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Uniquely clean 2×2 invertible integral matrices

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Abstract. While units in any unital ring are strongly clean by definition, which units are uniquely clean, is a far from being simple question, even in particular rings. In this paper, the question is solved for 2×2 integral matrices. It turns out that uniquely clean invertible matrices are scarce: only the matrices similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The study is splitted into three cases: the elliptic, the parabolic and the hyperbolic cases, according to the discriminant of their characteristic polynomial. In the first two cases, units are not uniquely clean.

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1. Introduction

Let R be a ring with identity. An element $r \in R$ is called *clean* if $r = e + u$ with idempotent e and unit u. It is called *uniquely* clean if it has only one clean decomposition, and *strongly* clean if the components of the decomposition commute.

Clean elements which use trivial idempotents (hereafter called *trivial* clean) are obviously strongly clean. That is, units and sums $1 + u$ with unit u are strongly clean.

However, when are such elements (also) uniquely clean turns out to be a difficult question even for particular unital rings.

In this paper we give a complete answer to this question for $R = \mathcal{M}_2(\mathbf{Z})$, that is, we show that only matrices U (with determinant -1 and trace 0) similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are uniquely clean invertible 2×2 integral matrices.

Since units already have the (strongly) clean 0_2-decomposition, a unit $U \in \mathcal{M}_2(\mathbf{Z})$ is uniquely clean iff U is not nontrivial clean and $U - I_2$ is not a unit. Notice that $\det(U - I_2) = \det U - \mathrm{Tr}(U) + 1$ and so

Lemma 1.1. *Suppose U is a unit. Then*

(a) for $\det U = 1$, *$U - I_2$ is a unit iff* $\mathrm{Tr}(U) \in \{1, 3\}$, *and*

(b) for $\det U = -1$, $U - I_2$ *is a unit iff* $\mathrm{Tr}(U) \in \{\pm 1\}$.

Any 2×2 integral matrix U has a characteristic polynomial $X^2 - \mathrm{Tr}(U)\cdot X + \det U$, whose discriminant is $\Delta = \mathrm{Tr}^2(U) - 4\det U$.

If U is a unit, then $\det U \in \{\pm 1\}$. In what follows we separately deal with the *elliptic, parabolic and hyperbolic cases* according to $\Delta < 0$, $\Delta = 0$ and $\Delta > 0$ respectively.

Definition 1.2. Two 2×2 matrices A, B over any unital ring R, are *similar* (or *conjugate*) if there is an invertible matrix U such that $B = U^{-1}AU$. Since similarity is obviously an equivalence relation, a partition of $\mathcal{M}_2(R)$ corresponds to it. The subsets in this partition are called *similarity classes*.

Such classes may consist only in one matrix, for instance, 0_2 respectively I_2. So is every scalar matrix (since it belongs to the center), and generally, a matrix A forms a singleton class iff $AU = UA$ for every invertible matrix U.

If A is idempotent (or unit) and B is similar to A then B is also idempotent (respectively unit). This similarity invariance clearly extends to clean matrices and it also restricts to uniquely or strongly clean matrices, respectively. Rephrasing, the notions of clean, uniquely clean and strongly clean are similarity invariants. So is the clean index.

Further, recall that for $R = \mathbf{Z}$, if $f(t) = t^n + a_1 t^{n-1} + \ldots + a_n$ is irreducible in $\mathbf{Q}[t]$ and ω is a root of $f(t) = 0$ then, according to Latimer and MacDuffee theorem (see e. g. [7]), in the elliptic case, there is a one-to-one correspondence between ideal classes in the ring of integers of the field $\mathbf{Q}[\omega]$ and $\mathbf{Z}$-similarity classes of $n \times n$ matrices A of integers which satisfy $f(A) = 0$. The common number is (finite and called) *the class number* of $\mathbf{Z}[\omega]$.

The answer to our question above amounts to several results from Number Theory related to (positive) quadratic forms. However, it was not necessary to use such results because of the transfer done directly to similarity classes of integral 2×2 matrices done in Behn, Van der Merwe paper (see [4]). From this paper we recall the following definitions and results.

Definition 1.3. A 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\Delta = \mathrm{Tr}(A)^2 - 4\det(A) < 0$ is *reduced* if $|d - a| \le c \le -b$ and, $d \ge a$ if at least one is equality, i.e. $|d-a| = c$ or $c = -b$. Notice that if $|d - a| < c < -b$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} d & b \\ c & a \end{bmatrix}$ are different reduced matrices.

An integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\Delta = \mathrm{Tr}^2(A) - 4\det(A) > 0$ but not a square in $\mathbf{Z}$ is *reduced* if $c > 0$ and $\left|\sqrt{\Delta} - 2c\right| < d - a < \sqrt{\Delta}$.

If Δ is a square (e.g. $\det(A) = 0$), that is, the characteristic polynomial of the matrix factors over the integers, say, $f(x) = (x-a)(x-d)$, where $a \ge d$, then, for $a \ne d$ the matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is *reduced* if $0 \le b < a - d$, and, for $a = d$, if $b \ge 0$. While our results are up to a similarity, in this case it is sufficient to define *upper triangular* *reduced* matrices because, if $f(x) = (x-a)(x-d)$ (and $a \geq d$) then A is similar to a matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $0 \leq b \leq |a-d|$, and for $a = d$, with $b \geq 0$.

For integers x, y and $y \neq 0$, $r(x,y)$ will denote the unique integer such that $r \equiv x \bmod 2y$ and $-|y| < r \leq |y|$ if $|y| > \sqrt{\Delta}$, and $\sqrt{\Delta} - 2|y| < r < \sqrt{\Delta}$ if $|y| < \sqrt{\Delta}$.

Theorem 1.4. ([4], *Theorem 3.3*) *Consider matrices in $\mathcal{M}_2(\mathbf{Z})$ with a fixed trace and determinant and $\Delta = \mathrm{Tr}^2 - 4\det < 0$. Then there is precisely one reduced matrix in each matrix class.*

Theorem 1.5. ([4], *Theorem 5.2*) *Let $M \in \mathcal{M}_2(\mathbf{Z})$, and assume that the characteristic polynomial of M factors over $\mathbf{Z}$. Then M is equivalent to a reduced matrix. Moreover, this class representative is unique thus no two different reduced matrices are equivalent.*

Theorem 1.6. ([4], *Theorem 4.3*) *Consider all matrices A in $\mathcal{M}_2(\mathbf{Z})$ with a fixed trace and determinant. If $\Delta = \mathrm{Tr}^2(A) - 4\det(A) > 0$ is not a square in $\mathbf{Z}$ then there is precisely one cycle of reduced matrices in each matrix class. Thus for each matrix class there is a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the class and a positive integer n such that $P^i \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for $0 \leq i \leq n$ are all the reduced matrices in the class and*

$$P^{n+1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where P denotes a reduction operator on the matrix, namely a conjugation *with* $\begin{bmatrix} 0 & -1 \\ 1 & -n \end{bmatrix}$, *if $-b > 0$ and with* $\begin{bmatrix} 0 & 1 \\ 1 & n \end{bmatrix}$, *if $b > 0$, where $n = \dfrac{r(a-d,b) + d - a}{2b}$.*

Finally recall the following characterization (partly hidden in [2]).

Theorem 1.7. *A 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nontrivial clean iff the system*

$$x^2 + x + yz = 0 \tag{1.1}$$

$$(a-d)x + cy + bz + \det(A) - d = \pm 1 \tag{1.2}$$

with unknowns x, y, z, has at least one solution over $\mathbf{Z}$. If $b \neq 0$ and (1.2) holds, then (1.1) is equivalent to

$$bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0. \tag{1.3}$$

The equation (1.3) is a quadratic Diophantine equation in x an y, and its type (elliptic, parabolic, or hyperbolic) is defined by its discriminant ([3, p.119-120]). In our case we have $\Delta = (a-d)^2 + 4bc = \mathrm{Tr}^2(A) - 4\det(A)$.

2. The elliptic case

Theorem 2.1. *Units in the elliptic case are not uniquely clean.*

Proof. First notice that in this case, $\mathrm{Tr}^2(U) - 4\det U < 0$. This happens only if $\det U = 1$ and $\mathrm{Tr}^2(U) < 4$.

Hence units U in the elliptic case have $\det U = 1$ and $\mathrm{Tr}(U) \in \{-1, 0, 1\}$. Comparing with Lemma 1.1, for $\det U = 1$ only $\mathrm{Tr}(U) \in \{-1, 0\}$ are suitable. Therefore we go into 2 cases.

(i) If $\mathrm{Tr}(U) = -1$, the characteristic polynomial for such matrices is $X^2 + X + 1$. Such matrices are of form

$$U = \begin{bmatrix} a & b \\ c & -a-1 \end{bmatrix} \tag{2.1}$$

with $a(a+1) + bc = -1$. The discriminant $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = -3$ which has class number 1 (see e.g. [5], p. 229).

To find the reduced matrix it suffices to reduce any representative of this similarity class, say $\begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix}$. All matrices of type (2.1) are (not) uniquely clean iff the reduced representative is so. This is (see [4], p. 7) $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ and it is readily seen that this matrix is **not** uniquely clean. It has 3 nontrivial clean decompositions:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Hence the matrices of type (2.1) are not uniquely clean.

(ii) If $\mathrm{Tr}(U) = 0$, by Cayley-Hamilton theorem, $U^2 + I_2 = 0_2$, i.e. $U^2 = -I_2$ and so

$$U = \begin{bmatrix} a & b \\ -\dfrac{a^2+1}{b} & -a \end{bmatrix}$$

for integers a, b with b a nonzero divisor of $a^2 + 1$.

Again, the discriminant $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = -4$ which has (also) class number 1, and we argue as in the previous case. The reduced representative of this similarity class is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Alternatively, it suffices to notice that matrices with $b \in \{\pm 1\}$ in this class, are **not** uniquely clean:

$$\begin{aligned} \begin{bmatrix} a & \pm 1 \\ \mp(a^2+1) & -a \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ \mp a & 0 \end{bmatrix} + \begin{bmatrix} a-1 & \pm 1 \\ \mp a^2 \pm a \mp 1 & -a \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ \pm a & 1 \end{bmatrix} + \begin{bmatrix} a & \pm 1 \\ \mp(a^2+a+1) & -a-1 \end{bmatrix}. \end{aligned}$$

This includes the reduced representative above. $\square$

3. The parabolic case

A unit u is called *unipotent* if $u = 1 + t$ with nilpotent t. The units, in the parabolic case, are precisely the unipotents (including I_2) and negatives of unipotents.

Indeed, in this case we have $\det U = 1$ and $\mathrm{Tr}(U) \in \{-2, 2\}$. The characteristic polynomial is now $X^2 \pm 2X + 1 = (X \pm 1)^2$ and so by Cayley-Hamilton theorem, we have to consider two cases: either $(U - I_2)^2 = 0_2$, i.e., $U = I_2 + T$ is unipotent (with nilpotent T), or else $(U + I_2)^2 = 0_2$, i.e., $-U = I_2 - T$ is unipotent.

Since we intend to prove that units in the parabolic case are not uniquely clean, in the proof of the next theorem, we deal with the first case, i.e. $\det = 1$ and $\mathrm{Tr}(U) = -2$. Matrices in this case are of form $\begin{bmatrix} a & b \\ c & -a-2 \end{bmatrix}$ with $a(a+2) + bc = -1$, i.e. $bc = -(a+1)^2$. The discriminant is now $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = 0$. The proof in the second case is analogous.

Theorem 3.1. *Units in the parabolic case are not uniquely clean.*

Proof. The proof follows the same lines as the proof of Theorem 2.1. The characteristic polynomial for such matrices is $(X-1)^2$, so factors over $\mathbf{Z}$ and it suffices to deal with the reduced representative, which is now $V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ (we just use the algorithm described by [4], in the proof of Theorem 1.5, for example for $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$). Thus

$$V = \begin{bmatrix} n+1 & n^2+n \\ -1 & -n \end{bmatrix} + \begin{bmatrix} -n-2 & -n^2-n+1 \\ 1 & n-1 \end{bmatrix}$$

for every integer n (infinite clean index) is not uniquely clean, and nor are all units in the parabolic case. □

4. The hyperbolic case

Theorem 4.1. *The only units in the hyperbolic case which are uniquely clean are the matrices similar to* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Proof. We have to distinguish two cases.

1. *For a unit U we have* $\det(U) = -1$.

Here also we go into 2 subcases.

(i) $\mathrm{Tr}(U) = 0$. In this subcase $\Delta = 2^2$ is a square, the characteristic polynomial factors over $\mathbf{Z}$ (i.e. $X^2 - 1 = (X-1)(X+1)$) and the proof follows the same lines as in the parabolic case. Again it suffices to deal with the reduced representatives which are now $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Since for S, equations (1.2) (see Theorem 1.7) are $2x = \pm 1$, with no integer solutions, this unit has no nontrivial clean decomposition. Since $S - I_2$ is not a unit, we deduce that S is indeed a uniquely clean matrix. So are all matrices similar to S.

Hence all units (with $\det(U) = -1$ and $\mathrm{Tr}(U) = 0$) similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are uniquely clean.

Notice that not all units U with $\det(U) = -1$ and $\mathrm{Tr}(U) = 0$ are uniquely clean. Indeed, the matrices similar to T have nontrivial clean decompositions and so, are not uniquely clean. An example:

$$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix},$$

is a nontrivial clean decomposition. 4).

(ii) $\mathrm{Tr}(U) \neq 0$. In this subcase $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = \mathrm{Tr}^2(U) + 4 > 0$ is never a square over $\mathbf{Z}$ (otherwise 2 would be component of a Pythagorean triple) and we use Theorem 1.6. In doing so, notice that it suffices to show that any reduced matrix (from the cycle) in any given similarity class is not uniquely clean. Denote $\mathrm{Tr}(U) = t$. If $t > 0$ then a reduced representative is

$$W_t = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix},$$

which is not uniquely clean since

$$W_t = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 1-t \\ 1 & t \end{bmatrix}.$$

If $t < 0$, a reduced representative is

$$V_t = \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix},$$

also not uniquely clean, having a symmetric nontrivial clean decomposition.

2. *For a unit U we have* $\det(U) = 1$ *and* $|\mathrm{Tr}(U)| > 2$.

Here $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = \mathrm{Tr}^2(U) - 4 > 0$ is never a square over $\mathbf{Z}$ (otherwise 2 would be component of a Pythagorean triple) and we use Theorem 1.6. We argue as in the previous subcase: now a reduced representative is $\begin{bmatrix} 0 & -1 \\ 1 & t-2 \end{bmatrix}$, if $t > 2$ and $\begin{bmatrix} t+2 & -1 \\ 1 & 0 \end{bmatrix}$, if $t < -2$. Both are not uniquely clean. Indeed, a nontrivial clean decomposition for the first is

$$\begin{bmatrix} 1 & t-4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -t+3 \\ 1 & t-2 \end{bmatrix},$$

and is

$$\begin{bmatrix} 0 & t+2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t+2 & -t-3 \\ 1 & -1 \end{bmatrix}$$

for the second. □

Therefore, the final conclusion of our paper is

Theorem 4.2. *An invertible* 2×2 *integral matrix* U *is uniquely clean iff it is similar to* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$*, i.e., there exists a unit* K *such that*

$$KU = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} K.$$

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