mathbb{E}. \tag{1.4}$$

Define the parabolic domain Ω as under:

$$\Omega = \{u + iv : u > \sqrt{(u-1)^2 + v^2}\}.$$

Note that the conditions (1.2) and (1.4) are equivalent to the condition that $\dfrac{zf'(z)}{f(z)}$ and $f'(z)$ take values in the parabolic domain Ω respectively.
Ronning [8] and Ma and Minda [4] showed that the function defined by

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \tag{1.5}$$

maps the unit disk $\mathbb{E}$ onto the parabolic domain Ω. Therefore, the condition (1.2) is equivalent to

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \prec q(z), z \in \mathbb{E}, \tag{1.6}$$

and condition (1.4) is same as

$$\Re\left(f'(z)\right) \prec q(z), z \in \mathbb{E}, \tag{1.7}$$

where q(z) is given by (1.5).
It has always been a matter of interest for the researchers to find sufficient conditions for uniformly starlike and close-to-convex functions. The operators $f'(z), \dfrac{zf'(z)}{f(z)}, 1 + \dfrac{zf''(z)}{f'(z)}$ have played an important role in the theory of univalent functions. Various classes involving the combinations of above differential operators have been introduced in literature by different authors. For $f \in \mathcal{A}$, define differential operator $J(\alpha; f)$ as follows:

$$J(\alpha; f)(z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right), \alpha \in \mathbb{R}.$$

In 1973, Miller et al. [5] studied the class $\mathcal{M}_\alpha$ (known as the class of α-convex functions) defined as follows:

$$\mathcal{M}_\alpha = \{f \in \mathcal{A} : \Re[J(\alpha; f)(z)] > 0, z \in \mathbb{E}\}.$$

They proved that if $f \in \mathcal{M}_\alpha$, then f is starlike in $\mathbb{E}$. In 1976, Lewandowski et al. [3] proved that if $f \in \mathcal{A}$ satisfies the condition

$$\Re \frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{E},$$

then f is starlike in $\mathbb{E}$. Further, Silverman [9] defined the class $\mathcal{G}_b$ by taking quotient of operators $1+\dfrac{zf''(z)}{f'(z)}$ and $\dfrac{zf'(z)}{f(z)}$:

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, z \in \mathbb{E} \right\}.$$

The class $\mathcal{G}_b$ had been studied by Tuneski ([7], [12]). For $f \in \mathcal{A}$, define differential operator $I(\alpha; f)$ as follows:

$$I(\alpha; f)(z) = (1-\alpha)f'(z) + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right), \alpha \in \mathbb{R}.$$

Let $\mathcal{H}_\alpha(\beta)$ be the class of normalized analytic functions defined in $\mathbb{E}$ which satisfy the condition

$$\Re[I(\alpha; f)(z)] > \beta, z \in \mathbb{E},$$

where α and β are pre-assigned real numbers. The class $\mathcal{H}_\alpha(0)$ was introduced and studied by Al-Amiri and Reade [1] in 1975. They proved that the members of $\mathcal{H}_\alpha(0)$ are univalent for $\alpha \le 0$. In 2005, Singh et al. [11] studied the class $\mathcal{H}_\alpha(\alpha)$ and proved that the functions in $\mathcal{H}_\alpha(\alpha)$ are univalent for $0 < \alpha < 1$. Recently, the class $\mathcal{H}_\alpha(\beta)$ has been studied by Singh et al. [10]. They established that members of $\mathcal{H}_\alpha(\beta)$ are univalent for $\alpha \le \beta < 1$. In the present paper, we use the technique of differential subordination to study differential operators $I(\alpha; f)(z)$ and $J(\alpha; f)(z)$ and we obtain certain sufficient conditions for uniformly close-to-convex and parabolic starlike functions in terms of differential subordinations involving the operators $I(\alpha; f)(z)$ and $J(\alpha; f)(z)$. To prove our main results, we shall use the following lemma of Miller and Mocanu [6].

Lemma 1.1. *Let q be a univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q(z)$ and suppose that either*
(i) h is convex, or
(ii) Q is starlike.
In addition, assume that
(iii) $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ for all z in $\mathbb{E}$. If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)], z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. Main result

Theorem 2.1. *If $f \in \mathcal{A}$, satisfies the differential subordination*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}$$

$$+\alpha\left\{1+\frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}, z \in \mathbb{E}, \tag{2.1}$$

for $0<\alpha\leq 1$, *then*

$$f'(z) \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, z \in \mathbb{E} \ i.e.\ f \in UCC.$$

Proof. Let us define the function θ and ϕ as follows:

$$\theta(w)=(1-\alpha)w+\alpha$$

and

$$\phi(w)=\frac{\alpha}{w}.$$

Obviously, the function θ and ϕ are analytic in domain $\mathbb{D}=\mathbb{C}\setminus\{0\}$ and $\phi(w)\neq 0$ in $\mathbb{D}$. Define the functions Q and h as follows:

$$Q(z)=zq'(z)\phi(q(z))=\frac{\alpha zq'(z)}{q(z)}$$

and

$$h(z)=\theta(q(z))+Q(z)=(1-\alpha)q(z)+\alpha\left(1+\frac{zq'(z)}{q(z)}\right).$$

Further, select the functions $p(z)=f'(z), f\in\mathcal{A}$ and $q(z)=1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, we obtain (2.1) reduces to

$$(1-\alpha)p(z)+\alpha\left(1+\frac{zp'(z)}{p(z)}\right) \prec (1-\alpha)q(z)+\alpha\left(1+\frac{zq'(z)}{q(z)}\right)=h(z). \tag{2.2}$$

Now,

$$Q(z)=\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} \tag{2.3}$$

and

$$\frac{zQ'(z)}{Q(z)}=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}-\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}. \tag{2.4}$$

It can easily be verified that $\Re\dfrac{zQ'(z)}{Q(z)} > 0$ in $\mathbb{E}$ and hence Q is starlike in $\mathbb{E}$.
Also we have

$$h(z) = (1-\alpha)\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\} + \alpha\left\{1 + \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} + \left(\frac{1-\alpha}{\alpha}\right)\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}.$$

For $0 < \alpha \leq 1$, we have $\Re\dfrac{zh'(z)}{Q(z)} > 0$.
The proof, now, follows from (2.2) by the use of Lemma 1.1. □

Theorem 2.2. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \alpha\left\{\frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}, z \in \mathbb{E}, \tag{2.5}$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in S_{\mathcal{P}}.$$

Proof. Let us define the function θ and ϕ as follows:

$$\theta(w) = w$$

and

$$\phi(w) = \frac{\alpha}{w}.$$

Obviously, the function θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Define Q and h as under:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\alpha zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \frac{\alpha zq'(z)}{q(z)}.$$

On writing $p(z) = \dfrac{zf'(z)}{f(z)}$, $f \in \mathcal{A}$ and $q(z) = 1 + \dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, (2.5) becomes

$$p(z) + \frac{\alpha z p'(z)}{p(z)} \prec q(z) + \frac{\alpha z q'(z)}{q(z)}. \tag{2.6}$$

Here Q is given by (2.3) and $\frac{zQ'(z)}{Q(z)}$ is given by (2.4). It can easily be verified that $\Re\dfrac{zQ'(z)}{Q(z)} > 0$ in $\mathbb{E}$ and hence Q is starlike in $\mathbb{E}$. Further

$$h(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \alpha\left\{\frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

and therefore, we have

$$\frac{zh'(z)}{Q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$
$$+\left(\frac{1}{\alpha}\right)\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}.$$

Since $\alpha > 0$, therefore, we have $\Re\dfrac{zh'(z)}{Q(z)} > 0$.

Thus, the proof follows from (2.6) by the use of Lemma 1.1. □

3. Deductions

Setting $\alpha = 1$ in Theorem 2.1, we get:

Corollary 3.1. *If $f \in \mathcal{A}$ satisfies*

$$\frac{zf''(z)}{f'(z)} \prec \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z \in \mathbb{E},$$

then

$$f'(z) \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in UCC.$$

Writing $\alpha = \frac{1}{2}$ in Theorem 2.1, we obtain:

Corollary 3.2. *Let $f \in \mathcal{A}$ satisfy the differential subordination*

$$f'(z) + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} = F(z), \tag{3.1}$$

then

$$f'(z) \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in UCC.$$

Remark 3.3. In 2011, Billing et al. [2] proved the following result:
If $f \in \mathcal{A}$ satisfies the condition

$$\left|f'(z) + \frac{zf''(z)}{f'(z)} - 1\right| < \frac{5}{6}, z \in \mathbb{E}, \tag{3.2}$$

then $f \in UCC$.

Note that, Corollary 3.2 is a particular case of Theorem 2.1 corresponding to the above result (given by (3.2)). For comparison, we plot the image of unit disk under the function F(z) given by (3.1) and this image is given by light shaded portion of Figure 3.1. We notice that, by virtue of Corollary 3.2 the differential operator $f'(z)+\frac{zf''(z)}{f'(z)}$ takes values in the whole shaded portion of the Figure 3.1 to conclude that $f \in UCC$, whereas by (3.2) the same operator can take values only in a disk of radius 5/6 centered at 1 (shown by dark portion of Figure 3.1) to conclude the same result. Thus, the region for variability of operator $f'(z) + \frac{zf''(z)}{f'(z)}$ is extended largely in Corollary 3.2.

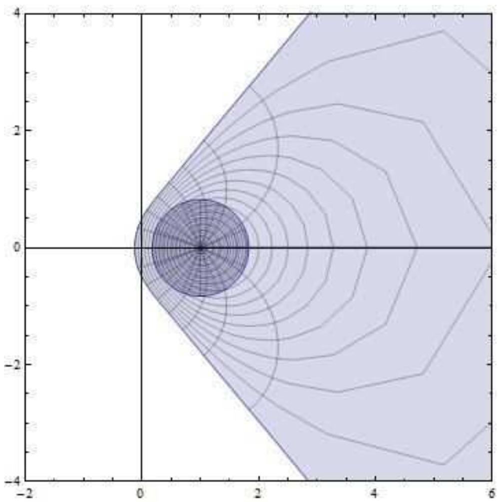

Figure 3.1

Taking $\alpha = 1$ in Theorem 2.2, we have the following result.

Corollary 3.4. *Suppose that $f \in \mathcal{A}$ satisfies*

$$\frac{zf''(z)}{f'(z)} \prec \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} = G(z), \tag{3.3}$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \;\; i.e.\ f \in S_{\mathcal{P}}.$$

Remark 3.5. In 2011, Billing et al. [2] also proved the following result which gives the parabolic starlikeness for the functions belonging to the class $\mathcal{A}$:
If $f \in \mathcal{A}$ satisfies the differential inequality

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{5}{6},\ z \in \mathbb{E}, \tag{3.4}$$

then $f \in S_{\mathcal{P}}$.

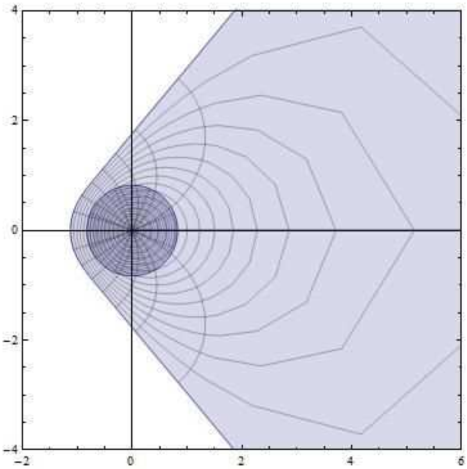

Figure 3.2

Clearly, Corollary 3.4 is a particular case of Theorem 2.2 corresponding to the above result given by (3.4). For comparison, we plot the image of unit disk under the function G(z) given by (3.3) and this image is shown in the light shaded portion of Figure 3.2. In the light of Corollary 3.4, the differential operator $\frac{zf''(z)}{f'(z)}$ takes values in the whole shaded portion of the Figure 3.2 to conclude that $f \in S_{\mathcal{P}}$, but (3.4) indicates that for the same conclusion, operator $\frac{zf''(z)}{f'(z)}$ can take values only in the disk of radius 5/6 centered at origin and this portion is shown by dark portion of Fig 3.2. Thus, the region for variability of operator $\frac{zf''(z)}{f'(z)}$ has been extended largely.

On writing $\alpha = \frac{1}{2}$ in Theorem 2.2, we get:

Corollary 3.6. *If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{4}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in S_{\mathcal{P}}.$$

References

[1] Al-Amiri, H.S., Reade, M.O., *On a linear combination of some expressions in the theory of univalent functions*, Monatshefto für Mathematik, **80**(1975), 257-264.

[2] Billing, S.S., Gupta, S., Dhaliwal, S.S., *Certain differential inequalities involving multivalent functions*, Int. J. Open Problems Complex Analysis, **3**(2011), no. 2, 43-49.

[3] Lewandowski, Z., Miller, S.S., Zlotkiewicz, E., *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc., **56**(1976), 111-117.

[4] Ma, W.C., Minda, D., *Uniformly convex functions*, Ann. Polon. Math., **57**(1992), no. 2, 165-175.

[5] Miller, S.S., Mocanu, P.T., Reade, M.O., *All α convex functions are univalent and starlike*, Proc. Amer. Math. Soc., **37**(1973), 553-554.

[6] Miller, S.S., Mocanu, P.T., *Differential subordination: Theory and applications*, Marcel Dekker, New York and Basel, 2000.

[7] Obradović, M., Tuneski, N., *On the starlike criteria defined by Silverman*, Zeszyty Nauk. Politech. Rzeszowskiej.Mat., **181**(2000), no. 24, 59-64.

[8] Ronning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118**(1993), no. 1, 189-196.

[9] Silverman, H., *Convex and starlike criteria*, Int. J. Math. and Math. Sci., **22**(1999), no. 1, 75-79.

[10] Singh, S., Gupta, S., Singh, S., *On a problem of univalence of functions satisfying a differential inequality*, Math. Inequal. Appl., **10**(2007), no. 1, 95-98.

[11] Singh, V., Singh, S., Gupta, S., *A problem in the theory of univalent functions*, Integral Transforms and Special Functions, **16**(2005), no. 2, 179-186.

[12] Tuneski, N., *On the quotient of the representations of convexity and starlikeness*, Math. Nachr., **248-249**(2003), no. 1, 200-203.

Richa Brar
"Sri Guru Granth Sahib World" University
Department of Mathematics
Fatehgarh Sahib, 140 407 Punjab, India
e-mail: richabrar4@gmail.com

Sukhwinder Singh Billing
"Sri Guru Granth Sahib World" University
Department of Mathematics
Fatehgarh Sahib, 140 407 Punjab, India
e-mail: ssbilling@gmail.com