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Multiplicity theorems involving functions with non-convex range

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Dedicated to the memory of Professor Csaba Varga, with nostalgia

Abstract. Here is a sample of the results proved in this paper: Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, let $\rho > 0$ and let $\omega : [0, \rho[\to [0, +\infty[$ be a continuous increasing function such that

$$\lim_{\xi \to \rho^-} \int_0^\xi \omega(x)dx = +\infty.$$

Consider $C^0([0,1]) \times C^0([0,1])$ endowed with the norm

$$\|(\alpha, \beta)\| = \int_0^1 |\alpha(t)|dt + \int_0^1 |\beta(t)|dt.$$

Then, the following assertions are equivalent:
(a) the restriction of f to $\left[-\frac{\sqrt{\rho}}{2}, \frac{\sqrt{\rho}}{2}\right]$ is not constant;
(b) for every convex set $S \subseteq C^0([0,1]) \times C^0([0,1])$ dense in $C^0([0,1]) \times C^0([0,1])$, there exists $(\alpha, \beta) \in S$ such that the problem

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2 dt\right) u'' = \beta(t) f(u) + \alpha(t) \text{ in } [0,1] \\ u(0) = u(1) = 0 \\ \int_0^1 |u'(t)|^2 dt < \rho \end{cases}$$

has at least two classical solutions.

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1. Introduction

Let H be a real Hilbert space. A very classical result of Efimov and Stechkin ([3]) states that if X is a non-convex sequentially weakly closed subset of H, then there exists $y_0 \in H$ such that the restriction to X of the function $x \to \|x - y_0\|$ has at least two global minima. A more precise version of such a result was obtained by I.G. Tsar'kov in [10]. Actually, he proved that any convex set dense in H contains a point y_0 with the above property.

In the present paper, as a by product of a more general result, we get the following:

Theorem 1.1. *Let $X \subset H$ be a non-convex sequentially weakly closed set and let $u_0 \in \mathrm{conv}(X) \setminus X$.*

Then, if we put

$$\delta := \mathrm{dist}(u_0, X)$$

and, for each $r > 0$,

$$\rho_r := \sup_{\|y\|<r} ((\mathrm{dist}(u_0 + y, X))^2 - \|y\|^2),$$

for every convex set $S \subseteq H$ dense in H, for every bounded sequentially weakly lower semicontinuous function $\varphi : X \to \mathbf{R}$ and for every r satisfying

$$r > \frac{\rho_r - \delta^2 + \sup_X \varphi - \inf_X \varphi}{2\delta},$$

there exists $y_0 \in S$, with $\|y_0 - u_0\| < r$, such that the function $x \to \|x - y_0\|^2 + \varphi(x)$ has at least two global minima in X.

So, with respect to the Efimov-Stechkin-Tsar'kov result, Theorem 1.1 gives us two remarkable additional informations: a precise localization of the point y_0 and the validity of the conclusion not only for the function $x \to \|x-y_0\|^2$, but also for suitable perturbations of it.

Let us recall the most famous open problem in this area: if X is a subset of H such that, for each $y \in H$, the restriction of the function $x \to \|x - y\|$ to X has a unique global minimum, is it true that the set X is convex? So, Efimov-Stechkin's result provides an affirmative answer when X is sequentially weakly closed. However, it is a quite common feeling that the answer, in general, should be negative ([1], [2], [5], [8]). In the light of Theorem 1.1, we posit the following problem:

Problem 1.1. Let X be a subset of H for which there exists a bounded sequentially weakly lower semicontinuous function $\varphi : X \to \mathbf{R}$ such that, for each $y \in H$, the function $x \to \|x - y\|^2 + \varphi(x)$ has a unique global minimum in X. Then, must X be convex?

What allows us to reach the advances presented in Theorem 1.1 is our particular approach which is entirely based on the minimax theorem established in [9]. So, also the present paper can be regarded as a further ring of the chain of applications and consequences of that minimax theorem.

2. Results

In the sequel, X is a topological space and E is real normed space, with topological dual E^*.

For each $S \subseteq E^*$, we denote by $\mathcal{A}(X, S)$ (resp. $\mathcal{A}_s(X, S)$) the class of all pairs (I, ψ), with $I : X \to \mathbf{R}$ and $\psi : X \to E$, such that, for each $\eta \in S$ and each $s \in \mathbf{R}$, the set

$$\{x \in X : I(x) + \eta(\psi(x)) \le s\}$$

is closed and compact (resp. sequentially closed and sequentially compact).

Let us start establishing the following useful proposition. E' denotes the algebraic dual of E.

Proposition 2.1. *Let $I : X \to \mathbf{R}$, let $\psi : X \to E$ and let $x_1, \ldots, x_n \in X$, $\lambda_1, \ldots, \lambda_n \in [0,1]$, with $\sum_{i=1}^{n} \lambda_i = 1$.*

Then, one has

$$\sup_{\eta \in E'} \inf_{x \in X} \left(I(x) + \eta \left(\psi(x) - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \right) \le \max_{1 \le i \le n} I(x_i).$$

Proof. Fix $\eta \in E'$. Clearly, for some $j' \in \{1, \ldots, n\}$, we have

$$\eta \left(\psi(x_{j'}) - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \le 0. \tag{2.1}$$

Indeed, if not, we would have

$$\eta(\psi(x_j)) > \sum_{i=1}^{n} \lambda_i \eta(\psi(x_i))$$

for each $j \in \{1, \ldots, n\}$. So, multiplying by λ_j and summing, we would obtain

$$\sum_{j=1}^{n} \lambda_j \eta(\psi(x_j)) > \sum_{i=1}^{n} \lambda_i \eta(\psi(x_i)),$$

a contradiction. In view of (2.1), we have

$$\begin{aligned} \inf_{x \in X} \left(I(x) + \eta \left(\psi(x) - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \right) &\le I(x_{j'}) + \eta \left(\psi(x_{j'}) - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \\ &\le I(x_{j'}) \le \max_{1 \le i \le n} I(x_i) \end{aligned}$$

and so we get the conclusion due to the arbitrariness of η. □

Our main result is as follows:

Theorem 2.1. *Let $I : X \to \mathbf{R}$, let $\psi : X \to E$, let $S \subseteq E^*$ be a convex set dense in E^* and let $u_0 \in E$.*

Then, for every bounded function $\varphi : X \to \mathbf{R}$ such that $(I+\varphi,\psi) \in \mathcal{A}(X,S)$ and for every r satisfying

$$\sup_X \varphi - \inf_X \varphi < \inf_{x\in X}(I(x) + \|\psi(x)-u_0\|r) - \sup_{\|\eta\|_{E^*}<r} \inf_{x\in X}(I(x)+\eta(\psi(x)-u_0)), \quad (2.2)$$

there exists $\tilde{\eta} \in S$, with $\|\tilde{\eta}\|_{E^*} < r$, such that the function $I + \tilde{\eta}\circ\psi + \varphi$ has at least two global minima in X.

Proof. Consider the function $g : X \times E^* \to \mathbf{R}$ defined by

$$g(x,\eta) = I(x) + \eta(\psi(x) - u_0)$$

for all $(x,\eta) \in X \times E^*$. Let B_r denote the open ball in E^*, of radius r, centered at 0. Clearly, for each $x \in X$, we have

$$\sup_{\eta\in B_r} \eta(\psi(x)-u_0)) = \|\psi(x)-u_0\|r. \quad (2.3)$$

Then, from (2.2) and (2.3), it follows

$$\sup_X \varphi - \inf_X \varphi < \inf_X \sup_{B_r} g - \sup_{B_r} \inf_X g. \quad (2.4)$$

Now, consider the function $f : X \times (S \cap B_r) \to \mathbf{R}$ defined by

$$f(x,\eta) = g(x,\eta) + \varphi(x)$$

for all $(x,\eta) \in X \times (S\cap B_r)$. Since S is dense in E^*, the set $S \cap B_r$ is dense in B_r. Hence, since $g(x,\cdot)$ is continuous, we obtain

$$\inf_X \sup_{S\cap B_r} g = \inf_X \sup_{B_r} g. \quad (2.5)$$

Then, taking (2.4) and (2.5) into account, we have

$$\begin{aligned}\sup_{S\cap B_r}\inf_X f \le \sup_{B_r}\inf_X f \le \sup_{B_r}\inf_X g + \sup_X\varphi < \inf_X\sup_{B_r} g + \inf_X \varphi \\ \le \inf_{x\in X}\left(\sup_{\eta\in S\cap B_r} g(x,\eta)+\varphi(x)\right) = \inf_X \sup_{S\cap B_r} f. \quad (2.6)\end{aligned}$$

Now, since $(I+\varphi,\psi) \in \mathcal{A}(X,S)$ and f is concave in $S \cap B_r$, we can apply Theorem 1.1 of [9]. Therefore, since (by (2.6)) $\sup_{S\cap B_r}\inf_X f < \inf_X \sup_{S\cap B_r} f$, there exists of $\tilde{\eta} \in S \cap B_r$ such that the function $f(\cdot,\tilde{\eta})$ has at least two global minima in X which, of course, are global minima of the function $I + \tilde{\eta}\circ\psi + \varphi$. □

If we renounce to the very detailed informations contained in its conclusion, we can state Theorem 2.1 in an extremely simplified form.

Theorem 2.2. *Let $I : X \to \mathbf{R}$, let $\psi : X \to E$ and let $S \subset E^*$ be a convex set weakly-star dense in E^*. Assume that $\psi(X)$ is not convex and that $(I,\psi) \in \mathcal{A}(X,S)$.*

Then, there exists $\tilde{\eta} \in S$ such that the function $I + \tilde{\eta}\circ\psi$ has at least two global minima in X.

Proof. Fix $u_0 \in \mathrm{conv}(\psi(X)) \setminus \psi(X)$ and consider the function $g : X \times E^* \to \mathbf{R}$ defined by

$$g(x,\eta) = I(x) + \eta(\psi(x) - u_0)$$

for all $(x, \eta) \in X \times E^*$. By Proposition 2.1, we know that

$$\sup_{E^*} \inf_{X} g < +\infty.$$

On the other hand, for each $x \in X$, since $\psi(x) \neq u_0$, we have

$$\sup_{\eta \in E^*} \eta(\psi(x) - u_0) = +\infty.$$

Hence, since S is weakly-star dense in E^* and $g(x, \cdot)$ is weakly-star continuous, we have

$$\sup_{\eta \in S} g(x, \eta) = +\infty.$$

Therefore

$$\sup_{S} \inf_{X} g < \inf_{X} \sup_{S} g. \tag{2.7}$$

Now, taken into account that $(I, \psi) \in \mathcal{A}(X, S)$, we can apply Theorem 1.1 of [9] to $g_{|X \times S}$. So, in view of (2.7), there exists $\tilde{\eta} \in S$ such that the function $g(\cdot, \tilde{\eta})$ (and so $I + \tilde{\eta} \circ \psi$) has at least two global minima in X, as claimed. □

The next result is a sequential version of Theorem 1.1 of [9].

Theorem 2.3. *Let X be a topological space, E a topological vector space, $Y \subseteq E$ a non-empty separable convex set and $f : X \times Y \to \mathbf{R}$ a function satisfying the following conditions:*
(a) for each $y \in Y$, the function $f(\cdot, y)$ is sequentially lower semicontinuous, sequentially inf-compact and has a unique global minimum in X;
(b) for each $x \in X$, the function $f(x, \cdot)$ is continuous and quasi-concave.
Then, one has

$$\sup_{Y} \inf_{X} f = \inf_{X} \sup_{Y} f.$$

Proof. The pattern of the proof is the same as that of Theorem 1.1 of [9]. We limit ourselves to stress the needed changes. First, for every $n \in \mathbf{N}$, one proves the result when $E = \mathbf{R}^n$ and $Y = S_n := \{(\lambda_1, \ldots, \lambda_n) \in ([0, +\infty[)^n : \lambda_1 + \ldots + \lambda_n = 1\}$. In this connection, the proof agrees exactly with that of Lemma 2.1 of [9], with the only difference of using the sequential version of Theorem 1.A of [9] instead of such a result itself (see Remark 2.1 of [9]). Next, we fix a sequence $\{x_n\}$ dense in Y. For each $n \in \mathbf{N}$, set

$$P_n = \mathrm{conv}(\{x_1, \ldots, x_n\}).$$

Consider the function $\eta : S_n \to P$ defined by

$$\eta(\lambda_1, \ldots, \lambda_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n$$

for all $(\lambda_1, \ldots, \lambda_n) \in S_n$. Plainly, the function $(x, \lambda_1, \ldots, \lambda_n) \to f(x, \eta(\lambda_1, \ldots, \lambda_n))$ satisfies in $X \times S_n$ the assumptions of Theorem A, and so, by the case previously proved, we have

$$\sup_{(\lambda_1, \ldots, \lambda_n) \in S_n} \inf_{x \in X} f(x, \eta(\lambda_1, \ldots, \lambda_n)) = \inf_{x \in X} \sup_{(\lambda_1, \ldots, \lambda_n) \in S_n} f(x, \eta(\lambda_1, \ldots, \lambda_n)).$$

Since $\eta(S_n) = P_n$, we then have

$$\sup_{P_n} \inf_X f = \inf_X \sup_{P_n} f.$$

Now, set

$$D = \bigcup_{n\in\mathbf{N}} P_n.$$

In view of Proposition 2.2 of [9], we have

$$\sup_D \inf_X f = \inf_X \sup_D f.$$

Finally, by continuity and density, we have

$$\sup_{y\in D} f(x,y) = \sup_{y\in Y} f(x,y)$$

for all $x \in X$, and so

$$\inf_X \sup_Y f = \inf_X \sup_D f = \sup_D \inf_X f \le \sup_Y \inf_X f \le \inf_X \sup_Y f$$

and the proof is complete. □

Reasoning as in the proof of Theorem 2.1 and using Theorem 2.3, we get

Theorem 2.4. *Let the assumptions of Theorem 2.1 be satisfied. In addition, assume that E^* is separable.*

Then, the conclusion of Theorem 2.1 holds with $\mathcal{A}_s(X,S)$ instead of $\mathcal{A}(X,S)$.

Analogously, the sequential version of Theorem 2.2 is as follows:

Theorem 2.5. *Let $I : X \to \mathbf{R}$, let $\psi : X \to E$ and let $S \subset E^*$ be a convex set weakly-star separable and weakly-star dense in E^*. Assume that $\psi(X)$ is not convex and that $(I,\psi) \in \mathcal{A}_s(X,S)$.*

Then, there exists $\tilde{\eta} \in S$ such that the function $I + \tilde{\eta} \circ \psi$ has at least two global minima in X.

Here is a consequence of Theorem 2.1:

Theorem 2.6. *Let E be a Hilbert space, let $\psi : X \to E$ be a weakly continuous function and let $S \subseteq E$ be a convex set dense in E. Assume that $\psi(X)$ is not convex and that the function $\|\psi(\cdot)\|$ is inf-compact. Let $u_0 \in \mathrm{conv}(\psi(X)) \setminus \psi(X)$.*

Then, for every bounded function $\varphi : X \to \mathbf{R}$ such that $\|\psi(\cdot)\|^2 + \varphi(\cdot)$ is lower semicontinuous and for every r satisfying

$$r > \frac{\sup_{\|y\|<r}((\mathrm{dist}(u_0 + y, \psi(X)))^2 - \|y\|^2) - (\mathrm{dist}(u_0, \psi(X)))^2 + \sup_X \varphi - \inf_X \varphi}{2\mathrm{dist}(u_0, \psi(X))}, \tag{2.8}$$

there exists $\tilde{y} \in S$, with $\|\tilde{y} - u_0\| < r$, such that the function $\|\psi(\cdot) - \tilde{y}\|^2 + \varphi(\cdot)$ has at least two global minima in X.

Proof. First, we observe that the set $\psi(X)$ is sequentially weakly closed (and so norm closed). Indeed, let $\{x_n\}$ be a sequence in X such that $\{\psi(x_n)\}$ converges weakly to $y \in E$. So, in particular, $\{\psi(x_n)\}$ is bounded and hence, since $\|\psi(\cdot)\|$ is inf-compact, there exists a compact set $K \subseteq X$ such that $x_n \in K$ for all $n \in \mathbf{N}$. Since ψ is weakly continuous, the set $\psi(K)$ is weakly compact and hence weakly closed. Therefore, $y \in \psi(K)$, as claimed. This remark ensures that $\mathrm{dist}(u_0, \psi(X)) > 0$. Now, we apply Theorem 2.1 identifying E with E^* and taking

$$I(x) = \frac{1}{2}\|\psi(x) - u_0\|^2$$

for all $x \in X$. Of course, we have

$$I(x) + \langle \psi(x) - u_0, y\rangle = \frac{1}{2}(\|\psi(x) - u_0 + y\|^2 - \|y\|^2) \tag{2.9}$$

for all $y \in E$. In view of (2.8) and (2.9), we have

$$\begin{aligned}\frac{1}{2}(\sup_X \varphi - \inf_X \varphi) &< \frac{1}{2}(\mathrm{dist}(u_0, \psi(X)))^2 + r\mathrm{dist}(u_0, \psi(X)) \\ &\quad - \frac{1}{2}\sup_{\|y\|<r}((\mathrm{dist}(u_0 - y, \psi(X)))^2 - \|y\|^2) \\ &\leq \inf_{x\in X}(I(x) + \|\psi(x) - u_0\|r) - \frac{1}{2}\sup_{\|y\|<r}((\mathrm{dist}(u_0 - y, \psi(X)))^2 - \|y\|^2) \\ &= \inf_{x\in X}(I(x) + \|\psi(x) - u_0\|r) - \sup_{\|y\|<r}\inf_{x\in X}(I(x) + \langle\psi(x) - u_0, y\rangle).\end{aligned} \tag{2.10}$$

Let us show that $(I + \frac{1}{2}\varphi, \psi) \in \mathcal{A}(X, E)$. So, fix $y \in E$. Since ψ is weakly continuous, $\langle\psi(\cdot), v\rangle$ is continuous in X for all $v \in E$. Observing that

$$I(x) + \frac{1}{2}\varphi(x) + \langle\psi(x), y\rangle = \frac{1}{2}(\|\psi(x)\|^2 + \varphi(x)) + \langle\psi(x), y - u_0\rangle + \frac{1}{2}\|u_0\|^2,$$

we infer that $I(\cdot) + \frac{1}{2}\varphi(\cdot) + \langle\psi(\cdot), y\rangle$ is lower semicontinuous since $\|\psi(\cdot)\|^2 + \varphi(\cdot)$ is so by assumption. Now, let $s \in \mathbf{R}$. We readily have

$$\begin{aligned}&\left\{x \in X : I(x) + \frac{1}{2}\varphi(x) + \langle\psi(x), y\rangle \leq s\right\} \\ &\subseteq \left\{x \in X : \|\psi(x)\|^2 - 2\|y - u_0\|\|\psi(x)\| \leq 2s - \inf_X \varphi\right\}.\end{aligned} \tag{2.11}$$

Since $\|\psi(\cdot)\|$ is inf-compact, the set in the right-hand side of (2.11) is compact and hence so is the set in left-hand right, as claimed. Since the set $u_0 - S$ is convex and dense in E, in view of (2.10), Theorem 2.1 ensures the existence of $\tilde{v} \in u_0 - S$, with $\|\tilde{v}\| < r$, such that the function $I(\cdot) + \langle\psi(\cdot), \tilde{v}\rangle + \frac{1}{2}\varphi(\cdot)$ has at least two global minima in X. Consequently, since

$$I(x) + \langle\psi(x), \tilde{v}\rangle + \frac{1}{2}\varphi(x) = \frac{1}{2}(\|\psi(x) + \tilde{v} - u_0\|^2 + \varphi(x)) - \frac{1}{2}(\|u_0\|^2 - \|\tilde{v} - u_0\|^2),$$

if we put

$$\tilde{y} := u_0 - \tilde{v},$$

we have $\tilde{y} \in S$, $\|\tilde{y} - u_0\| < r$ and the function $\|\psi(\cdot) - \tilde{y}\|^2 + \varphi(\cdot)$ has at least two global minima in X. The proof is complete. □

Remark 2.1. Of course, Theorem 1.1 is an immediate corollary of Theorem 2.6: take $E = H$, consider X equipped with the relative weak topology, take $\psi(x) = x$ and observe that if $\varphi : X \to \mathbf{R}$ is sequentially weakly lower semicontinuous, then $\|\cdot\|^2+\varphi(\cdot)$ is weakly lower semicontinuous in view of the Eberlein-Smulyan theorem.

Here is an application of Theorem 2.2. An operator T between two Banach spaces F_1, F_2 is said to be sequentially weakly continuous if, for every sequence $\{x_n\}$ in F_1 weakly convergent to $x \in F_1$, the sequence $\{T(x_n)\}$ converges weakly to $T(x)$ in F_2.

Theorem 2.7. *Let V be a reflexive real Banach space, let $x_0 \in V$, let $r > 0$, let X be the open ball in V, of radius r, centered at x_0, let $\gamma : [0, r[\to \mathbf{R}$, with $\lim_{\xi\to r^-} \gamma(\xi) = +\infty$, let $I : X \to \mathbf{R}$ and $\psi : X \to E$ be two Gâteaux differentiable functions. Moreover, assume that I is sequentially weakly lower semicontinous, that ψ is sequentially weakly continuous, that $\psi(X)$ is bounded and non-convex, and that*

$$\gamma(\|x - x_0\|) \le I(x)$$

for all $x \in X$.

Then, for every convex set $S \subseteq E^*$ weakly-star dense in E^*, there exists $\tilde\eta \in S$ such that the equation

$$I'(x) + (\tilde\eta \circ \psi)'(x) = 0$$

has at least two solutions in X.

Proof. We apply Theorem 2.2 considering X equipped with the relative weak topology. Let $\eta \in E^*$. Since $\psi(X)$ is bounded, we have $c := \inf_{x\in X} \eta(\psi(x)) > -\infty$. Let $s \in \mathbf{R}$. We have

$$\begin{aligned}\{x \in X : I(x) + \eta(\psi(x)) \le s\} &\subseteq \{x \in X : I(x) \le s - c\}\\ &\subseteq \{x \in X : \gamma(\|x - x_0\|) \le s - c\}.\end{aligned} \tag{2.12}$$

Since $\lim_{\xi\to r^-} \gamma(t) = +\infty$, there is $\delta \in]0, r[$, such that $\gamma(\xi) > s - c$ for all $\xi \in]\delta, r[$. Consequently, from (2.12), we obtain

$$\{x \in X : I(x) + \eta(\psi(x)) \le s\} \subseteq \{x \in V : \|x - x_0\| \le \delta\}. \tag{2.13}$$

From the assumptions, it follows that the function $I + \eta \circ \psi$ is sequentially weakly lower semicontinuous in X. Hence, from (2.13), since $\delta < r$ and V is reflexive, we infer that the set $\{x \in X : I(x) + \eta(\psi(x)) \le s\}$ is sequentially weakly compact and hence weakly compact, by the Eberlein-Smulyan theorem. In other words, $(I, \psi) \in \mathcal{A}(X, E^*)$. Therefore, we can apply Theorem 2.2. Accordingly, there exists $\tilde\eta \in S$ such that the function $I + \tilde\eta \circ \psi$ has at least two global minima in X which are critical points of it since X is open. □

Here is an application of Theorem 1.1:

Theorem 2.8. *Let H be a Hilbert space and let $I, J : H \to \mathbf{R}$ be two C^1 functionals with compact derivative such that $2I - J^2$ is bounded. Moreover, assume that $J(0) \ne 0$ and that there is $\hat x \in H$ such that $J(-\hat x) = -J(\hat x)$.*

Then, for every convex set $S \subseteq H \times \mathbf{R}$ dense in $H \times \mathbf{R}$ and for every r satisfying

$$r > \frac{\|\hat x\|^2 + |J(\hat x)|^2 - \inf_{x\in H}(\|x\|^2 + |J(x)|^2) + \sup_H(2I - J^2) - \inf_X(2I - J^2)}{2\inf_{x\in H} \sqrt{\|x\|^2 + |J(x)|^2}},$$

there exists $(y_0, \mu_0) \in S$, *with* $\|y_0\|^2 + |\mu_0|^2 < r^2$, *such that the equation*

$$x + I'(x) + \mu_0 J'(x) = y_0$$

has at least three solutions.

Proof. We consider the Hilbert space $E := H \times \mathbf{R}$ with the scalar product

$$\langle (x, \lambda), (y, \mu) \rangle_E = \langle x, y \rangle + \lambda\mu$$

for all $(x, \lambda), (y, \mu) \in E$. Take

$$X = \{(x, \lambda) \in E : \lambda = J(x)\}.$$

Since J' is compact, the functional J turns out to be sequentially weakly continuous ([11], Corollary 41.9). So, the set X is sequentially weakly closed. Moreover, notice that $(0, 0) \notin X$, while the antipodal points $(\hat{x}, J(\hat{x}))$ and $-(\hat{x}, J(\hat{x}))$ lie in X. So, $(0, 0) \in \text{conv}(X)$. Now, with the notations of Theorem 1.1, taking, of course, $u_0 = (0, 0)$, we have

$$\delta = \inf_{x \in X} \sqrt{\|x\|^2 + |J(x)|^2}$$

and

$$\rho_r = \sup_{\|y\|^2 + |\mu|^2 < r^2} \inf_{x \in X} (\|x\|^2 + |J(x)|^2 - 2\langle (x, J(x)), (y, \mu) \rangle_E).$$

Then, from Proposition 2.1, we infer that

$$\rho_r \leq \|\hat{x}\|^2 + |J(\hat{x})|^2.$$

Now, consider the function $\varphi : X \to \mathbf{R}$ defined by

$$\varphi(x, \lambda) = 2I(x) - \lambda^2$$

for all $(x, \lambda) \in X$. Notice that φ is sequentially weakly continuous and r satisfies the inequality of Theorem 1.1. Consequently, there exists $(y_0, \mu_0) \in S$ such that the functional

$$(x, \lambda) \to \|(x, \lambda)\|_E^2 - 2\langle (x, \lambda), (y_0, \mu_0) \rangle_E + 2I(x) - \lambda^2$$

has at least two global minima in X. Of course, if $(x, \lambda) \in X$, we have

$$\|(x, \lambda)\|_E^2 - 2\langle (x, \lambda), (y_0, \mu_0) \rangle_E + 2I(x) - \lambda^2$$
$$= \|x\|^2 + J^2(x) - 2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x) - J^2(x).$$

In other words, the functional

$$x \to \|x\|^2 - 2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x)$$

has two global minima in H. Since the functional

$$x \to -2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x)$$

has a compact derivative, a well know result ([11], Example 38.25) ensures that the functional

$$x \to \|x\|^2 - 2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x)$$

has the Palais-Smale property and so, by Corollary 1 of [6], it possesses at least three critical points. The proof is complete. □

Remark 2.2. In Theorem 2.8, apart from being C^1 with compact derivative, the truly essential assumption on J is, of course, that its graph is not convex. This amounts to say that J is not affine. The current assumptions are made to simplify the constants appearing in the conclusion. Actually, from the proof of Theorem 2.8, the following can be obtained:

Theorem 2.9. *Let H be a Hilbert space and let $I, J : H \to \mathbf{R}$ be two C^1 functionals with compact derivative such that $2I - J^2$ is bounded. Moreover, assume that J is not affine.*

Then, for every convex set $S \subseteq H \times \mathbf{R}$ dense in $H \times \mathbf{R}$, there exists $(y_0, \lambda_0) \in S$ such that the equation

$$x + I'(x) + \lambda_0 J'(x) = y_0$$

has at least three solutions.

Remark 2.3. For $I = 0$, the conclusion of Theorem 2.9 can be obtained from Theorem 4 of [7] (see also [4]) provided that, for some $r \in \mathbf{R}$, the set $J^{-1}(r)$ is not convex. Therefore, for instance, the fact that, for any non-constant bounded C^1 function $J : \mathbf{R} \to \mathbf{R}$, there are $a, b \in \mathbf{R}$ such that the equation

$$x + aJ'(x) = b$$

has at least three solutions, follows, in any case, from Theorem 2.9, while it follows from Theorem 4 of [7] only if J is not monotone.

We conclude presenting an application of Theorem 2.7 to a class of Kirchhoff-type problems.

Theorem 2.10. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, let $\rho > 0$ and let $\omega : [0, \rho[\to [0, +\infty[$ be a continuous increasing function such that*

$$\lim_{\xi \to \rho^-} \int_0^{\xi} \omega(x)dx = +\infty.$$

Consider $C^0([0,1]) \times C^0([0,1])$ endowed with the norm

$$\|(\alpha, \beta)\| = \int_0^1 |\alpha(t)|dt + \int_0^1 |\beta(t)|dt.$$

Then, the following assertions are equivalent:
(a) the restriction of f to $\left[-\frac{\sqrt{\rho}}{2}, \frac{\sqrt{\rho}}{2}\right]$ is not constant;
(b) for every convex set $S \subseteq C^0([0,1]) \times C^0([0,1])$ dense in $C^0([0,1]) \times C^0([0,1])$, there exists $(\alpha, \beta) \in S$ such that the problem

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2 dt\right) u'' = \beta(t)f(u) + \alpha(t) \text{ in } [0,1] \\ u(0) = u(1) = 0 \\ \int_0^1 |u'(t)|^2 dt < \rho \end{cases}$$

has at least two classical solutions.

Proof. Consider the Sobolev space $H_0^1(]0,1[)$ with the usual scalar product

$$\langle u, v\rangle = \int_0^1 u'(t)v'(t)dt.$$

Let $B_{\sqrt{\rho}}$ be the open ball in $H_0^1(]0,1[$, of radius $\sqrt{\rho}$, centered at 0. Let $g : [0,1]\times\mathbf{R} \to \mathbf{R}$ be a continuous function. Consider the functionals $I, J_g : B_{\sqrt{\rho}} \to \mathbf{R}$ defined by

$$I(u) = \frac{1}{2}\tilde{\omega}\left(\int_0^1 |u'(t)|^2 dt\right),$$

$$J_g(u) = \int_0^1 \tilde{g}(t, u(t))dt$$

for all $u \in B_{\sqrt{\rho}}$, where $\tilde{\omega}(\xi) = \int_0^\xi \omega(x)dx$, $\tilde{g}(t,\xi) = \int_0^\xi g(t,x)dx$. By classical results, taking into account that if $\omega(x) = 0$ then $x = 0$, it follows that the classical solutions of the problem

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2 dt\right) u'' = g(t,u) \text{ in } [0,1] \\ u(0) = u(1) = 0 \\ \int_0^1 |u'(t)|^2 dt < \rho \end{cases}$$

are exactly the critical points in $B_{\sqrt{\rho}}$ of the functional $I - J_g$.
Let us prove that $(a) \to (b)$. We are going to apply Theorem 2.7 taking $V = H_0^1(]0,1[)$, $x_0 = 0$, $r = \sqrt{\rho}$, I as above, $\gamma(\xi) = \frac{1}{2}\tilde{\omega}(\xi^2)$, $E = C^0([0,1]) \times C^0([0,1])$ and $\psi : B_{\sqrt{\rho}} \to E$ defined by

$$\psi(u)(\cdot) = (u(\cdot), \tilde{f}(u(\cdot)))$$

for all $u \in B_{\sqrt{\rho}}$, where $\tilde{f}(\xi) = \int_0^\xi f(x)dx$. Clearly, the functional I is continuous and strictly convex (and so weakly lower semicontinuous), while the operator ψ is Gâteaux differentiable and sequentially weakly continuous due to the compact embedding of $H_0^1(]0,1[)$ into $C^0([0,1])$. Recall that

$$\max_{[0,1]} |u| \le \frac{1}{2}\sqrt{\int_0^1 |u'(t)|^2 dt}$$

for all $u \in H_0^1(]0,1[)$. As a consequence, the set $\psi\left(B_{\sqrt{\rho}}\right)$ is bounded and, in view of (a), non-convex. Hence, each assumption of Theorem 2.7 is satisfied. Now, consider the operator $T : E \to E^*$ defined by

$$T(\alpha,\beta)(u,v) = \int_0^1 \alpha(t)u(t)dt + \int_0^1 \beta(t)v(t)dt$$

for all $(\alpha,\beta),(u,v) \in E$. Of course, T is linear and the linear subspace $T(E)$ is total over E. Hence, $T(E)$ is weakly-star dense in E^*. Moreover, notice that T is continuous with respect to the weak-star topology of E^*. Indeed, let $\{(\alpha_n,\beta_n)\}$ be a sequence in E converging to some $(\alpha,\beta) \in E$. Fix $(u,v) \in E$. We have to show that

$$\lim_{n\to\infty} T(\alpha_n,\beta_n)(u,v) = T(\alpha,\beta)(u,v). \tag{2.14}$$

Notice that

$$\lim_{n\to\infty}\left(\int_0^1 |\alpha_n(t)-\alpha(t)|dt+\int_0^1 |\beta_n(t)-\beta(t)|dt\right)=0. \tag{2.15}$$

On the other hand, we have

$$|T(\alpha_n,\beta_n)(u,v)-T(\alpha,\beta)(u,v)|=\left|\int_0^1 (\alpha_n(t)-\alpha(t))u(t)dt+\int_0^1 (\beta_n(t)-\beta(t))v(t)dt\right|$$

$$\leq \left(\int_0^1 |\alpha_n(t)-\alpha(t)|dt+\int_0^1 |\beta_n(t)-\beta(t)|dt\right)\max\left\{\max_{[0,1]}|u|,\max_{[0,1]}|v|\right\}$$

and hence (2.14) follows in view of (2.15).
Finally, fix a convex set $S\subseteq C^0([0,1])\times C^0([0,1])$ dense in $C^0([0,1])\times C^0([0,1])$. Then, by the kind of continuity of T just now proved, the convex set $T(-S)$ is weakly-star dense in E^* and hence, thanks to Theorem 2.7, there exists $(\alpha_0,\beta_0)\in -S$ such that, if we put

$$g(t,\xi)=\alpha_0(t)+\beta_0(t)f(\xi),$$

the functional $I-J_g$ has at least two critical points in $B_{\sqrt{\rho}}$ which are the claimed solutions of the problem in (b), with $\alpha=-\alpha_0$ and $\beta=-\beta_0$.

Now, let us prove that $(b)\to(a)$. Assume that the restriction of f to $\left[-\frac{\sqrt{\rho}}{2},\frac{\sqrt{\rho}}{2}\right]$ is constant. Let c be such a value. So, the classical solutions of the problem

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2dt\right)u''=c\beta(t)+\alpha(t) \text{ in } [0,1] \\ u(0)=u(1)=0 \\ \int_0^1 |u'(t)|^2dt<\rho \end{cases}$$

are the critical points in $B_{\sqrt{\rho}}$ of the functional

$$u\to\frac{1}{2}\tilde{\omega}\left(\int_0^1 |u'(t)|^2dt\right)-\int_0^1 (c\alpha(t)+\beta(t))u(t)dt.$$

But, since ω is increasing and non-negative, this functional is strictly convex and so it possesses a unique critical point. The proof is complete. □

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