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The level sets of functions with bounded critical sets and bounded Hess^+ complements

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. We denote by $\text{Hess}^+(f)$ the set of all points $p \in \mathbb{R}^n$ such that the Hessian matrix $H_p(f)$ of the C^2-smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is positive definite. In this paper we prove several properties of real-valued functions of several variables by showing the connectedness of their level sets for sufficiently high levels, under the boundedness assumption on the critical set. In the case of three variables we also prove the convexity of the levels surfaces for sufficiently high levels, under the additional boundedness assumption on the Hess^+ complement. The selection of the *a priori* convex levels, among the connected regular ones, is done through the positivity of the Gauss curvature function which ensure an ovaloidal shape of the levels to be selected. The ovaloidal shape of a level set makes a diffeomorphism out of the associated Gauss map. This outcome Gauss map diffeomorphism is then extended to a smooth homeomorphism which is used afterwards to construct one-parameter families of smooth homeomorphisms of Loewner chain flavor.

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1. Introduction

After analyzing in [12] the $\text{Hess}^+(f_a)$ region of the polynomial function

$$f_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ f_a(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2)$$

and noticing that its complement is bounded, we provided in [2] a class of norm-coercive polynomial functions with large Hess^+ regions, as their Hess^+ complements happen to be bounded as well. A detailed analysis of the Hess^+ regions for some particular polynomial functions which happen to have bounded $\text{Hess}^+(f)$ complements

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along with bounded critical sets is done there. Basic properties of their level curves, such as regularity, connectedness even convexity of their level sets, for sufficiently large levels, are also pointed out. These properties are then proved to hold true for the whole class of norm-coercive functions of two variables with bounded $\text{Hess}^+(f)$ complements. Since the convex hypersurfaces will be repeatedly used, let us mention that one way to consider convexity for regular hypersurfaces of $\mathbb{R}^n$ consists in their quality to stay on the same side of each of its tangent hyperplane [10, p. 174], [3, p. 37]. On the other hand a regular hypersurface could sometimes bound a convex set and such a hypersurface is also said to be convex [13] (see also [10, p. 175]). Apart from the mentioned source of examples, some sufficient conditions on two functions $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with bounded Hess^+ complements are provided in [2] in order for the product fg to keep having bounded Hess^+ complement.

In this paper we partially extend [2, Theorem 3.6] to real-valued functions of several variables by showing several properties of their level sets for sufficiently high levels, under the boundedness assumption on the critical set. Although we loose convexity of level hypersurfaces in the general case of several variables, we still prove the connectedness in this general case and recapture their convexity in the case of three variables for sufficiently high levels, under the additional boundedness assumption on the Hess^+ complement.

The paper is organized as follows: In the second section, we prove the connectedness of the level sets, for sufficiently large levels, of a real valued function with several variables having bounded critical set. Section 3 is still devoted to properties of the level sets but for functions with three variables under the additional requirements on the function to be norm-coercive and to have bounded Hess^+ complement. For this particular number of variables we recapture the convexity of the level sets for sufficiently large levels, a property we used to have in the case of two variables as well (see [2, 12]). In this case of three variables we still use the positivity test of the Gauss curvature function along a level set to select the *a priori* ovaloidal level sets, among the connected regular ones. For two variables we used the nonvanishing test of the curvature function. In section 4 we first use the outcome Gauss map diffeomorphism of the ovaloidal level sets of the norm-coercive functions with bounded critical set and bounded Hess^+ complement to construct a smooth homeomorphism from the disc D^3 to some sub-level set bounded by an ovaloidal level set. Such smooth homeomorphisms are than used to construct one-parameter families of Loewner chain flavor (compare Theorem 4.1(2)(4) with the definition of Loewner chains [4, 8, 6, 7, 9] and Theorem 4.1(4) with [4, Theorem 1.6 (iv)]) by using a homothetic perturbation of the gradient vector field of the function itself which permutes the, sufficiently high, regular levels.

2. Properties of the level sets of functions with bounded critical set

In order to state the first result of this paper, we shall quickly recall the critical/regular points and critical/regular sets of real-valued functions in a similar fashion with [2]. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a Fréchet differentiable map, then the *rank* of f at $x \in \mathbb{R}^n$ is defined as $\text{rank}_x f := \text{rank}(df)_x = \text{rank}(Jf)_x$. Observe that $\text{rank}_x f \leq \min\{m, n\}$ for every $x \in \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is said to be a *critical point* of f if $\text{rank}_x f < \min\{m, n\}$.

Otherwise x is said to be a *regular point* of f. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a C^1-smooth map, then each point $x \in \mathbb{R}^n$ has an open neighbourhood, say $V_x \subseteq \mathbb{R}^n$, such that $\text{rank}_y f \geq \text{rank}_x f$, for all $y \in V_x$. In particular, once a point x is regular, it has a whole neighbourhood of regular points. Indeed the Jacobian matrix $(Jf)_x$ has a non-zero minor of order $\text{rank}_x f$ and all minors of $(Jf)_x$ of superior order are zero. But the nonzero minor of $(Jf)_x$ is nonzero on a whole open neighbourhood of x since it is a continuous function. This shows that $\text{rank}_y f = \text{rank}_y (Jf)_y \geq \text{rank}(Jf)_x = \text{rank}_x f$, which are satisfied for y in a whole neighbourhood of x. Consequently the set $R(f)$, of regular points of f, is open in $\mathbb{R}^n$, while the set $C(f)$, of critical points of f, is closed in $\mathbb{R}^n$. We denote by $B(f)$ the set $f(C(f))$ of critical values of f. Note that for a real valued function $f : U \longrightarrow \mathbb{R}$, the critical set of f is the vanishing set of its gradient ∇f.

Theorem 2.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth norm-coercive function. If $C(f)$ is bounded, then the c-level set $f^{-1}(c)$ of f is a regular compact connected and orientable hypersurface for c sufficiently large.*

The proof of Theorem 2.1 works along the same lines with the proof of [2, Theorem 3.6].

Remark 2.2. ([14, Theorem 2.5.7]) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a C^1-smooth convex function, then its critical set $C(f)$ is convex. Indeed the critical points of f coincide with the global minimum points of f (see also [2]).

Apart from the examples of real-valued polynomial functions of two variables

1. $f_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ f_a(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2),\ (a > 0);$
2. $g_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ g_a(x, y) = (x^2 + y^2)^2 + 2a^2(x^2 - y^2),\ (a > 0);$
3. $f_a g_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (f_a g_a)(x, y) = (x^2 + y^2)^4 - 4a^4(x^2 - y^2)^2,\ (a > 0);$

which are not convex as their critical sets are discrete with at least two critical points, mentioned in [2], we consider here a polynomial function of three variables.

Example 2.3. $f : \mathbb{R}^3 \longrightarrow \mathbb{R},\ f_a(x, y, z) = (x^2 + y^2 + z^2)^2 - 8(x^2 - y^2 - z^2).$

This function is not convex as its critical set $C(f) = \{(-4, 0, 0), (0, 0, 0), (4, 0, 0)\}$ is obviously not convex. In the sections to come this function will be analyzed from the $\text{Hess}^+(f)$-region point of view.

3. Levels of functions whose Hess^+ complements are additionally bounded

Let D be a nonempty open convex subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a C^2-smooth function. The Hessian matrix of f at an arbitrary point $x \in D$ will be denoted by $H_x(f)$. Recall that $H_f(x)$ is a symmetric matrix and it defines a symmetric bilinear functional

$$\mathcal{H}_f(x) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},\ \mathcal{H}_f(x)(u, v) := u \cdot H_x(f) \cdot v^T.$$

We are interested about the region

$$\text{Hess}^+(f) = \{x \in D : H_x(f) \text{ is positive definite}\}.$$

Example 3.1. ([12]) For the polynomial function $f_a : \mathbb{R}^2 \longrightarrow \mathbb{R}$, given by

$$f_a(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2),\ (a > 0)$$

we have

$$\mathrm{Hess}^+(f_a) = \{(x, y) \in \mathbb{R}^2 | 3(x^2 + y^2)^2 + 2a^2(x^2 - y^2) > a^4\} = g^{-1}_{a/\sqrt{3}}\left(a^4/3, +\infty\right), \quad (3.1)$$

where $g_b : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by $g_b(x, y) = (x^2 + y^2)^2 + 2b^2(x^2 - y^2),\ (b > 0)$.

Recall that the nondiscrete level sets of f_a are the Cassini's ovals and its zero level curve, which is critical, is the Bernoulli's lemniscate.

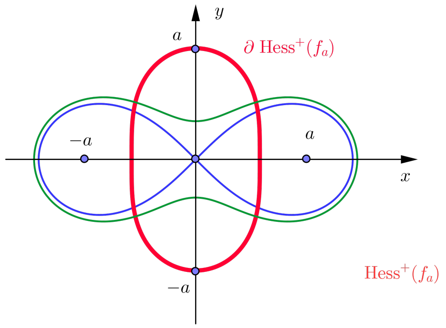

FIGURE 1. The boundary of $\mathrm{Hess}^+(f_a)$, the critical level $f_a^{-1}(0)$ and a positive regular value of f_a

Example 3.2. In this example we deal with a polynomial function of three variables whose restriction to $\mathbb{R}^2$ is f_2. More precisely, the Hess^+ region of the polynomial function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, given by $f(x, y, z) = (x^2 + y^2 + z^2)^2 - 8(x^2 - y^2 - z^2)$, is $\mathrm{Hess}^+(f) = \{(x, y, z) \in \mathbb{R}^3 | 3(x^2 + y^2 + z^2)^2 + 8(x^2 - y^2 - z^2) > 16\}$.

The characteristic polynomial $CP_{H(f)}$ of the Hessian matrix

$$H(f) = \begin{bmatrix} 4(3x^2 + y^2 + z^2 - 4) & 8xy & 8xz \\ 8yx & 4(x^2 + 3y^2 + z^2 + 4) & 8yz \\ 8zx & 8zy & 4(x^2 + y^2 + 3z^2 + 4) \end{bmatrix}$$

is invariant under the action

$$\mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3, (t, x) \mapsto t * x, \quad (3.2)$$

where $t * x := r(t) \cdot x^T$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is identified, on the right hand side, with the row matrix $[x_1\ x_2\ x_3]$ and $r(t)$ stands for the 3×3 matrix of the rotation around the x-axis of angle t.

$$r(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

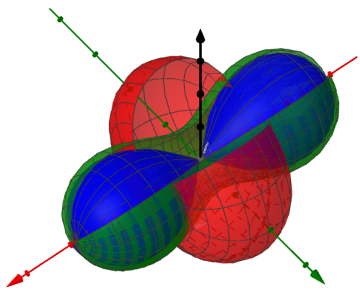

FIGURE 2. The 0-critical level of f, a nonconvex regular level of f and the boundary of $\mathrm{Hess}^+(f)$

Thus, the action (3.2) is

$$t * (x_1, x_2, x_3) := \begin{bmatrix} x_1 \\ x_2 \cos t - x_3 \sin t \\ x_2 \sin t + x_3 \cos t \end{bmatrix}.$$

Indeed, the characteristic polynomial $CP_{H(f)}$ of the Hessian matrix of the polynomial function $\mathrm{Tr}H_{(x,y,z)}(f) = 4(5(x^2+y^2+z^2)+4)$

$$\begin{aligned} A_{11} + A_{22} + A_{33} &= 16(7(x^2+y^2+z^2)^2 + 8(x^2+y^2+z^2) + 16x^2 - 16) \\ \frac{1}{4^3}\det H(f) &= 2x^2(x^2+y^2+z^2+4)^2 + (x^2+y^2+z^2-4)[(x^2+y^2+z^2)^2 \\ &\quad + 2(y^2+z^2+4)(x^2+y^2+z^2) + 8(y^2+z^2) + 16] \end{aligned}$$

are all invariant with respect to the action (3.2). Since $t * \left(x, \sqrt{y^2+z^2}, 0\right) = (x,y,z)$ for every $t \in \mathbb{R}$ such that $\cos t = y/\sqrt{y^2+z^2}$, $\sin t = z/\sqrt{y^2+z^2}$ we get by performing elemetary computations

$$\begin{aligned} CP_{H_{(x,y,z)}(f)}(\lambda) &= CP_{H_{\left(x,\sqrt{y^2+z^2},0\right)}(f)}(\lambda) \\ &= (4(x^2+y^2+3z^2+4) - \lambda) CP_{H_{\left(x,\sqrt{y^2+z^2}\right)}\left(f\big|_{\mathbb{R}^2}\right)} \\ &= (4(x^2+y^2+z^2+4) - \lambda) CP_{H_{\left(x,\sqrt{y^2+z^2}\right)}(f_2)}, \end{aligned}$$

where $f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f_2(u,v) = (u^2+v^2)^2 - 8(u^2-v^2)$. Consequently the Hessian matrix $H_f(x,y,z)$ is positive definite if and only if both eigenvalues of $H_{\left(x,\sqrt{y^2+z^2}\right)}(f_2)$, i.e. the roots of the characteristic polynomial,

$$CP_{H_{\left(x,\sqrt{y^2+z^2}\right)}\left(f\big|_{\mathbb{R}^2}\right)} = CP_{H_{\left(x,\sqrt{y^2+z^2}\right)}(f_2)}$$

are positive. Equivalently, the Hessian matrix $H_{f_2}\left(x, \sqrt{y^2+z^2}\right)$ must be positive definite, as one eigenvalue $\lambda = 4(x^2+y^2+z^2+4)$ of $H_{(x,y,z)}(f)$ is positive. In other words

$$\begin{aligned} (x,y,z) \in \mathrm{Hess}^+(f) &\iff \left(x, \sqrt{y^2+z^2}\right) \in H_{f_2}\left(x, \sqrt{y^2+z^2}\right) \\ &\overset{(3.1)}{\iff} \left(x, \sqrt{y^2+z^2}\right) \in \{(u,v) \in \mathbb{R}^2 | 3(u^2+v^2)^2 + 8(u^2-v^2) > 16\} \\ &\iff 3(x^2+y^2+z^2)^2 + 8(x^2-y^2-z^2) > 16. \end{aligned}$$

Definition 3.3. ([10, p. 174], [3, p. 322]) A compact connected surface $S \subset \mathbb{R}^3$ is said to be an *ovaloid* if its Gauss curvature is everywhere positive.

According with the Hadamard-Stoker Theorem [10, Theorem 6.1, p. 175], the interior of every ovaloid along with the closure of its interior are convex sets.

Theorem 3.4. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a C^2-smooth norm-coercive function. If $C(f)$ and $\mathbb{R}^3 \setminus \mathrm{Hess}^+(f)$ are bounded, then the level surface $f^{-1}(c)$ is a compact connected regular ovaloid for c sufficiently large.*

The compactness, connectedness and the regularity of the hypersurfaces $f^{-1}(c)$, for c sufficiently large, follow via Theorem 2.1. In order to prove the ovaloidal shape of the level surfaces $f^{-1}(c)$ for c sufficiently large, we need the following

Lemma 3.5. *The Gauss curvature function associated to the C^2-smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is positive over the* $\mathrm{Hess}^+(f)$ *region.*

Proof. Indeed, according to [5], the Gauss curvature function associated to f is

$$K_G = -\frac{\begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix}}{\|\nabla f\|^4} = \frac{\nabla f \cdot H^*(f) \cdot (\nabla f)^T}{\|\nabla f\|^4} = \det H(f) \frac{\nabla f \cdot H^{-1}(f) \cdot (\nabla f)^T}{\|\nabla f\|^4},$$

where $H^*(f)$ is the adjugate matrix of the Hessian matrix $H(f)$ of f. Its value at $x \in R(f) = \mathbb{R}^n \setminus C(f)$ is the Gauss curvature of the level surface $f^{-1}(f(x))$ at x. We next recall that $x \in \mathrm{Hess}^+(f)$ if and only if the eigenvalues of $H_x(f)$ are all positive, which implies that $\det H_x(f) > 0$ for $x \in \mathrm{Hess}^+(f)$. On the other hand the eigenvalues of $H_x^{-1}(f)$ are the inverse values of the eigenvalues of $H_x(f)$ and are therefore positive as well. Consequently the inverse matrix $H_x^{-1}(f)$ is also positive definite which implies that $\nabla_x f \cdot H_x^{-1}(f) \cdot (\nabla_x f)^T > 0$ and $K_G(x) > 0$ therefore. □

Corollary 3.6. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a C^2-smooth norm-coercive function. If $C(f)$ and $\mathbb{R}^3 \setminus \mathrm{Hess}^+(f)$ are bounded, then the level surface $f^{-1}(c)$ is diffeomorphic with the sphere S^2 and bounds an open convex set for c sufficiently large.*

Proof. It follows immediately by combining Theorem 3.4 with the Hadamard-Stoker Theorem [10, Theorem 6.5, p. 178]. □

Proof of Theorem 3.4. The boundedness of $\mathbb{R}^3 \setminus \mathrm{Hess}^+(f)$ combined with its closedness imply its compactness. For $c > h_{\max}(f) := \max\left\{f(x) \;\; x \in \mathbb{R}^3 \setminus \mathrm{Hess}^+(f)\right\}$, the level surface $f^{-1}(c)$ is completely contained in $\mathrm{Hess}^+(f)$ and for $c > \mu_{\max}(f)$ the level surface $f^{-1}(c)$ is regular. Therefore the level surface $f^{-1}(c)$ is additionally an ovaloid for $c > \max\{h_{\max}(f), \mu_{\max}(f)\}$, besides its compactness, connectedness and regularity ensured by Theorem 2.1. Indeed, for such a value of c, the compact connected regular level surface $f^{-1}(c)$ is completely contained in $\mathrm{Hess}^+(f)$, where the Gauss curvature function associated to f is, according to Lemma 3.5, positive. □

Corollary 3.7. *Every two level surfaces of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ subject to the hypothesis of Theorem* 3.4*, above the level $\mu_{\max}(f)$, are connected regular hypersurfaces diffeomorphic to the unit sphere S^2 which bounds a convex open set for $c > \max\{h_{\max}(f), \mu_{\max}(f)\}$.*

Proof. The statement follows by combining Corollary 3.6 with the Non-Critical Neck Principle (see e.g. [11, p. 194]) and the proof of Theorem 3.4. □

Corollary 3.8. *The sublevel set $f^{-1}(-\infty, h_{\max}(f)]$ of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ subject to the hypothesis of Theorem* 3.4 *is convex, whenever $h_{\max}(f) \geq \mu_{\max}(f)$.*

The two dimensional couterpart of Corollary 3.8 is [2, Corollary 3.8] whose proof works along the same lines.

Example 3.9. The first convex connected level curve of the function

$$f_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ f_a(x,y) = (x^2+y^2)^2 - 2a^2(x^2-y^2)\ (a>0),$$

is $f_a^{-1}(3a^4)$. It is the first positive regular level of f_a which is completely contained in

$$\mathrm{cl\ Hess}^+(f_a) = \{(x,y) \in \mathbb{R}^2 | 3(x^2+y^2)^2 + 2a^2(x^2-y^2) \geq a^4\} = g^{-1}_{a/\sqrt{3}}\left[a^4/3, +\infty\right),$$

and is illustrated in FIGURE 3 (see [12]).

The higher level curves of f_a are also contained in cl $\mathrm{Hess}^+(f_a)$ and they keep being convex. The convex level curves of f_a were selected through the nonvanishing requirement of the determinant

$$\delta_a := \begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix} = \begin{vmatrix} (f_a)_{xx} & (f_a)_{xy} & (f_a)_x \\ (f_a)_{yx} & (f_a)_{yy} & (f_a)_y \\ (f_a)_x & (f_a)_y & 0 \end{vmatrix} = -4^3 \frac{a^4+c}{2}\{3(x^2+y^2)^2 - c\}$$

as part of the curvature formula

$$\rho_C = -\frac{\begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix}}{\|\nabla f\|^3}$$

over the level curve $C = f_a^{-1}(c)$ of f. We can similarly select the level surfaces of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ through the positiveness requirement on the determinant

$$\Delta = \begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix}$$

as part of the Gauss curvature formula

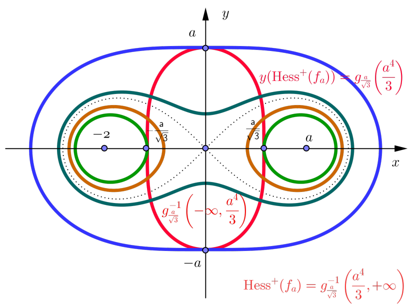

FIGURE 3. The boundary of $\mathrm{Hess}^+(f_a)$, the last negative regular level and the first positive convex regular level of f_a contained in cl $\mathrm{Hess}^+(f_a)$, Positive nonconvex regular level and disconnected negative regular level of f_a with convex components which are not contained in cl $\mathrm{Hess}^+(f_a)$

$$K_G = -\frac{\begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix}}{\|\nabla f\|^3} = -\frac{\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} & f_x \\ f_{yx} & f_{yy} & f_{yz} & f_y \\ f_{zx} & f_{zy} & f_{zz} & f_z \\ f_x & f_y & f_z & 0 \end{vmatrix}}{\|\nabla f\|^3}$$

of the level surface $S = f^{-1}(s)$. The theoretical basis for the convexity of the ovaloids is the Hadamard-Stoker [10, Theorem 6.5, p. 178].

Example 3.10. The first convex level of the function

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R},\ f(x,y,z) = (x^2+y^2+z^2)^2 - 8(x^2-y^2-z^2),$$

is $f^{-1}(3 \cdot 16)$. It is also the first positive regular level of f completely contained in

$$\mathrm{cl\ Hess}^+(f) = \{(x,y,z) \in \mathbb{R}^3 \mid 3(x^2+y^2+z^2)^2 + 8(x^2-y^2-z^2) \geq 16\}.$$

Indeed Δ is invariant with respect to the one parameter group of rotations (3.2) as

$$\frac{1}{16^2}\Delta = [(x^2+y^2+z^2)^2-16]\{-x^2(x^2+y^2+z^2-4)[x^2+3(y^2+z^2)] - (y^2+z^2)[(y^2+z^2+4)^2-x^4]\}.$$

The detailed computations of Δ were done in [1]. Therefore, the obvious equality

$$\Delta(x,y,z) = \Delta(x, \sqrt{y^2+z^2}, 0)$$

can now be exploited to obtain

$$\frac{1}{16^2}\Delta = (x^2+3y^2+3z^2+4)\delta_2\left(x, \sqrt{y^2+z^2}\right)$$

Therefore, the Gauss curvature

$$K_G(x,y,z) = \frac{-\Delta}{\left(f_x^2+f_y^2+f_z^2\right)^{3/2}} = \frac{16^2(x^2+3y^2+3z^2+4)\left[-\delta_2\left(x,\sqrt{y^2+z^2}\right)\right]}{\left(f_x^2+f_y^2+f_z^2\right)^{3/2}}$$

is positive over $f^{-1}(s)$ if and only if over $f_2^{-1}(s) = f^{-1}(s)\cap(z=0)$

$$\delta(x,\pm y) = \delta_2\left(x,\sqrt{y^2+z^2}\right) < 0.$$

But, according to [12], the determinant δ_2 is negative over $f_2^{-1}(s)$ if and only if one has $s^2 - 3\cdot 16 s > 0$, i.e. $s \in (-\infty, 0] \cup [3\cdot 16, +\infty)$.

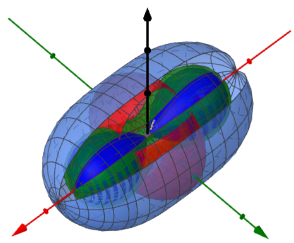

FIGURE 4. The 0-critical level of f, a nonconvex regular level of f, The boundary of $\text{Hess}^+(f)$ and the first convex regular level of f

4. One parameter families of one-to-one smooth maps

In this section we shall exploit the Theorem 3.4 to produce some one parameter families of one-to-one smooth maps of Loewner chain flavor. In this respect we first consider a smooth norm-coercive function $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$ with bounded critical set and bounded $\text{Hess}^+(f)$ complement region. Then the level surface $f^{-1}(c)$ is a compact connected regular ovaloid for every $c > \max\{h_{\max}(f), \mu_{\max}(f)\}$, one of which is fixed.

4.1. The Gauss map outcome smooth homeomorphism of an ovaloidal level set

Therefore, the inside of $f^{-1}(c)$, i.e. $\widehat{\overset{\circ}{M_c(f)}} = f^{-1}(-\infty, c) = f^{-1}[\min f, c)$, is convex open set and its closure $M_c(f) = f^{-1}(-\infty, c] = f^{-1}[\min f, c]$ is a compact convex set. Moreover, the restriction and co-restriction of the normalized gradient

$$G_c : f^{-1}(c) \longrightarrow S^2,\ G_c(x) = \frac{\nabla_x f}{\|\nabla_x f\|},$$

which is the Gauss map of the surface $f^{-1}(c)$ is, according with the Hadamard-Stoker theorem, a diffeomorphism (see [10, 178]). Therefore, the map

$$F_{x_0}^c : D^3 \longrightarrow M_c(f),\ F_{x_0}^c(x) = \begin{cases} x_0 + \exp\dfrac{\|x\|^4-1}{\|x\|^2}\left[-x_0 + G_c^{-1}\left(\dfrac{x}{\|x\|}\right)\right] & \text{if } x \neq 0 \\ x_0 & \text{if } x = 0. \end{cases}$$

is a smooth homeomorphism for every $x_0 \in \widehat{\overset{\circ}{M_c(f)}}$. In order to justify this statement, we first observe that the map

$$\varphi : D^3 \longrightarrow [0,1],\ \varphi(x) = \begin{cases} \exp\dfrac{\|x\|^4-1}{\|x\|^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

is well-defined, smooth and for each of its level sphere

$$\varphi^{-1}(l) = S\left(0, \sqrt{\frac{(\ln l)^2 + \sqrt{(\ln l)^2+4}}{2}}\right),\ 0 \leq l \leq 1$$

the restriction and co-restriction

$$\varphi^{-1}(l) \longrightarrow x_0 + l(-x_0 + f^{-1}(c)),\ x \longmapsto x_0 + l \cdot \left(-x_0 + G_c^{-1}\left(\frac{x}{\|x\|}\right)\right)$$

is obviously a diffeomorphism, as G_c^{-1} realizes a diffeomorphism between S^2 and $f^{-1}(c)$. In other words the restriction and co-restriction of $F_{x_0}^c$ to each of the leaves of the foliation

$$\{\varphi^{-1}(l)\}_{0<l\leq 1} \text{ of } D^3 \setminus \{0\} \tag{4.1}$$

to the corresponding leave of the foliation $\{x_0 + l(-x_0 + f^{-1}(c))\}_{0<l\leq 1}$ of $M_c(f) \setminus \{x_0\}$ is a diffeomorphism. On the other hand the restriction and co-restriction of $F_{x_0}^c$ to each of the leaves of the orthogonal foliation

$$\{]0x]\}_{x\in S^2} \tag{4.2}$$

is also a diffeomorphism onto the corresponding leave of the foliation

$$\left\{\left]x_0, G_c^{-1}\left(\frac{x}{\|x\|}\right)\right]\right\}_{x\in f^{-1}(c))},$$

of $M_c(f) \setminus \{x_0\}$, which is transversal to the foliation $\{x_0 + l(-x_0 + f^{-1}(c))\}_{0<l\leq 1}$, as for every $x \in S^2$ and $t \in]0,1]$ we have

$$\frac{d}{dt}\varphi(tx) = \frac{d}{dt}\left(t^2\|x\|^2 - \frac{1}{t^2\|x\|^2}\right)\exp\frac{t^4\|x\|^4-1}{t^2\|x\|^2} = \left(2t + 2\frac{2}{t^3}\right)\exp\frac{t^4\|x\|^4-1}{t^2\|x\|^2} > 0.$$

Therefore $F_{x_0}^c$ is bijective and its Fréchet differential $(dF_{x_0}^c)_x$ is an isomorphism at every point $x \in D^3 \setminus \{x_0\}$ as its restrictions to the orthogonal complement subpaces of $T_x(D^3)$, one of which is the tangent space to the leave through x of the foliation (4.1) and its orthogonal complement, i.e. the tangent space to the leave through x of the foliation (4.2), are one-to-one. Consequently $F_{x_0}^c$ is a global smooth homeomorphism on D^3 onto $M_c(f)$ and its restriction and co-restriction $D^3 \setminus \{0\} \longrightarrow M_c(f) \setminus \{x_0\},\ x \longmapsto F^c_{x_0}(x)$ is a diffeomorphism, but $(dF^c_{x_0})_0 = 0$.

4.2. Vector fields with bounded norm on $M_c(f)$ and their global flows

In this subsection we will point out quite a large family of diffeomorphisms from D^3 to $M_c(f)$ which is parametrized by $\mathbb{R} \times GL_3(\mathbb{R}) \times \widehat{\overset{\circ}{M_c(f)}}$ whenever $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a C^2-smooth function whose critical set $C(f)$ and $\text{Hess}^+(f)$ complement are bounded. For such a function we rely on Theorem 3.4 to conclude that the level surface $f^{-1}(c)$ is a compact connected regular ovaloid for c sufficiently large.

For every nonsingular linear operator $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, i.e. its matrix representation $[A]$ is nonsingular, and every point $x_0 \in \widehat{\overset{\circ}{M_c(f)}}$ we consider the vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ defiend by $X_x = A(x - x_0)$ as well as the smooth function $h := b \circ (F^c_{x_0})^{-1} : M_c(f) \longrightarrow \mathbb{R}$, where $x_0 \in M_c(f)$ and $b : D^3 \longrightarrow \mathbb{R}$ is a bump function such that $b\big|_{D^3_{1-\varepsilon}} \equiv 1$, $b\big|_{\mathbb{R}^3 \setminus D^3_{1-\varepsilon/2}} \equiv 0$ for some $\varepsilon > 0$ sufficiently small and D^3_r stands for the disc centered at $0 \in \mathbb{R}^3$ of radius $r > 0$. Now the vector field hX has obviously bounded norm, which make it completely integrable [11, Corollary 9.1.5, p. 183], i.e. there exists a global one-parameter group of diffeomorphisms $\Psi^{c,x_0} : \mathbb{R} \times M_c(f) \longrightarrow M_c(f)$ such that

$$\frac{d}{dt}\Psi^{c,x_0}_t(x) = X_{\Psi^{c,x_0}_t(x)},\ \forall x \in M_c(f),$$

where $\Psi^{c,x_0}_t : M_c(f) \longrightarrow M_c(f)$, $\Psi^{c,x_0}_t(x) = \Psi^{c,x_0}(t, x)$ is a diffeomorphism for every $t \in \mathbb{R}$, which implies that $(d\Psi^{c,x_0}_t)_x : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is an isomorphism for every $x \in M_c(f)$. Indeed, the following properties

1. $\Psi^{c,x_0}_0 = id_{M_c(f)}$;
2. $\Psi^{c,x_0}_s \circ \Psi^{c,x_0}_t = \Psi^{c,x_0}_{s+t}$ for all $s, t \in \mathbb{R}$

hold. In fact $\Psi^{c,x_0}_t(x) = x_0 + Ae^{tA}(x - x_0)$ for every $x \in (F^c_{x_0})^{-1}(D_{1-\varepsilon})$, as can be easily seen, and $d_x\Psi^{c,x_0}_t(\cdot) = Ae^{tA}$ for every such x. On the other hand, $\Psi^{c,x_0}_t(x) = x$, for all $x \in f^{-1}(c - \varepsilon/3, c)$ and all $t \in \mathbb{R}$ as $(hX)_x = 0$ for all $x \in f^{-1}(c - \varepsilon/3, c)$.

4.3. A gradient homothetic vector field whose flow permutes the sublevel sets

In this subsection we additionally assume that $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies the Palais-Smale condition, i.e. every sequence (x_n) such that $(df)_{x_n} \longrightarrow 0$ as $n \longrightarrow \infty$ has a convergent subsequence. For example every norm-coercive function such that $\|\nabla_x f\| \longrightarrow \infty$ as $x \longrightarrow \infty$ has this property. Since $(\nabla f)f = \|\nabla f\|^2$, on the set $\mathbb{R}^3 \setminus C(f)$ of regular points, where $\nabla f \neq 0$, the smooth vector field

$$Y = \pm\frac{\nabla f}{\|\nabla f\|^2}$$

satisfies $Yf = \pm 1$. More generally if $F : \mathbb{R} \longrightarrow \mathbb{R}$ is any smooth bounded function vanishing in a neighborhood of $B(f) = f(C(f))$, then, following [11, Section 9.3], we consider the smooth vector field $X = (F \circ f)Y$ on $\mathbb{R}^3$ that vanishes in a neighborhood of $C(f)$, and $X(f) = \pm(F \circ f)$. We denote by Φ_t the flow on $\mathbb{R}^3$ generated by X. Let us choose $F : \mathbb{R} \longrightarrow \mathbb{R}$ to be a smooth, non-negative function that is identically one on a neighborhood of $[c,d]$ and zero outside $(c-\varepsilon/2, d+\varepsilon/2)$, where d might be infinity. In the later case the intervals $[c,d]$ and $(c-\varepsilon/2, d+\varepsilon/2)$ are understood as $[c,+\infty)$ and $(c-\varepsilon/2,+\infty)$ respectively. Since $\|Y\| = 1/\|\nabla f\|$ and $\|X\| = \|\nabla f\| \cdot |F\circ f|$ along with the boundedness of F and the vanishing of $|F \circ f|$ outside $f^{-1}([c-\varepsilon/2, d+\varepsilon/2])$, one can show, following the same lines with those in the proof of [11, Proposition 9.3.1] that the vector field X has on $\mathbb{R}^3$ bounded length and hence its associated flow Φ_t generates a one-parameter group of diffeomorphisms of $\mathbb{R}^3$. We denote by $\gamma(t, c')$ the solution of the ordinary differential equation

$$\frac{d\gamma}{dt} = \pm F(\gamma) \tag{4.3}$$

with initial value c'. Since

$$\frac{d}{dt}(f \circ \Phi_t(x)) = X_{\Phi_t(x)} f = \pm(F \circ f)(\Phi_t(x)), \tag{4.4}$$

it follows that $f(\Phi_t(x)) = \gamma(t, f(x))$, and hence that $\Phi_t(f^{-1}(c')) = f^{-1}(\gamma(t,c'))$. On the other hand $\gamma(t,c')$ is bounded and its range is contained in the interval $[\min\{0, c-\varepsilon/2\}, \max\{0, d+\varepsilon/2\}]$, as F vanishes on $\mathbb{R}\setminus(c-\varepsilon/2, d+\varepsilon/2)$, and the flow Φ_t permutes the level sets of f on the other hand. From the definition of $\gamma(t,c')$ it follows that $\gamma(t,c') = c' \pm t$ for $c' \in [c,d]$ and $c' \pm t \in [c,d]$, while $\gamma(t,c') = c'$ if $c' > d+\varepsilon$ or $c' < c-\varepsilon$. Therefore, the range of $\gamma(t,c')$ is an interval with the endpoints i_c, s_c, where $i_c = \inf \gamma(\cdot, c)$ and $s_c = \sup \gamma(\cdot, c)$ for every $c' \in (c-\varepsilon/2, d+\varepsilon/2)$ and this range is the singleton $\{c'\}$ for every $c' \in \mathrm{Im}(f) \setminus (c-\varepsilon/2, d+\varepsilon/2)$, as $\gamma(t,c') = c'+t$ for $c' \in [c,d]$ and $c \le c'+t \le d$, while $\gamma(t,c') = c'$ if $c' \ge d+\varepsilon/2$ or $c' \le c-\varepsilon/2$.

4.4. One-parameter families of smooth homeomorphisms associated to the flow Φ_t

This subsection is devoted to the one-parameter family of smooth homeomorphisms

$$G_s^{c,x_0} : D^3 \longrightarrow \mathbb{R}^3, \; G_s^{c,x_0} = \Phi_s \circ g,$$

where $c > \max\{h_{\max}(f), \mu_{\max}(f)\}$, $x_0 \in \widehat{\overset{\circ}{M_c(f)}}$, $\mu_{\max}(f) := \max f\big|_{C(f)}$ and $g : D^3 \longrightarrow \mathbb{R}^3$ is a smooth homeomorphism such that $M_c(f) = g(D^3)$. Examples of such smooth homeomorphisms are $F_{x_0}^c$ or Ψ_t^{c,x_0} for some real parameter t. Note that for the later option G_s^{c,x_0} is a diffeomorphism. We also consider $d > c$ and $\varepsilon > 0$ such that $[c-\varepsilon, c+\varepsilon]$ remains an interval of regular values of f

Theorem 4.1. *The one-parameter family* $\{G_s^{c,x_0}\}_{0 \le s < +\infty}$ *has the following properties:*

1. *each function* G_s^{c,x_0} *is a smooth homeomorphism (diffeomorphism for* $g = \Psi_t^{c,x_0}$*)*
2. $G_s^{c,x_0}(D^3) \subseteq G_t^{c,x_0}(D^3)$, $\forall 0 \le s < t < d \le +\infty$ *for the* $''+''$ *option in* (4.3).
3. $G_s^{c,x_0}(D^3) \supseteq G_t^{c,x_0}(D^3)$, $\forall 0 \le s < t < d \le +\infty$ *for the* $''-''$ *option in* (4.3).
4. $|G_s^{c,x_0}(x) - G_t^{c,x_0}(x)| \le \displaystyle\int_s^t \|\nabla_{\Phi_r(g(x))} f\| dr.$
5. $\displaystyle\bigcup_{t\ge 0} G_t^{c,x_0}(D^3) = \mathbb{R}^3$ *for the* $''+''$ *option in* (4.3) *and* $d = +\infty$.
6. $\widehat{\overset{\circ}{M_{s_c}(f)}} \subseteq \displaystyle\bigcup_{t \ge 0} G_t^{c,x_0}(D^3) \subseteq M_{s_c}(f)$ *for the* $''+''$ *option in* (4.3) *and* $d < +\infty$.

7. $\overset{\circ}{\widehat{M_{i_c}(f)}} \subseteq \bigcap_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq M_{i_c}(f)$, *for the* $''-''$ *option in* (4.3) *and* $d < +\infty$.

Proof. (1) Obvious
(2) By using (4.4) for the $''+''$ option in (4.3) and taking into account that $F \equiv 1 > 0$ on $[c, +\infty)$ we deduce that the real-valued function of one real variable $f \circ \Phi_t$ is nondecreasing and for $t \geq 0$ we obtain that $(f \circ \Phi_{-t})(x) \leq (f \circ \Phi_0)(x) = f(x)$. This shows that $M_c(f)$ is invariant under the action of Φ_{-t}, as $f(x) \leq c \Longrightarrow f(\Phi_{-t}(x)) \leq c$. But $\Phi_{-t}(M_c(f)) \subseteq M_c(f)$ is equivalent with $M_c(f) \subseteq \Phi_t(M_c(f))$. Now, for the required inclusion we have

$$G_s^{c,x_0}(D^3) \subseteq G_t^{c,x_0}(D^3) \Longleftrightarrow \Phi_s(g(D^3)) \subseteq \Phi_t(g(D^3))$$
$$\Longleftrightarrow g(D^3) \subseteq \Phi_{t-s}(g(D^3)) \Longleftrightarrow M_c(f) \subseteq \Phi_{t-s}(M_c(f)),$$

which holds true as $t - s \geq 0$.
(3) Similar with (2).
(4) Since $\frac{d}{dt}\Phi_t(x) = X_{\Phi_t(x)} \Longleftrightarrow \Phi_t(x) = x + \int_0^t X_{\Phi_r(x)} dr$ we have successively:

$$|G_s^{c,x_0}(x) - G_t^{c,x_0}(x)| = \left| \int_0^s X_{\Phi_r(x)} dr - \int_0^t X_{\Phi_r(x)} dr \right| = \left| \int_s^t X_{\Phi_r(x)} dr \right| \leq \int_s^t \|X_{\Phi_r(x)} dr\|$$
$$= \int_s^t |(F \circ f)(\Phi_r(x))| \cdot \|\nabla_{\Phi_r(x)} f\| dr \leq \int_s^t \|\nabla_{\Phi_r(x)} f\| dr.$$

(5) Indeed, we have successively:

$$\bigcup_{t\geq 0} G_t^{c,x_0}(D^3) = \bigcup_{t\geq 0} \Phi_t\left(\overset{\circ}{\widehat{M_c(f)}} \cup f^{-1}(c)\right) = \bigcup_{t\geq 0} \Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{t\geq 0} \Phi_t(f^{-1}(c))$$
$$= \bigcup_{t\geq 0} \Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{t\geq 0} \Phi_t(f^{-1}(\gamma(t,c))$$
$$= \bigcup_{t\geq 0} \Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{t\geq 0} f^{-1}(c+t) = \bigcup_{t\geq 0} \overset{\circ}{\widehat{\Phi_t(M_c(f))}} \cup \bigcup_{t\geq 0} f^{-1}(c+t)$$

But $M_c(f) \subseteq \Phi_t(M_c(f))$ implies $\overset{\circ}{\widehat{M_c(f)}} \subseteq \overset{\circ}{\widehat{\Phi_t(M_c(f))}} = \Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right)$. Therefore $\overset{\circ}{\widehat{\Phi_t(M_c(f))}} = \overset{\circ}{\widehat{M_c(f)}} \cup \left(\overset{\circ}{\widehat{\Phi_t(M_c(f))}} \setminus \overset{\circ}{\widehat{M_c(f)}}\right)$, namely

$$\bigcup_{t\geq 0} \overset{\circ}{\widehat{M_c(f)}} \cup f^{-1}(c+t) = \overset{\circ}{\widehat{M_c(f)}} \cup \bigcup_{t\geq 0} f^{-1}(c+t) = M_c(f) \cup \bigcup_{t>0} f^{-1}(c+t)$$
$$= \bigcup_{\min(f)\leq r\leq c} f^{-1}(r) \cup \bigcup_{t>0} f^{-1}(c+t)$$
$$= \bigcup_{t\geq 0} \left(\overset{\circ}{\widehat{\Phi_t(M_c(f))}} \setminus \overset{\circ}{\widehat{M_c(f)}}\right) \cup \mathbb{R}^3 = \mathbb{R}^3.$$

Thus

$$\bigcup_{t\geq 0} G_t^{c,x_0}(D^3) = \bigcup_{t\geq 0} \widehat{\Phi_t\left(M_c(f)\right)}^{\circ} \cup \bigcup_{t\geq 0} f^{-1}(c+t)$$
$$= \bigcup_{t\geq 0}\left(\widehat{\Phi_t\left(M_c(f)\right)}^{\circ} \setminus \widehat{M_c(f)}^{\circ}\right) \cup \bigcup_{t\geq 0} \widehat{M_c(f)}^{\circ} \cup f^{-1}(c+t)$$
$$= \bigcup_{t\geq 0}\left(\widehat{\Phi_t\left(M_c(f)\right)}^{\circ} \setminus \widehat{M_c(f)}^{\circ}\right) \cup \mathbb{R}^3 = \mathbb{R}^3.$$

(6) Since $\dfrac{d}{dt}(f\circ\Phi_t(x)) = (F\circ f)(\Phi_t(x)) \geq 0$, it follows that $\gamma(t,f(x)) = f(\Phi_t(x))$ is nondecreasing. Thus

$$\bigcup_{t\geq 0} G_t^{c,x_0}(D^3) = \bigcup_{t\geq 0} \Phi_t\left(M_c(f)\right) = \bigcup_{t\geq 0} \Phi_t\left(\widehat{M_c(f)}^{\circ} \cup f^{-1}(c)\right)$$
$$= \bigcup_{t\geq 0} \Phi_t\left(\widehat{M_c(f)}^{\circ}\right) \cup \bigcup_{t\geq 0} \Phi_t(f^{-1}(c)) = \bigcup_{t\geq 0} \Phi_t\left(\widehat{M_c(f)}^{\circ}\right) \cup \bigcup_{t\geq 0} f^{-1}(\gamma(t,c))$$

Since $[c,s_c) \subseteq \{\gamma(t,c) \mid t\geq 0\} \subseteq [c,s_c]$ it follows that

$$\bigcup_{c\leq u<s_c} f^{-1}(u) \subseteq \bigcup_{t\geq 0} f^{-1}(\gamma(t,c)) \subseteq \bigcup_{c\leq v\leq s_c} f^{-1}(v) \tag{4.5}$$

The left hand side inclusion of (4.5) implies that

$$\bigcup_{t\geq 0} \Phi_t\left(\widehat{M_c(f)}^{\circ}\right) \cup \bigcup_{c\leq u<s_c} f^{-1}(u) \subseteq \bigcup_{t\geq 0} \Phi_t\left(\widehat{M_c(f)}^{\circ}\right) \cup \bigcup_{t\geq 0} f^{-1}(\gamma(t,c)) = \bigcup_{t\geq 0} G_t^{c,x_0}(D^3)$$

$$i.e. \bigcup_{t\geq 0}\left(\widehat{\Phi_t\left(M_c(f)\right)}^{\circ} \setminus \widehat{M_c(f)}^{\circ}\right) \cup \widehat{M_c(f)}^{\circ} \cup \bigcup_{c\leq u<s_c} f^{-1}(u) \subseteq \bigcup_{t\geq 0} G_t^{c,x_0}(D^3)$$

$$\Longleftrightarrow \bigcup_{t\geq 0}\left(\widehat{\Phi_t\left(M_c(f)\right)}^{\circ} \setminus \widehat{M_c(f)}^{\circ}\right) \cup \widehat{M_{s_c}(f)}^{\circ} \subseteq \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \Longrightarrow \widehat{M_{s_c}(f)}^{\circ} \subseteq \bigcup_{t\geq 0} G_t^{c,x_0}(D^3).$$

The right hand side inclusion of (4.5) implies that

$$\bigcup_{t\geq 0} G_t^{c,x_0}(D^3) = \bigcup_{t\geq 0} \Phi_t\left(\widehat{M_c(f)}^{\circ}\right) \cup \bigcup_{t\geq 0} f^{-1}(\gamma(t,c)) \subseteq \bigcup_{t\geq 0} \Phi_t\left(\widehat{M_c(f)}^{\circ}\right) \cup \bigcup_{c\leq v\leq s_c} f^{-1}(v)$$

$$i.e. \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq \bigcup_{t\geq 0} \Phi_t\left(\widehat{M_c(f)}^{\circ}\right) \cup \bigcup_{c\leq v\leq s_c} f^{-1}(v)$$

$$\Longleftrightarrow \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq \bigcup_{t\geq 0}\left(\widehat{\Phi_t\left(M_c(f)\right)}^{\circ} \setminus \widehat{M_c(f)}^{\circ}\right) \cup \widehat{M_c(f)}^{\circ} \cup \bigcup_{c\leq v\leq s_c} f^{-1}(v)$$

$$\Longleftrightarrow \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq \bigcup_{t\geq 0}\left(\widehat{\Phi_t\left(M_c(f)\right)}^{\circ} \setminus \widehat{M_c(f)}^{\circ}\right) \cup M_{s_c}(f) \Leftrightarrow \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq M_{s_c}(f),$$

as $\gamma(t,c) = f\left(\Phi_t\left(x\right)\right) \leq s_c$, for all $x\in M_c(f)$.

(7) Similar with (5). □

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