Stud. Univ. Babeş-Bolyai Math. 67(2022), No. 2, 369–382
DOI: 10.24193/subbmath.2022.2.13

Polynomial estimates for solutions of parametric elliptic equations on complete manifolds

Mirela Kohr, Simon Labrunie, Hassan Mohsen and Victor Nistor

We dedicate this paper to the memory of Professor Gabriela Kohr

Abstract. Let $P : \mathcal{C}^\infty(M;E) \to \mathcal{C}^\infty(M;F)$ be an order μ differential operator with coefficients a and $P_k := P : H^{s_0+k+\mu}(M;E) \to H^{s_0+k}(M;F)$. We prove polynomial norm estimates for the solution $P_0^{-1}f$ of the form

$$\|P_0^{-1}f\|_{H^{s_0+k+\mu}(M;E)} \leq C \sum_{q=0}^{k} \|P_0^{-1}\|^{q+1} \, \|a\|^q_{W^{|s_0|+k}} \, \|f\|_{H^{s_0+k-q}},$$

(thus in higher order Sobolev spaces, which amounts also to a parametric regularity result). The assumptions are that $E, F \to M$ are Hermitian vector bundles and that M is a complete manifold satisfying the Fréchet Finiteness Condition (FFC), which was introduced in (Kohr and Nistor, Annals of Global Analysis and Geometry, 2022). These estimates are useful for uncertainty quantification, since the coefficient a can be regarded as a vector valued random variable. We use these results to prove integrability of the norm $\|P_k^{-1}f\|$ of the solution of $P_k u = f$ with respect to suitable Gaussian measures.

Mathematics Subject Classification (2010): 35R01, 35J75, 46E35, 65N75.

Keywords: Parametric elliptic equations, complete manifolds, the Fréchet Finiteness Condition (FFC), Sobolev spaces, uncertainty quantification.

1. Introduction

1.1. A short summary

Let M be a Riemannian manifold and $E, F \to M$ be Hermitian vector bundles. We let ∇ denote a generic connection on vector bundles. On E and F, ∇ is given, whereas on TM we consider the Levi-Civita connection. Let

$$\nabla^j = \nabla \circ \nabla \circ \ldots \circ \nabla : \mathcal{C}^\infty(M;E) \to \mathcal{C}^\infty(M; T^{*\otimes j}M \otimes E)$$

Received 27 April 2022; Accepted 2 May 2022.

be the j-times composition of the covariant derivative on E, namely the composition of the maps $\nabla : \mathcal{C}^\infty(M; T^{*\otimes i}M \otimes E) \to \mathcal{C}^\infty(M; T^{*\otimes (i+1)}M \otimes E)$.

In this paper, we study an order $\mu \geq 1$ differential operator

$$P \ := \ a \cdot \nabla^{tot} \ := \ \sum_{j=0}^{\mu} a^{[j]} \nabla^j \tag{1.1}$$

acting on sections of E and with values sections of F. This setting allows us to consider *systems* of partial differential operators (PDEs). Let $s_0 \in \mathbb{Z}$ (fixed throughout the paper) and $k \in \mathbb{Z}_+$ and let

$$P_k \ := \ a \cdot \nabla^{tot} : H^{s_0+k}(M; E) \to H^{s_0+k+\mu}(M; F) \tag{1.2}$$

denote the operator induced by P on the indicated Sobolev spaces (which is defined and continuous provided that a is smooth enough, see Lemma 2.2). Our *main result* is to prove polynomial bounds for $\|P_k^{-1} f\|_{H^{s_0+k+\mu}}$ in terms of $\|P_0^{-1}\|$ and the norm of the coefficient a under suitable hypotheses on M, E, and F (Theorem 1.3).

We apply these estimates to the integrability of the norm of P_k^{-1} (in $\mathcal{L}(H^{s_0+k}(M; F), H^{s_0+k+\mu}(M; E))$) with respect to suitable Gaussian measures. This type of estimate is useful for uncertainty quantification, see [3, 7, 11, 13, 15, 16, 20].

Our results apply to complete manifolds M that satisfy the Fréchet Finiteness Condition (FFC), a condition that was introduced in [14] and will be recalled shortly. It was proved in Lemma 3.1 of [9] that manifolds with bounded geometry satisfy (FFC). Consequently, every open subset of a manifold with bounded geometry satisfies (FFC). In particular, all compact manifolds and all euclidean spaces satisfy (FFC).

1.2. Basic concepts

To formulate our result more precisely, we need to introduce some notation and terminology and to remind some basic definitions. If $E \to M$ is a vector bundle, then $\mathcal{M}(M; E)$ denotes the set of measurable sections of E. We shall use in the following ∇*-differential operators* [14]. To define them, let the *truncated Fock space* $\mathcal{F}_\mu^M(E)$ be defined by

$$\mathcal{F}_\mu^M(E) \ := \ \oplus_{j=0}^{\mu} T^{*\otimes j}M \otimes E\,. \tag{1.3}$$

Definition 1.1. Let $E, F \to M$ be vector bundles, with E endowed with a connection and let $a = (a^{[0]}, a^{[1]}, \ldots, a^{[\mu]})$ be a measurable section of $\mathrm{Hom}(\mathcal{F}_\mu^M(E); F)$, $\nabla^0 := id$. A ∇*–differential operator (on E with values in F)* is a map (see Equation (1.1))

$$P \ = \ a \cdot \nabla^{tot} \ := \ \sum_{j=0}^{\mu} a^{[j]} \nabla^j : \mathcal{C}^\infty(M; E) \to \mathcal{M}(M; F)\,.$$

The order of P, denoted $\mathrm{ord}(P)$, is the least μ for which such a writing exists.

If E is Hermitian, we let

$$W_\nabla^{k,\infty}(M; E) \ := \ \{u \in \mathcal{M}(M; E) \mid \nabla^j u \in L^p(M; E), 0 \leq j \leq k\}$$

be the space of sections of E whose first k covariant derivatives are bounded, as in [2, 12, 14]. For $k \leq 0$ and $k \notin \mathbb{Z}$, we proceed by duality and interpolation (see [14], for instance). We let $H^s(M;E) := W^{s,2}_\nabla(M;E)$ and $W^{\infty,\infty}_\nabla(M;E) := \cap_{k\geq 0} W^{k,\infty}_\nabla(M;E)$. Recall the Fréchet Finiteness Conditions (FFC), see [14, Definition 5.8].

Definition 1.2. Let M be a Riemannian manifold with a metric g. We say that (M,g) satisfies the *Fréchet finiteness condition* (FFC) if there exist $N \in \mathbb{N}$ and an isometric (vector bundle) embedding $\Phi : TM \to M \times \mathbb{R}^N$,, $\Phi \in W^{\infty,\infty}_\nabla(M; \mathrm{Hom}(TM;\mathbb{R}^N))$, where, in order to define the Sobolev space $W^{\infty,\infty}_\nabla(M; \mathrm{Hom}(TM;\mathbb{R}^N))$, we consider the trivial connection on the vector bundle $M \times \mathbb{R}^N \to M$.

1.3. Statement of the main result

It is known that the operator $P_k := a \cdot \nabla^{tot} : H^{s_0+k+\mu}(M;E) \to H^{s_0+k}(M;F)$ of Equation (1.2) is well-defined and continuous if $a \in W^{|s_0+k|,\infty}_\nabla(M; \mathrm{Hom}(\mathcal{F}^M_\mu(E);F))$ (see Lemma 2.2). A vector bundle E is said to have *totally bounded curvature* if its curvature tensor is in $W^{\infty,\infty}(M;\Lambda^2T^*M \otimes \mathrm{End}(E))$. We are ready to state our main result. Recall that, throughout this paper, we have fixed $s_0 \in \mathbb{Z}$.

Theorem 1.3. *Let us assume that M is a complete manifold satisfying (FFC) and that E and F have totally bounded curvature. Let $a \in W^{|s_0|+k,\infty}_\nabla(M; \mathrm{Hom}(\mathcal{F}^M_\mu(E);F))$ and $P_k := a \cdot \nabla^{tot} : H^{s_0+k+\mu}(M;E) \to H^{s_0+k}(M;F)$, $k \in \mathbb{Z}_+$. Let us assume that P_0 is invertible. Then $\mathfrak{C} := \|P_0^{-1}\| \, \|a\|_{W^{|s_0|+k}} \geq 1$. Let $f \in H^{s_0+k}(M;F)$, so $P_0^{-1}f \in H^{s_0+\mu}(M;E)$ is defined, then, in fact, $P_0^{-1}f \in H^{s_0+k+\mu}(M;E)$, and*

$$\|P_0^{-1}f\|_{H^{s_0+k+\mu}} \lesssim \|P_0^{-1}\| \sum_{q=0}^{k} \mathfrak{C}^q \, \|f\|_{H^{s_0+k-q}} \,. \qquad (I_k)$$

Consequently, P_k is an isomorphism with $\|P_k^{-1}\| \leq \|P_0^{-1}\|^{k+1} \, \|a\|^k_{W^{|s_0|+k}}$.

For operators in divergence form, we obtain a slightly better result in that we may allow lower regularity for a, as in [18]. A consequence of our results is the integrability of $\|P_k^{-1}f\|_{H^{k+\mu}}$ for operators of divergence form of *order* $2m$ with respect to certain measures of Gaussian type on the set of coefficients a, see Theorem 5.3. We stress that a particular, but important, special case of our results is when M is compact without boundary. The case of bounded domains is discussed in [19].

1.4. Contents of the paper

The main result is stated in the Introduction states, where we also recall some needed concepts. Section 2 contains some preliminary material, including a version of Nirenberg's trick following [6, 18]. The third section is devoted to proving that a totally bounded vector field (i.e. one in $W^{\infty,\infty}_\nabla(M;TM)$) integrates to a global one-parameter groups of diffeomorphisms of M and of automorphisms of our Sobolev spaces. The forth section is devoted to the proof of the main result (Theorem 1.3) following the method from [18]. The integrability of $\|P_k^{-1}f\|_{H^{s_0+k+\mu}}$ with respect to suitable Gaussian measures on the space of coefficients is proved in the last section.

2. Operators and Nirenberg's trick

Here we describe the ingredients needed to formulate our main result in more detail. We also recall some needed results, including an extension of Nirenberg's trick. See [4, 5, 8, 17, 21] for concepts and results that are not discussed in this article.

2.1. Operators and their norms

The following notation will be used throughout the paper. It was already used in the statement of the main result. We fix $\mu \in \mathbb{N} = \{1, 2, \ldots\}$, which will be the order of the operator $P := a \cdot \nabla^{tot}$ that we study. We also fix throughout this paper $s_0 \in \mathbb{Z}$, which will be the order of the minimal regularity Sobolev spaces where we assume the invertibility of P. We let $\sigma := s_0 + k$, to simplify the notation. We shall usually write $\|u\|_{W^k}$ (or even $\|a\|_k$) instead of $\|u\|_{W^{k,\infty}(M;E)}$ and $\|u\|_{H^s}$ instead of $\|u\|_{H^s(M;E)}$. Recall that if X and Y are two normed spaces, then $\mathcal{L}(X, Y)$ denotes the space of linear, continuous operators $X \to Y$. Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

Notation 2.1. Let $k \in \mathbb{Z}_+$ and s_0, μ, and $\sigma := s_0 + k$ as above. We shall write $\|T_1\|_k := \|T_1\|_{\mathcal{L}(H^{\sigma+\mu}(M;E);H^{\sigma}(M;F))}$ and $|||T_2|||_k := \|T_2\|_{\mathcal{L}(H^{\sigma}(M;F);H^{\sigma+\mu}(M;E))}$. We shall write $D_1 \lesssim D_2$ if there is $C_{\Phi,k,M,E,F} > 0$ such that $D_1 \le C_{\Phi,k,M,E,F} D_2$, where $C_{\Phi,k,M,E,F}$ depends only on Φ, k, M, E, and F, where Φ is as in the Definition 1.2, k some other parameter, usually related to the order of the Sobolev spaces involved, M is our manifold and $E, F \to M$ are the vector bundle involved. The next result follows from [14, Proposition 3.7].

Lemma 2.2. *Let E, F, E_j be Hermitian vector bundles, $j = 1, 2, 3$. Let $b_j \in W^{k,\infty}_{\nabla}(M; \mathrm{Hom}(E_j, E_{j+1}))$, $j = 1, 2$, and $f \in H^s(M; E_1)$.*

(i) $\|b_2 b_1\|_{W^k} \lesssim \|b_2\|_{W^k} \|b_1\|_{W^k}$.

(ii) $\|b_1 f\|_{H^s} \lesssim \|b_1\|_{W^k} \|f\|_{H^s}$, *if* $|s| \le k$.

(iii) Let $P_k := a \cdot \nabla^{tot} : H^{\sigma+\mu}(M; E) \to H^{\sigma}(M; F)$. Then P_k is continuous with norm $\|P_k\|_k \lesssim \|a\|_{W^{|s_0|+k}}$, if $a \in W^{|s_0|+k,\infty}_{\nabla}(M; \mathrm{Hom}(\mathcal{F}^M_m(E); F))$.

Here $k, \mu \in \mathbb{Z}_+$, $s, s_0 \in \mathbb{Z}$, and $\sigma := s_0 + k$.

As in [18], we obtain the following simple lemma.

Lemma 2.3. *We use the notation of Lemma 2.2 and assume P_k is invertible. Then* $1 \lesssim |||P_k^{-1}|||_k \, \|a\|_{W^{|s_0|+k}}$.

2.2. Nirenberg's trick

In this section we recall a version of Nirenberg's trick, as it is formalized in [6, 18]. We write $t \searrow 0$ if $t \to 0$ and $t > 0$. Let X and Y be two Banach spaces, recall that a family $(T_t)_{t\ge0}$ in $\mathcal{L}(X, Y)$ converges strongly to T for $t \searrow 0$ if $\lim_{t\searrow0} \|T_t u - Tu\|_Y = 0$, for all $u \in X$. We shall need also the following basic concept.

Definition 2.4. A family of operators $(S(t))_{t\ge0}$ of $\mathcal{L}(X) := \mathcal{L}(X, X)$ is *a strongly continuous semigroup* on X if the following conditions are satisfied: $S(0) = id_X$, for all $t \ge 0$ and $r \ge 0$, $S(t + r) = S(t)S(r)$, and, for all $x \in X$, $\lim_{t\to0} \|S(t)x - x\|_X = 0$. Then *the infinitesimal generator of* $(S(t))_{t\ge0}$ is the operator $(L_S, \mathcal{D}(L_S))$ defined by

$$\mathcal{D}(L_S) := \{x \in X \mid L_S x := \lim_{t\to0} t^{-1}\big(S(t)x - x\big) \text{ exists in } X\}.$$

The following lemma [6, 18] will play an essential role in what follows. The version here is a simplified one compared to the ones in the aforementioned articles.

Proposition 2.5. *Let $T : X \to Y$ be an invertible bounded operator between two Banach spaces and let $S_X(t) \in \mathcal{L}(X)$ and $S_Y(t) \in \mathcal{L}(Y)$ be two strongly continuous semigroups of operators. We assume that, for each $t \in \mathbb{R}$, there exists $T_t \in \mathcal{L}(X, Y)$ such that $T_t S_X(t) = S_Y(t)T$. Suppose that $t^{-1}(T_t - T)$ converges strongly to $Q \in \mathcal{L}(X, Y)$ for $t \searrow 0$. Then, for all v in $\mathcal{D}(L_Y)$, we have*

$$T^{-1}v \in \mathcal{D}(L_{S_X}) \quad \text{and} \quad L_{S_X} T^{-1} v = T^{-1} L_{S_Y} v - T^{-1} Q T^{-1} v.$$

In our case, at least one of the assumptions of this result will be easy to check.

Remark 2.6. In our applications, both S_X and S_Y will extend to *groups* of operators. Thus, $S_X(t)$ and $S_Y(t)$ are defined for $t \in \mathbb{R}$ with the usual group laws:

$$S_X(t)S_X(t') = S_X(t+t') \text{ and } S_Y(t)S_Y(t') = S_Y(t+t').$$

Therefore, the existence of T_t satisfying $T_t S_X(t) = S_Y(t)T$ is guaranteed simply by taking $T_t := S_Y(t) T S_X(-t)$.

3. Groups of diffeomorphisms and Sobolev spaces

In this section, we systematically use vector fields to define our Sobolev spaces.

3.1. Vector fields and Sobolev spaces

Assume M satisfies (FFC). Let $\Phi : TM \to M \times \mathbb{R}^N$ be as in Definition 1.2 and e_j, $j = 1, \dots, N$, be the canonical basis of $\mathbb{R}^N$. Then $Z_1, Z_2, \dots, Z_N \in \mathcal{W}_b(M) := W_\nabla^{\infty,\infty}(M; TM)$ will continue to denote a Fréchet system of generators of $\mathcal{W}_b(M)$ as $\mathcal{C}_{\mathrm{b}}^\infty$-module, as in [14], that is,

$$Z_j \ := \ \Phi^T(e_j)\,, \tag{3.1}$$

We shall need the following proposition from [14].

Proposition 3.1. *Let us assume that M satisfies (FFC) and let $\{Z_j\}$ be a Fréchet system of generators of $\mathcal{W}_b(M)$, as in Equation (3.1). Let $\ell \in \mathbb{N}$ and $1 \le p \le \infty$. Then*

$$W_\nabla^{\ell,p}(M; E) \,=\, \{\, u \mid \nabla_{Z_{k_1}}^E \nabla_{Z_{k_2}}^E \dots \nabla_{Z_{k_j}}^E u \in L^p(M; E),\ j \le \ell,\ 1 \le k_i \le N \,\}.$$

We shall need the following standard consequence. (See also [18].)

Lemma 3.2. *Let $Z_0 u := u$ and $Z_k u := \nabla_{Z_k}(u)$, for simplicity, with Z_j as in Equation (3.1). Let $s \in \mathbb{Z}_+$ and*

$$\|u\|' \ := \ \sum_{i=0}^{N} \|Z_i u\|_{H^s}\,.$$

Then $\|u\|'$ defines an equivalent norm on $H^{s+1}(M; E)$.

Proof. This follows right away from Proposition 3.1. □

3.2. Diffeomorphism groups

We shall need the fact that vector fields $X \in W_{\nabla}^{\infty,\infty}(M;TM)$ integrate to global diffeomorphisms groups and then that these diffeomorphism groups lift to automorphism groups of vector bundles.

Proposition 3.3. *Let $X \in W_{\nabla}^{\infty,\infty}(M;TM)$ (that is, X is a totally bounded smooth vector field on M). Assume that M is complete, then X generates a one-parameter group of diffeomorphisms $\phi_t : M \to M$, $t \in \mathbb{R}$.*

A one-parameter group of diffeomorphisms (of M) will also be called a "flow (on M)."

Proof. The proof is almost the same as the one of the existence of global geodesics on complete Riemannian manifolds. First, the existence of a local family ϕ_t is a classical result in ordinary differential equations and differential geometry. Then, for each $x \in M$, the curve $\phi_t(x)$ is an integral curve of the vector field X and is defined at least on some interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$ may depend on x. We need to show that this curve extends indefinitely for each x under the assumption that our manifold M is complete. We shall proceed by contradiction. For any given $x \in M$, let $I \subset \mathbb{R}$ be a maximal interval on which the integral curve $\phi_t(x)$, $t \in I$, is defined. Let us assume $I \neq \mathbb{R}$ and let $a \in \overline{I} \smallsetminus I$. Let also $t_n \in I$, $t_n \to a$. As X is bounded, we have $dist(\phi_{t_n}(x), \phi_{t_m}(x)) \leq \|X\|_{L^\infty}|t_n - t_m|$, for all $n, m \in \mathbb{N}$. Hence, the sequence $\phi_{t_n}(x)$ is a Cauchy sequence. Since we have assumed that M is complete, this sequence has a limit $y \in M$. Therefore $\lim_{t\to a, t\in I} \phi_t(a) = y$ exists. Then the local existence of the flow generated by X in a neighborhood of y will allow us to extend the flow $\phi_t(x)$ for t past a by setting $\phi_{a+t}(x) = \phi_t(y)$ for $|t|$ small. This is a contradiction and hence our result is proved. □

Lemma 3.4. *We use the notation and the assumptions of Proposition* 3.3*, in particular, $X \in W_{\nabla}^{\infty,\infty}(M;TM)$ generates the flow $\phi_t : M \to M$, $t \in \mathbb{R}$. For any vector bundle E endowed with a connection, the parallel transport τ_t along the integral curves of X generates a one-parameter group of automorphisms of $\mathcal{C}^\infty(M;E)$.*

Proof. This is a classical result. Indeed, the definition of the parallel transport τ_t amounts to solving a linear system of ordinary differential equations (ODEs) along each of the integral curves $\phi_t(x)$ of ϕ. We know by Proposition 3.3 that the integral curves of X extend indefinitely, so $\phi_t(x)$ is defined for all $t \in \mathbb{R}$ and $x \in M$. Hence we have global solutions for the system of ODEs defining the parallel transport (since the integral curves of ϕ extend indefinitely and the ODE system is linear). □

In what follows, one should distinguish between the parallel transport τ_t and the map $\phi_{t*} : \mathcal{C}^\infty(M;T) \to \mathcal{C}^\infty(M;T)$, where T is a tensor bundle on M (tensor product of TM and T^*M or canonical subbundles) and ϕ_{t*} is the map induced by the diffeomorphism $\phi_t : M \to M$. We shall use this construction for $T = TM$ and $T = \mathbb{R}$ (plain functions). Of course, if α is a function, then $\tau_t(\alpha) = \phi_{t*}(\alpha)$.

Theorem 3.5. *We use the notation and the assumptions of Proposition* 3.3 *and Lemma* 3.4*. Let $k \in \mathbb{Z}_+$. Let us assume also that $E \to M$ is a Hermitian vector bundle endowed with a metric-preserving connection with totally bounded curvature. Then the parallel* *transport* $\tau = (\tau_t^*)_{t\in\mathbb{R}}$ *defines a one-parameter group of continuous operators on all spaces* $W_\nabla^{k,p}(M;E)$, $1 \le p \le \infty$, *such that, if* $k \ge 1$ *and* $Z \in W_\nabla^{k-1,\infty}(M;TM)$, *then*

$$
\begin{aligned}
A(t;X,Z) &:= \tau_t(Z) - \phi_{t*}(Z) \in W_\nabla^{k-1,\infty}(M;TM)\,,\\
&\phi_{t*} \text{ is bounded on } W^{k-1,\infty}(M;TM) \text{ and} \qquad (3.2)\\
B(t;X,Z) &:= \tau_t \nabla_Z \tau_{-t} - \nabla_{\phi_t^*(Z)} \in W_\nabla^{k-1,\infty}(M;\mathrm{End}(E))\,,
\end{aligned}
$$

with $\mathcal{C}^\infty$*-dependence for* A *and* B *on* $t \in \mathbb{R}$ *if* $Z \in W_\nabla^{\infty,\infty}(M;TM)$. *If* $p < \infty$, *the resulting group* $(\tau_t)_{t\in\mathbb{R}}$ *is strongly continuous. The infinitesimal generator* L_τ *of* τ *acting on* $W^{k+1,p}(M;E)$ *satisfies* $L_\tau\xi = \nabla_X \xi$ *for* $\xi \in W^{k+1,p}(M;E)$.

Proof. We shall prove our result by induction on k. Let us assume $k = 0$. Since the connection on E is metric-preserving, the parallel transport will be isometric between the fibers of E. To obtain the desired result on the boundedness of the induced operator, it is enough to notice that the volume form is increased by at most a bounded factor since $\mathrm{div}(X)$ is bounded (i.e. in L^∞), by the results of [14]. (For $k = 0$ there is nothing to check about ϕ_{t*} or the functions A and B of Equation (3.2).)

Let us assume now that the result is true for $k-1 \ge 0$ and let us prove it for k. We will first prove the result for A, then the boundedness of ϕ_{t*} on the $W_\nabla^{k-1,\infty}(M;TM)$ spaces, then the result for B and, finally, we will check the boundedness of τ_t on the $W_\nabla^{k,p}$ spaces.

Let us prove the result for $A(t;X,Z)$. If, furthermore, $Z \in W^{k,p}(M;TM)$ (slightly better regularity than in the statement), then

$$
\begin{aligned}
\partial_t A(t;X,Z) &= \partial_t\big(\tau_t(Z) - \phi_{t*}(Z)\big)\\
&= \tau_t(\nabla_X(Z)) - \phi_{t*}([X,Z])\\
&= \tau_t([X,Z]) - \phi_{t*}([X,Z]) + \tau_t(\nabla_Z(X))\\
&= A(t;X,[X,Z]) + \tau_t(\nabla_Z(X))\,.
\end{aligned}
$$

We shall use this relation for all $Z = Z_j$, $j = 1,\dots,N$, where $\{Z_j\}$ is a Fréchet system of generators of $W_\nabla^{\infty,\infty}(M;TM)$, as in Equation (3.1). We have $X, Z_j \in W_\nabla^{\infty,\infty}(M;TM)$, and hence $\nabla_{Z_j}(X) \in W_\nabla^{\infty,\infty}(M;TM)$ [14]. The induction hypothesis tells us that τ_t is bounded on the space $W_\nabla^{k-1,\infty}(M;TM)$, which gives then that $\tau_t(\nabla_{Z_j}(X)) \in W_\nabla^{k-1,\infty}(M;TM)$. We then express each $[X,Z_j] = \sum_{ji} C_{ji} Z_i$, with $C_{ji} \in W_\nabla^{\infty,\infty}(M)$, as in [14]. This yields an inhomogeneouse linear system of ODEs in $W^{k-1,p}(M;TM)$ for $A(t;X,Z_j)$ with free term $\tau_t(\nabla_{Z_j}(X)) \in W_\nabla^{k-1,\infty}(M;TM)$. Since $A(0;X,Z) = 0$ and since τ_t preserves $W_\nabla^{k-1,\infty}(M;TM)$, we obtain the desired result that $A(t;X,Z_j) \in W_\nabla^{k-1,\infty}(M;TM)$ for $Z = Z_j$. Next, we use use the linearity of $A(t;X,Z)$ in $Z \in W^{k-1,p}(M;TM)$ (same regularity now as in the statement) and express $Z = \sum_{j=1}^N \alpha_j Z_j$ as a linear combination of the Fréchet system of generators $\{Z_j\}$, $j = 1,\dots,N$ with coefficients $\alpha_j \in W_\nabla^{k-1,\infty}(M)$. We also notice that

$A(t; X, \alpha Z) = \tau_t(\alpha)A(t; X, \alpha Z)$ for α a function, by the definition of A. This gives

$$A(t; X, Z) \;=\; \sum_{j=1}^{N} A(t; X, \alpha_j Z_j) \;=\; \tau_t(\alpha_j) \sum_{j=1}^{N} A(t; X, Z_j) \in W_{\nabla}^{k-1,\infty}(M; TM)\,,$$

where we have used again the induction hypothesis for τ_t acting on $W_{\nabla}^{k-1,\infty}(M)$.

Let us check next that ϕ_{t*} is bounded on the spaces $W_{\nabla}^{k-1,\infty}(M; TM)$. Let $Z \in W_{\nabla}^{k-1,\infty}(M; TM)$. The result we have just proved for A gives

$$\phi_{t*}(Z) \;:=\; \tau_t(Z) - A(t; X, Z) \in W_{\nabla}^{k-1,\infty}(M; TM)$$

Hence ϕ_{t*} maps $W_{\nabla}^{k-1,\infty}(M; TM)$ to itself. Since ϕ_{t*} is continuous in the sense of distributions (by Lemma 3.4) it has closed graph, and hence it is continuous.

Let us now turn to the study of the term $B(t; X, Z)$, which will be similar. Let Ω be the curvature of E. Our assumption that E has totally bounded curvature amounts to $\Omega \in W_{\nabla}^{\infty,\infty}(M; \Lambda^2 T^*M \otimes \mathrm{End}(E))$. Again for Z with a little bit more regularity, we have

$$\begin{aligned} \partial_t B(t; X, Z) \;&=\; \partial_t \big[\tau_t \nabla_Z \tau_{-t} - \nabla_{\phi_{t*}(Z)}\big] \\ &=\; \tau_t \big(\nabla_X \nabla_Z - \nabla_Z \nabla_X\big)\tau_{-t} - \nabla_{\phi_{t*}([X,Z])} \\ &=\; \tau_t \big(\nabla_{[X,Z]} + \Omega(X,Z)\big)\tau_{-t} - \nabla_{\phi_{t*}([X,Z])} \\ &=\; B(t; X, [X,Z]) + \tau_t \Omega(X,Z)\tau_{-t}\,. \end{aligned}$$

We know by the induction hypothesis (boundedness of τ_t on $W_{\nabla}^{k-1,\infty}$ and E with totally bounded curvature) that $\tau_t \Omega(X,Z)\tau_{-t} \in W_{\nabla}^{k-1,\infty}(M; \mathrm{End}(E))$. We complete the discussion for B as we did for A. That is, we notice first that $B(t; X, Z)$ is linear in Z and that $B(t; X, \alpha Z) = \tau_t(\alpha)B(t; X, Z)$ for α a smooth enough function. We then use the last displayed equation for Z ranging through the vectors of a Fréchet system of generators $\{Z_j\}$. We next express $[X, Z_j] = \sum_j \alpha_j Z_j$, and we use linearity to obtain an inhomogeneous linear system of ODEs for $B(t; X, Z_j)$. Since $B(0, X, Z) = 0$, since τ_t preserves $W_{\nabla}^{k-1,\infty}(M; TM)$, and since $\tau_t \Omega(X,Z)\tau_{-t} \in W_{\nabla}^{k-1,\infty}(M; \mathrm{End}(E))$, we obtain $B(t; X, Z_j) \in W_{\nabla}^{k-1,\infty}(M; \mathrm{End}(E))$. By using this relation, the linearity of $B(t; X, Z)$ in Z, and by expressing Z as a linear combination with $W_{\nabla}^{k-1,\infty}(M)$ coefficients of the Fréchet basis $\{Z_j\}$, $j = 1, \ldots, N$, we obtain the desired result that $B(t; X, Z) \;:=\; \tau_t \nabla_Z \tau_{-t} - \nabla_{\phi_{t*}(Z)} \in W_{\nabla}^{k-1,\infty}(M; \mathrm{End}(E))$.

It remains to prove that τ_t is maps continuously $W_{\nabla}^{k,p}(M; E)$ to itself. Let us prove this without checking the continuity in t. Let $\xi \in W_{\nabla}^{k,p}(M; E)$. By Lemma 3.2, it is enough to check that $\nabla_{Z_j}\tau_{-t}(\xi) \in W_{\nabla}^{k-1,p}(M; E)$ for all $j = 1, \ldots, N$. We have just proved that $\phi_{t*}(Z_j) \in W_{\nabla}^{k-1,\infty}(M; TM)$. Hence $\phi_{t*}(Z_j)$ can be expressed as a linear combination of the vectors Z_i with coefficients in $W_{\nabla}^{k-1,\infty}(M)$ and therefore $\nabla_{\phi_{t*}(Z_j)}\xi \in W_{\nabla}^{k-1,p}(M; E)$. The relation we proved for B, namely $B(t; X, Z) \in W_{\nabla}^{k-1,\infty}(M; \mathrm{End}(E))$ gives also $B(t; X, Z)\xi \in W_{\nabla}^{k-1,p}(M; E)$. Putting all this together we obtain

$$\tau_t \nabla_{Z_j} \tau_{-t} \xi \;=\; \nabla_{\phi_{t*}(Z_j)} \xi + B(t; X, Z)\xi \in W^{k-1,p}_{\nabla}(M; E)\,.$$

The desired result follows by multiplying the last equation to the left with τ_{-t} (which is bounded on $W^{k-1,p}_{\nabla}(M; E)$ by induction). Hence τ_t maps $W^{k,\infty}_{\nabla}(M; TM)$ to itself. Since τ_t is continuous in the sense of distributions (Lemma 3.4) it has closed graph, and hence $\tau_t : W^{k,\infty}_{\nabla}(M; TM) \to W^{k,\infty}_{\nabla}(M; TM)$ is continuous.

The smoothness of A and B as functions of t follows from the fact that the free term of the linear equations defining them are smooth functions of t and from the smoothness of $\tau_t(\alpha)$ if $\alpha \in W^{\infty,\infty}_{\nabla}(M)$.

Finally, the continuity of the group τ_t follows from the continuity on smooth sections with compact support and the density of those in the spaces $W^{k,p}_{\nabla}$ for $p < \infty$. The statement on the infinitesimal generator is proved by the same argument. □

The following consequence will allow us to check the hypotheses of Proposition 2.5.

Proposition 3.6. *Let $a \in W^{k+1,\infty}_{\nabla}(M; \mathrm{Hom}(\mathcal{F}^M_\mu(E); F))$ and $X \in W^{\infty,\infty}_{\nabla}(M; TM)$. We assume E to have totally bounded curvature and we use the notation in Theorem 3.5. Let $\tau = (\tau_t)_{t\in\mathbb{R}}$ be the one-parameter group defined by parallel transport on any given Sobolev space H^k or $W^{k,\infty}_{\nabla}$.*

(i) $[\nabla_X, a \cdot \nabla^{tot}] = \nabla_X(a) \cdot \nabla^{tot} + a \cdot [\nabla_X, \nabla^{tot}]$ *is a differential operator of order $\leq \mu$ with coefficients in $W^{k,\infty}_{\nabla}(M; \mathrm{Hom}(\mathcal{F}^M_\mu(E); F))$.*

(ii) $\|[\nabla_X, a \cdot \nabla^{tot}]\|_{\mathcal{L}(H^{k+\mu}(M;E);\mathcal{L}(H^k(M;F))} \lesssim \|a\|_{W^{k+1}_{\nabla}}$.

(iii) $t^{-1}\big[\tau_t(a \cdot \nabla^{tot})\tau_{-t} - a \cdot \nabla^{tot}\big]$ *converges strongly to* $[\nabla_X, a \cdot \nabla^{tot}]$ *in* $\mathcal{L}(H^{k+\mu}(M; E); \mathcal{L}(H^k(M; F))$.

Proof. (i) is a straightforward calculation. To prove (ii), we notice that $\nabla_X(a) \in W^{k,\infty}_{\nabla}(M; \mathrm{Hom}(\mathcal{F}^M_\mu(E); F))$ and that $[\nabla_X, \nabla^{tot}]$ is a ∇-differential operator with $W^{\infty,\infty}_{\nabla}$ coefficients. Therefore (ii) follows right away from (i) and Lemma 2.2. To prove (iii), let us first notice that the operators τ_t are bounded in view of Theorem 3.5. We then compute for $\xi \in H^{k+\mu}(M; E)$:

$$\frac{1}{t}\big[\tau_t(a \cdot \nabla^{tot})\tau_{-t} - a \cdot \nabla^{tot}\big]\xi \;=\; \frac{1}{t}\big[\tau_t(a \cdot \nabla^{tot})\tau_{-t} - \tau_t a \tau_{-t} \cdot \nabla^{tot}$$
$$+ \tau_t a \tau_{-t} \cdot \nabla^{tot} - a \cdot \nabla^{tot}\big]\xi \;=\; \tau_t a \tau_{-t} \cdot \frac{1}{t}\big[\tau_t \nabla^{tot} \tau_{-t} - \nabla^{tot}\big]\xi + \frac{1}{t}\big[\tau_t a \tau_{-t} - a\big] \cdot \nabla^{tot}\xi$$
$$\to a \cdot [\nabla_X, \nabla^{tot}]\xi + \nabla_X(a) \cdot \nabla^{tot}\xi\,.$$

The proof that $\lim_{t\to 0} \frac{1}{t}\big[\tau_t a \tau_{-t} - a\big]\xi = \nabla_X(a)\xi$, $\xi \in H^{k+\mu}(M; E)$ is as in [18]. □

These results give the action of certain diffeomorphism groups on Sobolev spaces on manifolds with bounded geometry [1, 2, 9, 10].

Corollary 3.7. *We use the notation of Proposition* 3.6*. Suppose that $a \in W^{|s_0|+k+1,\infty}_{\nabla}(M; \mathrm{Hom}(\mathcal{F}^M_\mu(E); F))$ and that $P = a \cdot \nabla^{tot} : H^{\sigma+\mu}(M; E) \to H^\sigma(M; F)$ is an isomorphism. Then, for all $f \in H^{\sigma+1}(M; E)$ we have*

$$\nabla_X(P^{-1}f) \;=\; P^{-1}(\nabla_X f) - P^{-1}[\nabla_X, P]P^{-1}f\,.$$

Proof. As in [18], the proof of this result is a direct and immediate consequence of Propositions 2.5 and 3.6 and of Theorem 3.5. Indeed, let us consider the spaces $H^{\sigma+\mu}(M;E)$ and $H^{\sigma}(M;F)$ and the operator $T := P = a \cdot \nabla^{tot}$ and the groups of automorphisms $\tau = (\tau_t)_{t\in\mathbb{R}}$ generated by the parallel transport along the flow defined by X. It was assumed that $T := P := a \cdot \nabla$ is bijective. Proposition 3.6, parts (i) and (iii) shows that the other two hypotheses of Proposition 2.5 are satisfied with $Q := [\nabla_X, a \cdot \nabla^{tot}]$. Proposition 2.5 then gives the result. □

4. Proof of the main theorem

Let us now give the proof of the main result (Theorem 1.3). Our proof follows the method of [18]. See also [19] for other similar results.

Proof of Theorem 1.3. In this proof, we shall write H^j instead of $H^j(M;E)$ or $H^j(M;F)$, and $W^{j,\infty}_{\nabla}$ instead of $W^{j,\infty}_{\nabla}(M;\mathrm{Hom}(E;F))$, to simplify the notation. Thus we shall write $\|f\|_{H^j} = \|f\|_{H^j(M;E)}$ and so on. Moreover, we shall write $\|a\|_j = \|a\|_{W^{j,\infty}(M;E)}$. The relation $\mathfrak{C} \geq 1$ follows from Lemma 2.3. To prove the relation (I_k) we shall proceed by induction on $k \geq 0$ using Corollary 3.7 (which is Proposition 2.5 applied to the setting that we need).

The case $k = 0$ (that is, the relation (I_0)) follows right away from the definition of the norm of P_0^{-1}. Let us now prove the result for $k+1$ assuming that it is true for k. As before, we shall write $\sigma := s_0+k$ for the sake of brevity, and we shall use the result of Lemma 3.2 on norm equivalences. Let then $f \in H^{\sigma+1} = H^{s_0+k+1}(M;F)$. Since $f \in H^{\sigma}$ as well, the induction hypothesis gives that $P_0^{-1}f \in H^{\sigma+\mu}$. The induction step is then to prove that $P_0^{-1}f \in H^{\sigma+1+\mu}$ and that (I_{k+1}) is satisfied.

Let $Z_1, Z_2, \ldots, Z_N \in \mathcal{W}_b(M) := W^{\infty,\infty}_{\nabla}(M;TM)$ be a Fréchet system of generators of $\mathcal{W}_b(M)$ as $\mathcal{C}^{\infty}_{\mathrm{b}}$-module, for instance the system given in Equation (3.1). Let also $Z_0 := id$, to simplify the notation, as before. Also, we shall write Z_ℓ instead of ∇_{Z_ℓ}. Corollary 3.7 for $P_0 : H^{\sigma+\mu} \to H^{\sigma}$ and the parallel transport automorphisms groups generated by Z_ℓ, $\ell \geq 1$, (which exist due to Theorem 3.5) give

$$\|Z_\ell(P_0^{-1}f)\|_{H^{\sigma+\mu}} \leq \|P_0^{-1}(Z_\ell f)\|_{H^{\sigma+\mu}} + \|Qf\|_{H^{\sigma+\mu}}\,, \tag{4.1}$$

where $Q := P_0^{-1}[Z_\ell, P_0]P_0^{-1} : H^{\sigma}(M) \to H^{\sigma+\mu}(M)$. For every $\ell = 0, \ldots, N$, we have $Z_\ell f \in H^{\sigma}$, and hence we can use by the induction hypothesis the relation (I_k) with f replaced with $Z_\ell f$ to obtain for $\ell \geq 1$:

$$\begin{aligned}\|P_0^{-1}(Z_\ell f)\|_{H^{\sigma+\mu}} &\lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1}\, \|a\|^q_{|s_0|+k} \|Z_\ell f\|_{H^{\sigma-q}} \\ &\lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1}\, \|a\|^q_{|s_0|+k} \|f\|_{H^{\sigma+1-q}}\,. \end{aligned} \tag{4.2}$$

To estimate the term $\|Qf\|_{H^{\sigma+\mu}}$, we shall use the relation (I_k) twice. First, for $g := [Z_\ell, P]P^{-1}f \in H^\sigma(M)$, the induction hypothesis (I_k) gives the relation

$$\|Qf\|_{H^{\sigma+\mu}} := \|P_0^{-1}(g)\|_{H^{\sigma+\mu}} \lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1} \, \|a\|_{|s_0|+k}^q \|g\|_{H^{\sigma-q}} \,. \tag{4.3}$$

Moreover, for q fixed in $\{0, 1, \dots, k\}$, using Proposition 3.6(ii), we get:

$$\begin{aligned} \|g\|_{H^{\sigma-q}} &= \|[Z_\ell, P_0]P_0^{-1}f\|_{H^{\sigma-q}} \lesssim \|a\|_{|s_0|+k+1-q}\|P_0^{-1}(f)\|_{H^{\sigma-q+\mu}} \\ &\lesssim \|a\|_{|s_0|+k+1-q} \sum_{s=0}^{k-q} |||P_0^{-1}|||_0^{s+1} \, \|a\|_{|s_0|+k-q}^s \|f\|_{H^{\sigma-q-s}} \\ &\lesssim \sum_{s=0}^{k-q} |||P_0^{-1}|||_0^{s+1} \, \|a\|_{|s_0|+k+1}^{s+1} \|f\|_{H^{\sigma-q-s}} \,. \end{aligned} \tag{4.4}$$

Consequently, the equations (4.3) and (4.4) give,

$$\|Qf\|_{H^{\sigma+\mu}} \lesssim \sum_{q=0}^{k} \sum_{s=0}^{k-q} |||P_0^{-1}|||_0^{q+s+2} \, \|a\|_{|s_0|+k+1}^{q+s+1} \|f\|_{H^{\sigma-q-s}} \,. \tag{4.5}$$

Then, by substituting $p = q + s + 1$, we get that

$$\|Qf\|_{H^{\sigma+\mu}} \lesssim \sum_{p=1}^{k+1} |||P_0^{-1}|||_0^{p+1} \, \|a\|_{|s_0|+k+1}^p \|f\|_{H^{\sigma+1-p}} \,. \tag{4.6}$$

Then, by using the equations (4.2) and (4.6) to estimate the two right-hand side terms in (4.1), we obtain that:

$$\|Z_\ell(P^{-1}f)\|_{H^{\sigma+\mu}} \lesssim \sum_{q=0}^{k+1} |||P_0^{-1}|||_0^{q+1} \, \|a\|_{|s_0|+k+1}^q \|f\|_{H^{\sigma+1-q}} \,. \tag{4.7}$$

To use the estimate of Lemma 3.2, need to estimate $\|Z_0 P_0^{-1} f\|_{H^{\sigma+\mu}} = \|P_0^{-1}f\|_{H^{\sigma+\mu}}$, which we will do by using (I_k) to obtain

$$\begin{aligned} \|Z_0(P_0^{-1}f)\|_{H^{\sigma+\mu}} &\lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1} \, \|a\|_{|s_0|+k}^q \|f\|_{H^{\sigma-q}} \\ &\lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1} \, \|a\|_{|s_0|+k+1}^q \|f\|_{H^{\sigma+1-q}} \,. \end{aligned}$$

We then take the sum of this last equation with all the equations 4.7, for $\ell = 1, \dots, N$. As desired, Lemma 3.2 gives

$$\|P_0^{-1}f\|_{H^{\sigma+1+\mu}} \lesssim \sum_{q=0}^{k+1} |||P_0^{-1}|||_0^{q+1} \, \|a\|_{|s_0|+k+1}^q \|f\|_{H^{\sigma+1-q}} \,, \tag{4.8}$$

which is exactly the relation (I_{k+1}) we were also looking for. This reasoning also gives $P_0^{-1} f \in H^{\sigma+1+\mu}$. Using $\mathfrak{C} \geq 1$ and bounding $\|f\|_{H^{\sigma-q}}$ with $\|f\|_{H^\sigma}$, we also obtain the desired inequality for $|||P_{k+1}^{-1}|||$. This completes the proof of Theorem 1.3. □

5. Remarks and applications

The method of our theorem gives a stronger result than the more straightforward estimates expounded below.

Remark 5.1. As in [18], Corollary 3.7 and the Lemmas 2.2 and 3.2 give

$$|||P_0^{-1}|||_{k+1} \leq |||P_0^{-1}|||_k^2 \|a\|_{|s_0|+k+1} \,. \tag{5.1}$$

An induction argument in k then gives

$$|||P_0^{-1}|||_k \ := \ \|P_k^{-1}\| \leq \|P_0^{-1}\|^{2^k} \|a\|_{|s_0|+k}^{2^k-1} \,, \tag{5.2}$$

which is, obviously, much weaker than the result of Theorem 1.3. Nevertheless, this type of result (which follows the method of [6]) is also sufficient for many applications.

Let $A \in W_\nabla^{k+2m,\infty}(M; \mathrm{End}(\mathcal{F}_m^M(E)))$, $k, m \geq 0$. It is known then from [14] that $(\nabla^{tot})^* A \nabla^{tot}$ is a ∇-differential operator of order $2m$ with coefficients in $W_\nabla^{k+2m,\infty}(M; \mathrm{End}(\mathcal{F}_m^M(E)))$. We will let $\mathrm{Re}\, A := \frac{1}{2}\big(A + A^*\big)$, with A^* the adjoint of A. We write $A \geq \gamma I$ if, for any complex vector ξ on which A acts, we have $(A\xi, \xi) \geq \gamma \|\xi\|^2$, pointwise (that is, as functions on M). From now on, we shall assume that $s_0 = -m$. Theorem 1.3 applied to the operator $(\nabla^{tot})^* A \nabla^{tot}$ (and $s_0 = -m$) then yields the following result.

Theorem 5.2. *Let* $A \in W_\nabla^{k+2m,\infty}(M; \mathrm{End}(\mathcal{F}_\mu^M(E)))$ *be such that* $\mathrm{Re}\, A \geq \gamma I$, $\gamma > 0$. *Let* $\mathcal{P}_k = (\nabla^{tot})^* A \nabla^{tot} : H^{k+m}(M; E) \to H^{k-m}(M; E)$. *Then* $\mathcal{P}_0$ *is invertible with norm* $|||\mathcal{P}_0^{-1}|||_0 \leq \gamma^{-1}$. *Moreover, given* $f \in H^{k-m}(M; E)$, *we have that*

$$\|\mathcal{P}_0^{-1} f\|_{H^{k+m}} \lesssim \sum_{q=0}^{k} \gamma^{-(q+1)} \, \|A\|_{W^{k+2m}}^q \|f\|_{H^{k-m}} \,.$$

In particular, $\mathcal{P}_k$ *is invertible and* $|||\mathcal{P}_k^{-1}|||_k \leq \gamma^{-k-1} \, \|A\|_{W^{k+2m}}^k$.

Proof. Let $u \in H^m(M; E)$. Then

$$\mathrm{Re}(\mathcal{P}_0 u, u) \ := \ \mathrm{Re}(A\nabla^{tot} u, \nabla^{tot} u) \geq \gamma(\nabla^{tot} u, \nabla^{tot} u) \geq \gamma \|u\|_{H^m}^2 \,.$$

So $\mathcal{P}_0$ is invertible and $|||\mathcal{P}^{-1}|||_0 \leq \gamma^{-1}$, by the Lax-Milgram lemma. We also notice that $1 \leq \mathfrak{C} \lesssim \gamma^{-1} \|A\|_{W^{k+2m}}$. The proof is then completed by using Theorem 1.3. □

We obtain the following consequence.

Theorem 5.3. *We use the setting of Theorem* 1.3. *Let* $X = (X_1, X_2, \ldots, X_K)$ *be a vector Gaussian random variable with covariance* $\sigma = (\sigma_{ij}) > 0$. *Let*

$$A_1, A_2, \ldots, A_K \in W_\nabla^{k+2m,\infty}(M; \mathrm{End}(\mathcal{F}_\mu^M(E))) \,,$$

with $A_j \geq \gamma I$, $\gamma > 0$. *We note* $A := \sum_{j=1}^K e^{X_j} A_j$. *Then* $|||\mathcal{P}_k^{-1}|||_k$ *is integrable.*

Proof. We proceed as in [18]. We have

$$\mathrm{Re}(A) := \mathrm{Re}\left(\sum_{j=1}^{K} e^{X_j} A_j\right) \geq \gamma \sum_{j=1}^{K} e^{X_j} .$$

Likewise $\|A\|_{W^{k+2m}} \leq \sum_{j=1}^K e^{X_j} \|A_j\|_{W^{k+2m}}$. According to Theorem 5.2, we get

$$\begin{aligned}|||\mathcal{P}_k^{-1}|||_k &\leq \left(\gamma \sum_{j=1}^K e^{X_j}\right)^{-k-1} \|A\|_{W^{k+2m}}^K \\ &\leq \left(\gamma \sum_{j=1}^K e^{X_j}\right)^{-k-1} \left(\sum_{j=1}^K e^{X_j} \|A_j\|_{W^{k+2m}}\right)^k \\ &\leq C\Big(\sum_{j=1}^K e^{X_j}\Big)^{-1},\end{aligned}$$

which is integrable, because $\left(\sum_{j=1}^K e^{x_j}\right)^{-1} \leq e^{|x_1|+\ldots+|x_K|}$ is integrable with respect to the measure of density $e^{-(\sigma x,x)} \leq e^{-\epsilon\|x\|^2}$, where C depends on $\|A_j\|_{W^{k+2m}}$, k and K. □

References

[1] Amann, H., *Function spaces on singular manifolds.* Math. Nachr. **286**(2013), no. 5-6 , 436–475.

[2] Ammann, B., Grosse, N., Nistor, V., *Well-posedness of the Laplacian on manifolds with boundary and bounded geometry*, Math. Nachr., **292**(2019), no. 6, 1213-1237.

[3] Banjai, L., Otárola, E., *A PDE approach to fractional diffusion: a space-fractional wave equation*, Numer. Math., **143**(2019), no. 1, 177-222.

[4] Brezis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[5] Brüning, J., Geyler, V., Pankrashkin, K., *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators*, Rev. Math. Phys., **20**(2008), no. 1, 1-70.

[6] Băcuţă, C., Li, H., Nistor, V., *Differential operators on domains with conical points: precise uniform regularity estimates*, Rev. Roumaine Math. Pures Appl., **62**(2017), no. 3, 383-411.

[7] Cohen, A., Devore, R., Schwab, C., *Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's*, Anal. Appl. (Singap.), **9**(2011), no. 1, 11-47.

[8] Dauge, M., *Elliptic Boundary Value Problems on Corner Domains*, vol. 1341 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1988. Smoothness and asymptotics of solutions.

[9] Große, N., Nistor, V., *Uniform Shapiro-Lopatinski conditions and boundary value problems on manifolds with bounded geometry*, Potential Anal., **53**(2020), no. 2, 407-447.

[10] Große, N., Schneider, C., *Sobolev spaces on Riemannian manifolds with bounded geometry: general coordinates and traces*, Math. Nachr., **286**(2013), no. 16, 1586-1613.

[11] Harbrecht, H., Schmidlin, M., *Multilevel methods for uncertainty quantification of elliptic PDEs with random anisotropic diffusion*, Stoch. Partial Differ. Equ. Anal. Comput., **8**(2020), no. 1, 54-81.

[12] Hebey, E., *Sobolev Spaces on Riemannian Manifolds*, vol. 1635 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1996.

[13] Herrmann, L., Keller, M., Schwab, C., *Quasi-Monte Carlo Bayesian estimation under Besov priors in elliptic inverse problems*, Math. Comp., **90**(330)(2021), 1831-1860.

[14] Kohr, M., Nistor, V., *Sobolev spaces and r-differential operators on manifolds I: Basic properties and weighted spaces*, Ann. Global Anal. Geom., (2022).

[15] Le Maître, O.P., Knio, O.M., Najm, H.N., Ghanem, R.G., *Uncertainty propagation using Wiener-Haar expansions*, J. Comput. Phys., **197**(2004), no. 1, 28-57.

[16] Li, H., Nistor, V., Qiao, Y., *Uniform shift estimates for transmission problems and optimal rates of convergence for the parametric finite element method*, In: Numerical Analysis and its Applications, vol. 8236 of Lecture Notes in Comput. Sci. Springer, Heidelberg, 2013, 12-23.

[17] Lions, J.-L., Magenes, E., *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der Mathematischen Wissenschaften, Band 181.

[18] Mohsen, H., Labrunie, S., Nistor, V., *Estimations polynomiales pour les solutions des problèmes de transmission sur cylindres*, preprint, 2022.

[19] Mohsen, H., Labrunie, S., Nistor, V., *Polynomial bounds for the solutions of transmission problems on smooth, bounded domains*, in final preparation.

[20] Nistor, V., Schwab, C., *High-order Galerkin approximations for parametric second-order elliptic partial differential equations*, Math. Models Methods Appl. Sci., **23**(2013), no. 9, 1729-1760.

[21] Taylor, M., *Partial Differential Equations, I. Basic Theory*, second ed., vol. 115 of Applied Mathematical Sciences, Springer, New York, 2011.

Mirela Kohr
Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, Cluj-Napoca, Romania
e-mail: mkohr@math.ubbcluj.ro

Simon Labrunie
Université de Lorraine, Nancy, France
e-mail: simon.labrunie@univ-lorraine.fr

Hassan Mohsen
Université de Lorraine, Nancy, France
e-mail: hassan.mohsen@univ-lorraine.fr

Victor Nistor
Université de Lorraine, Metz, France
e-mail: victor.nistor@univ-lorraine.fr