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Generalized Szász-Mirakian type operators

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Abstract. In this paper we propose certain modifications of Szász-Mirakian type operators and study their approximation properties. We also give a Voronovskaya type theorem for these operators.

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1. Introduction

Classical Szász-Mirakian operator is defined as

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{(k)!} f\left(\frac{k}{n}\right) \tag{1.1}$$

with $x \in R_0 = [0, \infty), n \in N = \{1, 2, 3, \dots\}, k \in N_0 = N \cup 0$ and $f \in C(R_0)$. The approximation behaviour of this operator for bounded functions is well known (see, e.g. [2, 8]). Hermann considered this operator on a much wider class, growing faster than exponentially. In 2005, Schurer[6, 7] type generalization was given by Moreno [4] for this operator 1.1.

$$S_{n,p}(f;x) = e^{-(n+p)x} \sum_{k=0}^{\infty} \frac{((n+p)x)^k}{(k)!} f\left(\frac{k}{n}\right), x \in R_0, n \in N, p \in N_0.$$

Later Firlej and Rempulska [3] introduced a modified Szász-Mirakian operator:

$$\bar{S}_n(f;x) = \frac{1}{\cosh(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(2k)!} f\left(\frac{2k}{n}\right), x \in R_0, n \in N.$$

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Motivated by the above two modifications now we consider Szász-Mirakian type operators for $f \in C_B$

$$\widehat{S}_{n,p}(f;x) = \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} f\left(\frac{2k}{n}\right), x \in R_0, n \in N, p \in N_0. \quad (1.2)$$

and $coshx, tanhx$ are elementary hyperbolic functions. Let

$$C_B^2 = \{f \in C_B \cap C^2(R_0) : f'; f'' \in C_B\}$$

be the space of real-valued functions uniformly continuous and bounded on $R_0 = [0;\infty)$ and let the norm in C_B be given by the formula

$$\|f\| = \sup_{x \in R_0} |f(x)|.$$

In the year 2008, Deo et al. [1] studied Voronovskaya type results for modified Bernstein operators. Now the purpose of this study is to give Voronovskaya type theorems for Schurer ([6, 7]) as well Firlej and Rempulska [3] type modification of Szász-Mirakian operators.

2. Auxiliary results

In this section we prove some results on $\widehat{S}_{n,p}$ that will help in establishing the main result.

Lemma 2.1. *For each $n \in N$ and $x \in R_0$ we have*

$$\widehat{S}_{n,p}(1;x) = 1, \quad (2.1)$$

$$\widehat{S}_{n,p}(t;x) = \frac{(n+p)x}{n} \tanh((n+p)x), \quad (2.2)$$

$$\widehat{S}_{n,p}(t^2;x) = \frac{((n+p)x)^2}{n^2} + \frac{(n+p)x}{n^2} \tanh((n+p)x), \quad (2.3)$$

$$\widehat{S}_{n,p}(t^3;x) = \frac{1}{n^3}\left[\{((n+p)x)^3 + (n+p)x\} \tanh((n+p)x) + 3((n+p)x)^2\right], \quad (2.4)$$

and

$$\widehat{S}_{n,p}(t^4;x) = \frac{1}{n^4}\left[((n+p)x)^4 + \{6((n+p)x)^3 + (n+p)x\} \tanh((n+p)x) + 7((n+p)x)^2\right]. \quad (2.5)$$

Proof. From (1.2) we can easily obtain (2.1) and

$$\widehat{S}_{n,p}(t;x) = \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} \left(\frac{2k}{n}\right) = \frac{(n+p)x}{n} \tanh((n+p)x).$$

$$\widehat{S}_{n,p}(t^2;x) = \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} \left(\frac{2k}{n}\right)^2$$
$$= \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} \frac{\{(2k)(2k-1)+2k\}}{n^2}$$
$$= \frac{((n+p)x)^2}{n^2} + \frac{(n+p)x}{n^2} \tanh((n+p)x)$$

$$\widehat{S}_{n,p}(t^3;x) = \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} \left(\frac{2k}{n}\right)^3$$
$$= \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} \frac{\{(2k)(2k-1)(2k-2)+6k(2k-1)+2k\}}{n^3}$$
$$= \frac{1}{\cosh((n+p)x)} \Big[\frac{((n+p)x)^3 + (n+p)x}{n^3} \sinh((n+p)x)$$
$$+ \frac{3((n+p)x)^2}{n^3} \cosh((n+p)x) \Big]$$
$$= \frac{1}{n^3} \left[((n+p)x)^3 \tanh((n+p)x) + 3((n+p)x)^2 + (n+p)x \tanh((n+p)x) \right].$$

$$\widehat{S}_{n,p}(t^4;x) = \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} \left(\frac{2k}{n}\right)^4$$
$$= \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!n^4} \Big[(2k)(2k-1)(2k-2)(2k-3)$$
$$+ 6(2k)(2k-1)(2k-2) + 7(2k)(2k-1) + 2k \Big]$$
$$= \frac{1}{n^4} \left[((n+p)x)^4 + (6((n+p)x)^3 + (n+p)x) \tanh((n+p)x) + 7((n+p)x)^2 \right].$$

□

Using above Lemma 2.1, we shall prove the following Lemma.

Lemma 2.2. *The following equalities hold for all $x \in R_0$ and $n \in N$:*

$$\widehat{S}_{n,p}(t-x;x) = \frac{(n+p)x}{n} [\tanh((n+p)x) - 1] + \frac{px}{n}$$

$$\widehat{S}_{n,p}((t-x)^2;x) = \frac{(n+p)}{n} \left(2x^2 - \frac{x}{n}\right) (1 - \tanh(n+p)x) + \frac{(n+p)x + p^2x^2}{n^2}$$

$$\widehat{S}_{n,p}((t-x)^3;x) \qquad = \frac{(n+p)}{n} tanh((n+p)x) \left[\frac{(n+p)^2}{n^2} x^3 + \frac{x}{n^2} - \frac{3x^2}{n} + 3x^3 \right]$$
$$+ \frac{3(n+p)^2x^2}{n^3} - \frac{3x(n+p)^2x^2}{n^2} - x^3$$

$$\widehat{S}_{n,p}((t-x)^4;x)$$
$$= \frac{(n+p)}{n}\tanh((n+p)x)\Big[\frac{6(n+p)^2x^3}{n^3} - \frac{4(n+p)^2x^3}{n^2} + \frac{x}{n^3} - \frac{4x^2}{n^2} + \frac{6x^3}{n} - 4x^4\Big]$$
$$+ \frac{((n+p)x)^4 + 7((n+p)x)^2 - 12xn((n+p)x)^2 + 6x^2n^2((n+p)x)^2 + x^4n^4}{n^4}$$

Proof. Using Lemma 2.1 we have,

$$\widehat{S}_{n,p}(t-x;x) = \frac{(n+p)x}{n}\tanh((n+p)x) - x$$
$$= \frac{(n+p)x}{n}\left[\tanh((n+p)x) - 1\right] + \frac{px}{n} \tag{2.6}$$

$$\widehat{S}_{n,p}((t-x)^2;x) = \frac{((n+p)x)^2}{n^2} + \frac{(n+p)x}{n^2}\tanh((n+p)x)$$
$$- 2x\frac{(n+p)x}{n}\tanh((n+p)x) + x^2$$
$$= \tanh((n+p)x)\left[\frac{(n+p)x}{n^2} - 2x\frac{(n+p)x}{n}\right] + x^2 + \frac{((n+p)x)^2}{n^2} \tag{2.7}$$
$$= \frac{(n+p)}{n}\left(2x^2 - \frac{x}{n}\right)(1 - \tanh(n+p)x) + \frac{(n+p)x + p^2x^2}{n^2}$$

$$\widehat{S}_{n,p}((t-x)^3;x)$$
$$= \frac{1}{n^3}\left[((n+p)x)^3\tanh((n+p)x) + 3((n+p)x)^2 + (n+p)x\tanh((n+p)x)\right]$$
$$- 3x\left[\frac{((n+p)x)^2}{n^2} + \frac{(n+p)x}{n^2}\tanh((n+p)x)\right]$$
$$+ 3\frac{(n+p)x^3}{n}\tanh((n+p)x) - x^3 \tag{2.8}$$
$$= \frac{(n+p)}{n}tanh((n+p)x)\left[\frac{(n+p)^2}{n^2}x^3 + \frac{x}{n^2} - \frac{3x^2}{n} + 3x^3\right]$$
$$+ \frac{3(n+p)^2x^2}{n^3} - \frac{3x(n+p)^2x^2}{n^2} - x^3$$

$$\widehat{S}_{n,p}((t-x)^4;x) = \widehat{S}_{n,p}(t^4;x) - 4x\widehat{S}_{n,p}(t^3;x) + 6x^2\widehat{S}_{n,p}(t^2;x) - 4x^3\widehat{S}_{n,p}(t;x) + x^4\widehat{S}_{n,p}(1;x)$$
$$= \tanh((n+p)x)\Big[\frac{(n+p)x + 6((n+p)x)^3}{n^4} - \frac{4x\{((n+p)x)^3 + (n+p)x\}}{n^3}$$
$$+ \frac{6x^3(n+p) - 4x^4n(n+p)}{n^2}\Big]$$
$$+ \frac{((n+p)x)^4 + 7((n+p)x)^2 - 12xn((n+p)x)^2 + 6x^2n^2((n+p)x)^2 + x^4n^4}{n^4}$$
$$= \frac{(n+p)}{n}\tanh((n+p)x)\Big[\frac{6(n+p)^2x^3}{n^3} - \frac{4(n+p)^2x^3}{n^2} + \frac{x}{n^3} - \frac{4x^2}{n^2} + \frac{6x^3}{n} - 4x^4\Big]$$

$$+\frac{((n+p)x)^4+7((n+p)x)^2-12xn((n+p)x)^2+6x^2n^2((n+p)x)^2+x^4n^4}{n^4} \quad (2.9)$$

□

In order to prove a Voronovskaya type theorem we need the following lemmas.

Lemma 2.3. *For $n, r \in N, p \in N_0$ and $x \geq 0$, the following results hold*

$$(a)\ 0 \leq x^r(1-tanh(n+p)x) \leq 2^{1-r}r!(n+p)^{-r}$$

$$(b) \lim_{n\to\infty}\left[n\left\{\tanh((n+p)x)-1\right\}\right]=0$$

$$(c) \lim_{n\to\infty}\left\{\tanh((n+p)x)\right\}=1,$$

and

$$(d) \lim_{n\to\infty}\frac{\left\{\tanh((n+p)x)-1\right\}}{n}=0$$

Proof. We shall use an inequality considered in (eq.(22), [5]) which says that, for $m, r \in N$ and $x \geq 0$,

$$0 \leq x^r(1-tanhmx) \leq 2^{1-r}r!m^{-r}$$

Now on replacing m by $n+p, n \in N, p \in N_0$ we get the desired result.
Following the technique used in Lemma 2 of [5],we easily obtain (b),(c) and (d). □

Lemma 2.4. *For every fixed $x \in R_0$, we have*

$$(i) \lim_{n\to\infty} n\widehat{S}_{n,p}(t-x;x)=px,$$

$$(ii) \lim_{n\to\infty} n\widehat{S}_{n,p}((t-x)^2;x)=x,$$

Proof. Using Lemma 2.2 we obtain,

$$\begin{aligned}(i)\widehat{S}_{n,p}(t-x;x) &= \frac{(n+p)x}{n}\tanh((n+p)x)-x\\ &= \frac{(n+p)x}{n}\left[\tanh((n+p)x)-1\right]+\frac{px}{n}\\ &= x\left[\tanh((n+p)x)-1\right]+\frac{px}{n}\tanh((n+p)x).\end{aligned}$$

therefore

$$n\widehat{S}_{n,p}(t-x;x)=xn\left[\tanh((n+p)x)-1\right]+px\tanh((n+p)x).$$

Using Lemma 2.3 we get

$$\begin{aligned}\lim_{n\to\infty} n\widehat{S}_{n,p}(t-x;x) &= \lim_{n\to\infty}\left[xn\left\{\tanh((n+p)x)-1\right\}+px\tanh((n+p)x)\right]\\ &= px.\end{aligned}$$

$$(ii)\widehat{S}_{n,p}((t-x)^2;x)=\frac{(n+p)}{n}\left(2x^2-\frac{x}{n}\right)(1-\tanh(n+p)x)+\frac{(n+p)x+p^2x^2}{n^2}$$

$$n\widehat{S}_{n,p}((t-x)^2;x) = 2x^2[n\{1-\tanh((n+p)x)\}]$$
$$+2px^2\,[1-\{\tanh((n+p)x)\}]-x\,[\{1-\tanh((n+p)x)\}]$$
$$-\frac{px}{n}\,[1-\{\tanh((n+p)x)\}]+x+\frac{px}{n}+\frac{p^2x^2}{n}$$

Again using Lemma 2.3 we obtain

$$\lim_{n\to\infty} n\widehat{S}_{n,p}((t-x)^2;x) = x \qquad \square$$

Lemma 2.5. *The following inequalities are satisfied for all* $n \in N, p \in N_0$ *and* $x \in R_0$:

$$|\widehat{S}_{n,p}(t-x;x)| \le \frac{px+1}{n}$$

$$|\widehat{S}_{n,p}((t-x)^2;x)| \le \frac{3+(p+1)x+p^2x^2}{n}$$

$$|\widehat{S}_{n,p}((t-x)^4;x)| \le \frac{47+(1+p)x+(3+10p+7p^2)x^2+6p^2x^3+p^4x^4}{n^2} \tag{2.10}$$

Proof. For $n, r \in N, p \in N_0$ *and* $x \ge 0$ we have

$$0 \le x^r(1-tanh(n+p)x) \le 2^{1-r}r!(n+p)^{-r}.$$

So from Lemma 2.2,

$$|\widehat{S}_{n,p}(t-x;x)| = \left|\frac{(n+p)x}{n}\,[\tanh((n+p)x)-1]\right| + \frac{px}{n}$$
$$\le \frac{px+1}{n}$$

$$\widehat{S}_{n,p}((t-x)^2;x) = \frac{((n+p)x)^2}{n^2} + \frac{(n+p)x}{n^2}\tanh((n+p)x)$$
$$-2x\frac{(n+p)x}{n}\tanh((n+p)x)+x^2$$
$$= \tanh((n+p)x)\left[\frac{(n+p)x}{n^2} - 2x\frac{(n+p)x}{n}\right] + x^2 + \frac{((n+p)x)^2}{n^2}$$
$$= (\tanh(n+p)x-1)\left[\frac{(n+p)x)}{n^2} - 2x\frac{(n+p)x}{n}\right] + \frac{(n+p)x+p^2x^2}{n^2}$$

$$|\widehat{S}_{n,p}((t-x)^2;x)| = \left|(\tanh(n+p)x-1)\left[\frac{(n+p)x)}{n^2} - 2x\frac{(n+p)x}{n}\right] + \frac{(n+p)x+p^2x^2}{n^2}\right|$$
$$\le \frac{1+(n+p)x+p^2x^2}{n^2} + \frac{2}{n(n+p)}$$
$$\le \frac{3+px+p^2x^2}{n^2} + \frac{x}{n}$$
$$\le \frac{3+px+x+p^2x^2}{n}$$
$$\le \frac{3+(p+1)x+p^2x^2}{n}$$

$$
\begin{aligned}
&\widehat{S}_{n,p}((t-x)^4;x)\\
&=\frac{(n+p)}{n}\tanh((n+p)x)\Big[\frac{6(n+p)^2x^3}{n^3}-\frac{4(n+p)^2x^3}{n^2}+\frac{x}{n^3}-\frac{4x^2}{n^2}+\frac{6x^3}{n}-4x^4\Big]\\
&\quad+\frac{((n+p)x)^4+7((n+p)x)^2-12xn((n+p)x)^2+6x^2n^2((n+p)x)^2+x^4n^4}{n^4}\\
&=\tanh((n+p)x)\Big[\frac{6(n+p)^3x^3}{n^4}-\frac{4(n+p)^3x^4}{n^3}+\frac{(n+p)x}{n^4}-\frac{4(n+p)x^2}{n^3}+\frac{6(n+p)x^3}{n^2}\\
&-4\frac{(n+p)}{n}x^4\Big]+\frac{(n+p)^4x^4}{n^4}+\frac{7(n+p)^2x^2}{n^4}-\frac{12(n+p)^2x^3}{n^3}+\frac{6(n+p)^2x^4}{n^2}+x^4\\
&=\Big(\tanh((n+p)x)-1\Big)\Big[\frac{6(n+p)^3x^3}{n^4}-\frac{4(n+p)^3x^4}{n^3}+\frac{(n+p)x}{n^4}-\frac{4(n+p)x^2}{n^3}\\
&+\frac{6(n+p)x^3}{n^2}-4\frac{(n+p)}{n}x^4\Big]+\left(\frac{(n+p)^4}{n^4}-\frac{4(n+p)^3}{n^3}+\frac{6(n+p)^2}{n^2}-4\frac{(n+p)}{n}+1\right)x^4\\
&+\Big(\frac{6(n+p)^3}{n^4}+\frac{6(n+p)}{n^2}-\frac{12(n+p)^2}{n^3}\Big)x^3+\Big(\frac{7(n+p)^2}{n^4}-\frac{4(n+p)}{n^3}\Big)x^2+\frac{(n+p)}{n^4}x
\end{aligned}
$$

$$
\begin{aligned}
|\widehat{S}_{n,p}((t-x)^4;x)|&\le\frac{10}{n^4}+\frac{16}{n^3(n+p)}+\frac{9}{n^2(n+p)^2}+\frac{12}{n(n+p)^3}+\\
&\frac{p^4x^4}{n^4}+\frac{6p^2x^3}{n^3}+\frac{(3+10p+7p^2)x^2}{n^2}+\frac{(1+p)x}{n^2}\\
&\le\frac{47}{n^4}+\frac{p^4x^4+6p^2x^3+(3+10p+7p^2)x^2+(1+p)x}{n^2}\\
&\le\frac{47+(1+p)x+(3+10p+7p^2)x^2+6p^2x^3+p^4x^4}{n^2}
\end{aligned}
$$

□

3. Voronovskaya type theorems

In this section we give a Voronovskaya-type theorem for the operators $\widehat{S}_{n,p}$ with the help of properties of $\widehat{S}_{n,p}$, which are already mentioned in the above lemmas.

Lemma 3.1. *Suppose that x_0 is a fixed point in R_0 and $\varphi(t;x_0)$is a given function belonging to C_B and such that*

$$\lim_{t\to x_0}\varphi(t;x_0)=0\left(\lim_{t\to p}\varphi(t;0)=0\right)$$

Then for a fixed $p\in N$

$$\lim_{n\to\infty}\widehat{S}_{n,p}(\varphi(t;x_0);x_0)=0. \tag{3.1}$$

Proof. By (1.2) we have for $n\in N$ and a fixed point $x_0\ge 0$

$$\widehat{S}_{n,p}(\varphi(t;x_0);x_0)=\frac{1}{cosh((n+p)x)}\sum_{k=0}^{\infty}\frac{((n+p)x)^{2k}}{(2k)!}\varphi\left(\frac{2k}{n};x_0\right) \tag{3.2}$$

Choose $\epsilon > 0$. Since $\varphi(\cdot; x_0) \in C_B$, there exists a positive constant $\delta \equiv \delta(\epsilon)$ such that

$$|\varphi(t; x_0)| < \frac{\epsilon}{2}, \quad \text{if } |t - x_0| < \delta, \ t \geq 0$$

Moreover there exists a positive constant M such that $|\varphi(t; x_0)| \leq M$ for all $t > 0$. Hence, from (3.2) we get for every $n \in N$

$$\begin{aligned} |\widehat{S}_{n,p}(\varphi(t; x_0); x_0)| &\leq \frac{1}{cosh((n+p)x)} \sum_{k \in Q_{1,n}} \frac{((n+p)x)^{2k}}{(2k)!} \left| \varphi\left(\frac{2k}{n}; x_0\right) \right| \\ &+ \frac{1}{cosh((n+p)x)} \sum_{k \in Q_{2,n}} \frac{((n+p)x)^{2k}}{(2k)!} \left| \varphi\left(\frac{2k}{n}; x_0\right) \right| \\ &= E_1 + E_2 \end{aligned} \tag{3.3}$$

where

$$Q_{1,n} = k \in N_0 : \left| \frac{2k}{n} - x_0 \right| < \delta$$

and

$$Q_{2,n} = k \in N_0 : \left| \frac{2k}{n} - x_0 \right| \geq \delta.$$

From (2.1) we get,

$$E_1 < \frac{\epsilon}{2} \frac{1}{cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} = \frac{\epsilon}{2} \tag{3.4}$$

$$E_2 \leq M \frac{1}{cosh((n+p)x)} \sum_{k \in Q_{2,n}} \frac{((n+p)x)^{2k}}{(2k)!} \tag{3.5}$$

Since $|\frac{2k}{n} - x_0| \geq \delta$ implies $1 \leq \delta^{-2} \left(\frac{2k}{n} - x_0\right)^2$, we can write

$$\begin{aligned} E_2 \quad &\leq \quad M\delta^{-2} \frac{1}{cosh((n+p)x))} \sum_{k \in Q_{2,n}} \frac{((n+p)x)^{2k}}{(2k)!} \left(\frac{2k}{n} - x_0\right)^2 \\ &\leq M\delta^{-2} \widehat{S}_{n,p}((t - x_0)^2; x_0) \end{aligned}$$

which by Lemma 2.5 gives,

$$E_2 \leq \frac{M(3 + px_0 + x_0 + p^2 x_0^2)}{n\delta^2}$$

It is obvious that for given $\epsilon > 0, \delta > 0, M > 0$ *and* $x_0 \geq 0$ we can choose $n_0 \equiv n_0(\epsilon; \delta; M; x_0) \in N$ such that for all natural numbers $n > n_0$ one gets

$$\frac{M(3 + px_0 + x_0 + p^2 x_0^2)}{n\delta^2} < \frac{\epsilon}{2}$$

Hence,

$$E_2 < \frac{\epsilon}{2} \ for \ n > n_0. \tag{3.6}$$

Using equations (3.4) and(3.6) to (3.3) we get

$$\lim_{n \to \infty} \widehat{S}_{n,p}(\varphi(t; x_0); x_0) = 0$$

and this completes the proof. □

Theorem 3.2. *If $f \in C_B^2$, then for every fixed $x \in R_0$ one gets*

$$\lim_{n\to\infty} n\{\widehat{S}_{n,p}(f;x) - f(x)\} = pxf'(x) + \frac{x}{2}f''(x). \tag{3.7}$$

Proof. Let $x_0 \in R_0$ be a fixed point.Then by Taylor formula we can write for every $t \in R_0$,

$$f(t) = f(x_0) + f'(x_0)(t-x_0) + \frac{1}{2}f''(x_0)(t-x_0)^2 + \psi(t;x_0)(t-x_0)^2, \tag{3.8}$$

where $\psi(t;x_0) \in C_B$ and $\lim_{t\to x_0} \psi(t;x_0) = \left(\lim_{t\to 0+} \psi(t;0) = 0\right)$

On applying the operator $\widehat{S}_{n,p}$ on both sides of (3.8), we have for every $n \in N$,

$$\widehat{S}_{n,p}(f;x_0) - f(x_0) = f'(x_0)\widehat{S}_{n,p}(t-x_0;x_0) + \frac{1}{2}f''(x_0)\widehat{S}_{n,p}((t-x_0)^2;x_0) + \widehat{S}_{n,p}(\psi(t;x_0)(t-x_0)^2;x_0) \tag{3.9}$$

In view of (3.9) and by Holder's inequality we get for $n \in N$

$$|\widehat{S}_{n,p}(\psi(t;x_0)(t-x_0)^2;x_0)| \leq \left\{\widehat{S}_{n,p}(\psi^2(t;x_0);x_0)\right\}^{1/2}\left\{\widehat{S}_{n,p}((t-x_0)^4;x_0)\right\}^{1/2} \tag{3.10}$$

Since the function $\varphi(t;x_0) = \psi^2(t;x_0), t \geq 0$ satisfies the assumption of Lemma 3.1 we have,

$$\lim_{n\to\infty} \widehat{S}_{n,p}(\psi^2(t;x_0);x_0) = 0$$

From (2.10) it follows that there exists a constant $M_1 = M_1(p,x_0)$ depending on p *and* x_0 such that

$$n^2\widehat{S}_{n,p}((t-x_0)^4;x_0) \leq M_1 \ for \ n \in N$$

Consequently we obtain

$$\lim_{n\to\infty} n\widehat{S}_{n,p}(\psi(t;x_0)(t-x_0)^2;x_0) = 0 \tag{3.11}$$

from (3.10). Using Lemma 2.4 and (3.11), we derive immediately (3.7) from (3.9). Thus the proof is complete. □

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