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A criterion of univalence in $\mathbb{C}^n$ in terms of the Schwarzian derivative

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. Using the Loewner Chain Theory, we obtain a new criterion of univalence in C^n in terms of the Schwarzian derivative for locally biholomorphic mappings. We as well derive explicitly the formula of this Schwarzian derivative using the numerical method of approximation of zeros due by Halley.

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1. Introduction

The *Schwarzian derivative* of a locally univalent analytic function f in a simply connected domain Ω of the complex plane $\mathbb{C}$ is

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 . \tag{1.1}$$

The quotient f''/f', denoted by Pf, is the *pre-Schwarzian derivative* of the function f.

These two operators come up naturally as the values of the derivatives of the generating functions of two particular methods for approximating zeros, as we now explain.

It is well know that the Newton (or the Newton-Raphson method) is a technique to approximate the zero of a function f via the sequence

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad n \geq 1 , \tag{1.2}$$

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starting with a guess z_0, say.

The function $g(z) = z - f(z)/f'(z)$ is called the *generating function* of the Newton iteration method. It is not difficult to prove that the following identities hold, assuming that $f(\alpha) = 0$:

$$g(\alpha) = \alpha, \quad g'(\alpha) = 0\,, \quad g''(\alpha) = Pf(\alpha)\,.$$

The Halley Method can be derived by applying the Newton Method to the function $\dfrac{f}{\sqrt{|f'|}} = f|f'|^{-1/2}$. In this case, (1.2) becomes

$$z_{n+1} = z_n - \frac{2f(z_n)f'(z_n)}{2f'(z_n)^2 - f(z_n)f''(z_n)}\,, \quad n \geq 1\,.$$

The generating function h of the method is given by

$$h(z) = z - \frac{2f(z)f'(z)}{2f'(z)^2 - f(z)f''(z)},$$

which satisfies

$$h(\alpha) = \alpha,\, h'(\alpha) = h''(\alpha) = 0,\, h'''(\alpha) = -Sf(\alpha)\,,$$

where Sf is the Schwarzian derivative (1.1).

In this paper, we analyze the extension of the two methods mentioned to several complex variables and, in particular, we show that the definition of the Schwarzian derivative for locally biholomorphic mappings introduced in [3] (which derive from Oda's definition given in [8]) is precisely the value of the third coefficient of the generating function of the corresponding Halley method in several variables. That is, this operator is a third order differential tensor $Sf(z) \in \mathcal{L}^3(\mathbb{C}^n)$.

In addition to this, and using the Loewner chain theory, we obtain a criterion of univalence for locally biholomorphic mappings in the unit ball of $\mathbb{C}^n$ in terms of the Schwarzian derivative.

2. Preliminaries

2.1. Several complex variables

As usual, $\mathbb{C}^n$ is set of points $z = (z_1, \ldots, z_n)$, where $z_i \in \mathbb{C}, i = 1, \ldots, n$. The inner (dot) product and the norm are defined, respectively, by $z \cdot w = \sum_{i=1}^n z_i w_i$ and $|z| = (z \cdot \overline{z})^{1/2}$.

We denote by $\mathcal{L}^k(\mathbb{C}^n)$ the space of continuous k-linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$. For $T \in \mathcal{L}^k(\mathbb{C}^n)$, we write $T\langle\cdot, \ldots, \cdot\rangle$ to denote its placeholders. When $k = 1$ we simply write $\mathcal{L}(\mathbb{C}^n)$ for the space of linear maps and also write Tu instead of $T\langle u\rangle$ for any linear map T. The identity linear operator in $\mathbb{C}^n$ is denoted by I_n.

The standard operator norm in $\mathcal{L}^k(\mathbb{C}^n)$ is given by

$$\|T\| = \max_{u_1,\ldots,u_k \in \mathbb{C}^n} \left| T\left\langle \tfrac{u_1}{|u_1|}, \ldots, \tfrac{u_k}{|u_k|} \right\rangle \right|.$$

Let Ω be a domain in $\mathbb{C}^n$ and let f be a mapping defined in Ω with values in $\mathcal{L}(\mathbb{C}^n)$. If f is k-times (Fréchet) differentiable with respect to $z \in \Omega$ then its k-th derivative, denoted by $D^k f(z)$, is a symmetric mapping in $\mathcal{L}^{k+1}(\mathbb{C}^n)$, meaning that the value $D^k f(z)\langle u_1, \ldots, u_{k+1}\rangle$ remains unchanged after any permutation of the entries $u_1, \ldots, u_{k+1}$ (see Theorem 14.6 in [6]).

The *product rule* for the derivative of the product fg of two differentiable mappings $f, g : \Omega \to \mathcal{L}(\mathbb{C}^n)$, equals

$$D(fg)(z)\langle \cdot, \cdot\rangle \;=\; Df(z)\langle \cdot, g(z)\,\cdot\rangle + f(z)\,Dg(z)\langle \cdot, \cdot\rangle, \quad z \in \Omega. \tag{2.1}$$

Using the product rule, it is easy to check that if $f(z)$ is invertible for every $z \in \Omega$ then the derivative of $g(z) = f(z)^{-1}$ is given by

$$Dg(z)\langle \cdot, \cdot\rangle \;=\; -g(z)Df(z)\langle \cdot, g(z)\,\cdot\rangle, \qquad z \in \Omega. \tag{2.2}$$

If $f : \Omega \to \mathcal{L}(\mathbb{C}^n)$ is holomorphic in Ω, then f can be writen in terms of Taylor's formula centered at some $\alpha \in \Omega$:

$$f(z) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(\alpha)\langle z-\alpha\rangle^k, \qquad |z-\alpha| < \delta(\alpha),$$

where $\delta(\alpha)$ denotes the distance from α to the boundary of Ω (see Theorem 7.13 in [6]). Here, the notation $D^k f(\alpha)w^k$ should be understood as $D^k f(\alpha)\langle w, \ldots, w, \cdot\rangle$, with the point w repeated k times and one placeholder left without being evaluated.

2.2. Schwarzian derivative in several complex variables

Let f be a holomorphic mapping in the simply connected domain $\Omega \subset \mathbb{C}^n$. It is well known that f is locally univalent (biholomorphic) in Ω if and only if its Jacobian, $J_f = \det(Df)$, has no zeros (see Lewy [5]). For such functions, the *pre-Schwarzian derivative* and the *Schwarzian derivative* are linear operators defined, for any u and v in $\mathbb{C}^n$, as follows:

$$P_f(z)\langle u, v\rangle = Df(z)^{-1} D^2 f(z)\langle u, v\rangle \tag{2.3}$$

and

$$\begin{aligned} S_f(z)\langle u, v\rangle \;&=\; P_f(z)\langle u, v\rangle \\ &-\; \frac{1}{n+1}\left((-\nabla \log J_f(z)\cdot u)v + (\nabla \log J_f(z)\cdot v)u\right). \end{aligned} \tag{2.4}$$

The pre-Schwarzian derivative (2.3) was introduced by J. Pfaltzgraff in [10]. The Schwarzian derivative (2.4) was presented in [3]. This higher dimensional Schwarzian derivative splits into two operators S_f and S_f^0 of order two and three, respectively. The fact that, unlike in one single variable, an operator of purely order two must appear is consistent with the fact that the dimension of the group to be anihilated by the Schwarzian, namely the special linear projective group in dimension one or higher, is not big enough to prescribe all jets up to order 2 of a given mapping.

More concretely, we would like to mention that T. Oda in [8] defined the Schwarzian derivative S_{ij}^k of a locally biholomorphic mapping $f(z) = (f_1, \ldots, f_n)$

by

$$S^k_{ij}f = \sum_{l=1}^{n} \frac{\partial^2 f_l}{\partial z_i \partial z_j}\frac{\partial z_k}{\partial f_l} - \frac{1}{n+1}\left(\delta^k_i \frac{\partial}{\partial z_j} + \delta^k_j \frac{\partial}{\partial z_i}\right)\log J_f\,, \tag{2.5}$$

where $i, j, k = 1, 2, \ldots, n$, and δ^k_i are the Kronecker symbols. For $n > 1$ the Schwarzian derivatives have the following properties:

$$S^k_{ij}f = 0 \quad \text{for all} \quad i, j, k = 1, 2, \ldots, n \quad \text{iff} \quad f(z) = M(z)$$

for some Möbius transformation

$$M(z) = \left(\frac{l_1(z)}{l_0(z)}, \ldots, \frac{l_n(z)}{l_0(z)}\right),$$

where $l_i(z) = a_{i0} + a_{i1}z_1 + \cdots + a_{in}z_n$ with $det(a_{ij}) \neq 0$. Furthermore, for a composition

$$S^k_{ij}(f \circ g)(z) = S^k_{ij}g(z) + \sum_{l,m,r=1}^{n} S^r_{lm}f(w)\frac{\partial w_l}{\partial z_i}\frac{\partial w_m}{\partial z_j}\frac{\partial z_k}{\partial w_r}\,,\; w = g(z)\,.$$

Thus, if f is a Mobius transformation then $S^k_{ij}(f \circ g) = S^k_{ij}g$. The $S^0_{ij}f$ coefficients are given by

$$S^0_{ij}f(z) = J_f^{1/(n+1)}\left(\frac{\partial^2}{\partial z_i \partial z_j}J_f^{-1/(n+1)} - \sum_{k=1}^{n}\frac{\partial}{\partial z_k}J_f^{-1/(n+1)}S^k_{ij}f(z)\right). \tag{2.6}$$

By using these Schwarzian derivatives in Oda's paper [8], the following definition, which coincides with (2.4), was presented in [3].

Definition 2.1. *The Schwarzian derivative S_f of a locally biholomorphic mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ equals*

$$S_f(z)(v, v) = \left(v^t\mathbb{S}^1 f(z)v\,, \ldots, v^t\mathbb{S}^n f(z)v\right),$$

where $\vec{v} \in \mathbb{C}^n$ and the $n \times n$ matrix operator $\mathbb{S}^k f$, $k = 1, \ldots, n$, are given by

$$\mathbb{S}^k f = (S^k_{ij}f)\,, \quad i, j = 1, \ldots, n.$$

It was proved in [3] that it is possible to recover the mapping f from its Schwarzian components. More explicitly, consider the following overdetermined system of partial differential equations,

$$\frac{\partial^2 u}{\partial z_i \partial z_j} = \sum_{k=1}^{n} P^k_{ij}(z)\frac{\partial u}{\partial z_k} + P^0_{ij}(z)u\,, \quad i, j = 1, 2, \ldots, n\,, \tag{2.7}$$

where $z = (z_1, z_2, ..., z_n) \in \Omega$ and $P^k_{ij}(z)$ are holomorphic functions in Ω, for $i, j, k = 0, \ldots, n$. The system (2.7) is called *completely integrable* if there are $n+1$ (maximun) linearly independent solutions, and is said to be in *canonical form* (see [11]) if the coefficients satisfy

$$\sum_{j=1}^{n} P^j_{ij}(z) = 0\,, \quad i = 1, 2, \ldots, n.$$

T. Oda proved that (2.7) is a completely integrable system in canonical form if and only if $P_{ij}^k = S_{ij}^k f$ for a locally biholomorphic mapping $f = (f_1, \dots, f_n)$, where $f_i = u_i/u_0$ for $1 \le i \le n$ and $u_0, u_1, \dots, u_n$ is a set of linearly independent solutions of the system. For a given mapping f, $u_0 = J_f^{-\frac{1}{n+1}}$ is always a solution of (2.7) with $P_{ij}^k = S_{ij}^k f$.

3. On the Schwarzian derivative and the Halley method of approximation of zeros

As was mentioned in the introduction, the Halley Method, to find zeros of a function $f : \mathbb{C} \to \mathbb{C}$, can be obtained by applying the Newton Method to $f \cdot |f'|^{-\frac{1}{2}}$.

By considering a function $f : \mathbb{C}^n \to \mathbb{C}^n$, now with $n \ge 1$, whis is locally biholomorphic in a simply connected domain Ω with $f(\alpha) = 0$ for some $\alpha \in \Omega$, the *Newton Iteration Method* is given by

$$z_{n+1} = z_n - Df(z_n)^{-1}\langle f(z_n)\rangle, \quad n \ge 0.$$

The generating function of this method, then, equals

$$F(z) = z - Df(z)^{-1}\langle f(z)\rangle\,. \tag{3.1}$$

By applying the Newton iteration method to $g = f \cdot J_f^{-\frac{1}{n+1}}$, the corresponding function in (3.1) becomes

$$G(z) = z - Dg(z)^{-1}\langle g(z)\rangle\,. \tag{3.2}$$

The next theorem shows how this function in (3.2) is related to the Schwarzian derivative in several complex variables.

Theorem 3.1. *Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be a locally biholomorphic mapping defined in the simply connected domian Ω, such that $f(\alpha) = 0$ for some $\alpha \in \Omega$. Then*

$$G(\alpha) = \alpha, \quad DG(\alpha) = 0, \quad D^2G(\alpha) = S_f(\alpha)\,,$$

where S_f is given by equation (2.4).

Proof. By (3.2), we have that $G(\alpha) = \alpha$ (since $g(\alpha) = 0$). Moreover, a straightforward calculation shows that (suppressing the variable z),

$$DG = \mathrm{Id} + D(Dg^{-1})\langle g, \cdot\rangle - \mathrm{Id} = -Dg^{-1}D^2g\langle Dg^{-1}\langle g\rangle, \cdot\rangle,$$

which gives $DG(\alpha) = 0$.

Notice that

$$Dg = J_f^{-\frac{1}{n+1}} Df - \frac{J_f^{-\frac{1}{n+1}} f}{(n+1)J_f}(\nabla J_f)^t = J_f^{-\frac{1}{n+1}} Df - \frac{g}{n+1}(\nabla \log J_f)^t.$$

Now let u and v be two vectors in $\mathbb{C}^n$. Then

$$\begin{aligned} D^2g\langle u, v\rangle \quad &= \quad J_f^{-\frac{1}{n+1}} D^2 f\langle u, v\rangle - \frac{J_f^{-\frac{1}{n+1}} Df(u)}{n+1}\nabla \log J_f \cdot v \\ &- \quad \frac{J_f^{-\frac{1}{n+1}} Df(v)}{n+1}\nabla \log J_f \cdot u - \frac{J_f^{-\frac{1}{n+1}} f}{n+1} u(\mathrm{Hess} \log J_f)v^t. \end{aligned}$$

On the other hand, differentiating DF and using equations (2.1) and (2.2), we obtain

$$\begin{aligned} D^2F\langle\cdot,\cdot\rangle &= Dg^{-1}D^2g\langle\cdot, Dg^{-1}D^2g\langle Dg^{-1}\langle g\rangle,\cdot\rangle\rangle - Dg^{-1}D^3g\langle Dg^{-1}\langle g\rangle,\cdot,\cdot\rangle \\ &+ Dg^{-1}D^2g\langle Dg^{-1}D^2g\langle Dg^{-1}\langle g\rangle,\cdot\rangle,\cdot\rangle - Dg^{-1}D^2g\langle\cdot,\cdot\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} D^2F(\alpha)\langle u,v\rangle &= -Dg^{-1}(\alpha)D^2g(\alpha)\langle u,v\rangle \\ &= -Df(\alpha)^{-1}D^2f\langle u,v\rangle - \frac{\nabla\log J_f\cdot u}{n+1}v - \frac{\nabla\log J_f\cdot v}{n+1}u. \\ &= -S_f(\alpha)\langle u,v\rangle. \end{aligned}$$

This ends the proof of the theorem. □

4. Univalence Criterion on the unit ball of $\mathbb{C}^n$

In this section we obtain the main result in this paper. Namely, we get a new sufficient condition for the univalence of a locally biholomorphic mapping defined in the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ in terms of the Schwarzian derivative defined by (2.4).

Recall that in the setting of functions in the complex plane, there are different applications of the Loewner chain theory to get univalence criteria. In particular, we can mention that for a given locally univalent function f defined in the unit disk $\mathbb{D}$ in the complex plane, the conditions

$$|P_f(z)| \le \frac{1}{1-|z|^2} \quad \text{or} \quad |S_f(z)| \le \frac{2}{(1-|z|^2)^2}, \quad \forall z\in\mathbb{D},$$

are sufficient to guarantee the univalence of f in $\mathbb{D}$.

These sufficient conditions for the global univalence of a function defined on the unit disk are due to Becker and Nehari, respectively (see [1] and [7]). Actually, these results can be proved by using the Loewner chain theory, as is shown in the great survey book "*Geometric Function Theory in one and higher dimension*" by Gabriela Kohr and Ian Graham [2].

The generalization of the classical Loewner chain theory to several complex variables was first introduced by J. Pfaltzgraff in [9]. We again refer the reader to [2, Ch. 8] for a beautiful review of this theory. Pfaltzgraff himself generalized the analogous of the Becker criterion of univalence to several variables in [10]. Specifically, it is proved in [10] that given a locally univalent function $f:\mathbb{B}^n\to\mathbb{C}^n$ normalized by the conditions $f(0)=0$ and $Df(0)=$Id, if the inequality

$$(1-\|z\|^2)\|Df(z)^{-1}D^2f\langle z,\cdot\rangle\| \le 1$$

holds for all $z\in\mathbb{B}^n$, then f is univalent in the unit ball. The reader can be found in [4] another univalence criterion that involves the Schwarzian derivative in several complex variables.

Here is the new criterion of univalence, now in terms of the Schwarzian derivative of functions in several complex variables.

Theorem 4.1. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a locally biholomorphic mapping normalized by $f(0) = 0$ and $Df(0) =Id$. Assume that f satisfies the following inequality (where S_f and S_f^0 are evaluated in z) for all $z \in \mathbb{B}^n$, where $p = \nabla \log J_f(z)$:*

$$\begin{aligned}\frac{\|S_f\langle z,\cdot\rangle\|}{(1-\|z\|^2)} &+ \left\|\frac{(z\cdot p)S_f\langle z,\cdot\rangle - (S_f\langle z,z\rangle\cdot p)\,Id}{n+1} + S_f^0\langle z,z\rangle Id\right\| \\ &\leq \frac{1}{(1-\|z\|^2)^2}.\end{aligned} \tag{4.1}$$

Then, f is univalent in $\mathbb{B}^n$.

Proof. We shall prove that

$$f(z,t) = \frac{u(e^{-t}z) + (e^{2t}-1)Du(e^{-t}z)\langle e^{-t}z\rangle}{u_0(e^{-t}z) + (e^{2t}-1)\nabla u_0(e^{-t}z)\cdot e^{-t}z}, \tag{4.2}$$

where $u = (u_1, \ldots, u_n)$ and u_i, $i = 0, \ldots, n$, are the independent solutions of (2.7) with $P_{ij}^k = S_{ij}^k f$ (where $S_{ij}^k f$ are defined by (2.5) and (2.6)) and $u_0 = J_f^{-\frac{1}{n+1}}$, is a Loewner chain in $\mathbb{B}^n$. To do so, we will show that $f(z,t)$ satisfies the hypothesis of Theorem 8.1.6. in [2, p. 308] by following the same arguments as in the proof of [2, Thm. 8.4.1].

By differentiating $f(z,t)$ with respect to the variable z, we have

$$\begin{aligned}Df(z,t) &= \frac{e^{-t}Du + (e^{2t}-1)(e^{-t}Du + e^{-t}D^2u\langle e^{-t}z,\cdot\rangle)}{u_0 + (e^{2t}-1)\nabla u_0\cdot e^{-t}z} \\ &- \frac{(u+(e^{2t}-1)Du\langle e^{-t}z\rangle)(e^t\nabla u_0 + e^{-t}(e^{2t}-1)\text{Hess } u_0(e^{-t}z,\cdot)}{(u_0 + (e^{2t}-1)\nabla u_0\cdot e^{-t}z)^2},\end{aligned}$$

where all the functions u_0, u, ∇u_0, Du, and D^2u are evaluated at $e^{-t}z$.

By (2.7) we have that $D^2u\langle\cdot,\cdot\rangle = Du\cdot S_f\langle\cdot,\cdot\rangle + S_f^0\langle\cdot,\cdot\rangle u$ and Hess $u_0(\cdot,\cdot) = S_f\langle\cdot,\cdot\rangle\cdot\nabla u_0 + S_f^0\langle\cdot,\cdot\rangle u_0$. Therefore, we get

$$e^tDf(z,t) = \frac{u_0Du - u\nabla u_0 + (e^{2t}-1)A + (e^{2t}-1)^2B}{(u_0+(e^{2t}-1)\nabla u_0\cdot e^{-t}z)^2}, \tag{4.3}$$

where A is the differential operator (evaluated in $e^{-t}z$) given by

$$\begin{aligned}A &= (e^{-t}z\cdot\nabla u_0)\,Du + u_0[Du + DuS_f\langle e^{-t}z,\cdot\rangle] \\ &- u[\nabla u_0 + S_f\langle e^{-t}z,\cdot\rangle\cdot\nabla u_0\rangle] - Du\langle e^{-t}z\rangle\nabla u_0.\end{aligned}$$

Notice that

$$A\langle e^{-t}z\rangle = [u_0Du - u(\nabla u_0)^t]\langle e^{-t}z + S_f\langle e^{-t}z, e^{-t}z\rangle\rangle.$$

In the same way, the linear operator B given by

$$\begin{aligned}B &= [Du + DuS_f\langle e^{-t}z,\cdot\rangle + uS_f^0\langle e^{-t}z,\cdot\rangle]\nabla u_0\cdot e^{-t}z \\ &- Du\langle e^{-t}z\rangle[\nabla u_0\cdot(\cdot) + \nabla u_0\cdot S_f\langle e^{-t}z,\cdot\rangle + S_f^0\langle e^{-t}z,\cdot\rangle u_0]\end{aligned},$$

satisfies

$$\begin{aligned} B\langle e^{-t}z\rangle &= (\nabla u_0\cdot e^{-t}z)Du\langle S_f\langle e^{-t}z,e^{-t}z\rangle\rangle \\ &- (S_f\langle e^{-t}z,e^{-t}z\rangle\cdot\nabla u_0)Du\langle e^{-t}z\rangle \\ &- S_f^0\langle e^{-t}z,e^{-t}z\rangle[u_0Du-u(\nabla u_0)^t]\langle e^{-t}z\rangle. \end{aligned}$$

On the other hand, $u_0f=u$, and then, $u_0Du-u\nabla u_0=u_0^2Df$. Therefore, the derivative $Df(z,t)$ in (4.3) of the function $f(z,t)$ in (4.2) satisfies

$$e^tDf(z,t)\langle e^{-t}z\rangle=\frac{u_0^2Df\langle e^{-t}z\rangle\rangle+(e^{2t-1})A\langle e^{-t}z\rangle+(e^{2t}-1)^2B\langle e^{-t}z\rangle}{(u_0+(e^{2t}-1)\nabla u_0\cdot e^{-t}z)^2}.$$

Using that

$$A\langle e^{-t}z\rangle=u_0^2Df\langle e^{-t}z+S_f\langle e^{-t}z,e^{-t}z\rangle\rangle,$$

and

$$\begin{aligned} B\langle e^{-t}z\rangle &= (\nabla u_0\cdot e^{-t}z)u_0Df\langle S_f\langle e^{-t}z,e^{-t}z\rangle\rangle \\ &- S_f^0\langle e^{-t}z,e^{-t}z\rangle u_0^2Df\langle e^{-t}z\rangle \\ &- \nabla u_0\cdot S_f\langle e^{-t}z,e^{-t}z\rangle u_0Df\langle e^{-t}z\rangle\,, \end{aligned}$$

it follows that

$$\begin{aligned} e^tDf(z,t)\langle e^{-t}z\rangle &= Df\,\langle\ e^{-t}z+(e^{2t}-1)(e^{-t}z+S_f\langle e^{-t}z,e^{-t}z\rangle) \\ &+ (e^{2t}-1)^2(\nabla\log u_0\cdot e^{-t}zS_f\langle e^{-t}z,e^{-t}z\rangle \\ &- \nabla\log u_0\cdot S_f\langle e^{-t}z,e^{-t}z\rangle e^{-t}z \\ &- S_f^0\langle e^{-t}z,e^{-t}z\rangle e^{-t}z)\ \rangle\,(1+(e^{2t}-1)\langle\nabla\log u_0,e^{-t}z\rangle)^{-2} \\ &= e^{2t}Df\,[\ \mathrm{Id}+(1-e^{-2t})S_f\langle e^{-t}z,\cdot\rangle \\ &+ e^{2t}(1-e^{-2t})^2(\nabla\log u_0\cdot e^{-t}zS_f\langle e^{-t}z,\cdot\rangle \\ &- \nabla\log u_0\cdot S_f\langle e^{-t}z,e^{-t}z\rangle\mathrm{Id}) \\ &- S_f^0(e^{-t}z,e^{-t}z)\mathrm{Id}\ ]\,\langle e^{-t}z\rangle/(1+(e^{2t}-1)\nabla\log u_0\cdot e^{-t}z)^2 \\ &= \frac{e^{2t}Df[\mathrm{Id}-E(z,t)]\langle e^{-t}z\rangle}{(1+(e^{2t}-1)\nabla\log u_0\cdot e^{-t}z)^2}, \end{aligned}$$

where

$$\begin{aligned} E(z,t) &= (e^{-2t}-1)S_f\langle e^{-t}z,\cdot\rangle-e^{2t}(1-e^{2t})^2(\nabla\log u_0\cdot e^{-t}zS_f\langle e^{-t}z,\cdot\rangle \\ &- \nabla\log u_0\cdot S_f\langle e^{-t}z,e^{-t}z\rangle\mathrm{Id}-S_f^0\langle e^{-t}z,e^{-t}z\rangle\mathrm{Id}). \end{aligned}$$

Since

$$\frac{\partial e^{-t}z}{dz}=e^{-t}\quad\text{and}\quad\frac{\partial e^{-t}z}{dt}=-e^{-t}z,$$

we have

$$\frac{\partial f}{\partial t}(z,t)=\frac{e^{2t}Df\,[\mathrm{Id}+E(z,t)]\,\langle e^{-t}z\rangle}{(1+(e^{2t}-1)\nabla\log u_0\cdot e^{-t}z)^2}.$$

Notice that $(n+1)\log u_0 = -\log J_f$ and that $(1-e^{-2t}) < 1 - \|e^{-t}z\|^2$. Hence, using (4.1), we conclude that $\|E(z,t)\| < 1$. As a consequence, we see that $\mathrm{Id} - E(z,t)$ is an invertible operator. Therefore, its follows that

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)\,(\mathrm{Id} - E(z,t))^{-1}\,(\mathrm{Id} + E(z,t))\,\langle z\rangle.$$

Thus, $f(z,t)$ satisfies the differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad z \in \mathbb{B}^n, \quad t \geq 0,$$

where $h(z,t) = [\mathrm{Id} - E(z,t)]^{-1}[\mathrm{Id} + E(z,t)]\langle z\rangle$. This shows that $f(z,t)$ is a Loewner chain with the intial value $f(z,0) = f(z)$, which completes the proof. $\square$

Remark 4.2. If f is a locally univalent analytic function defined in the unit disk $\mathbb{D} \subset \mathbb{C}$, the correspondig $S^k_{ij}f$ and $S^0_{ij}f$ given, respectively, by (2.5) and (2.6), satisfy that $S^k_{ij}f = 0$ and

$$S^0_{11}f = -\frac{1}{2}Sf, \quad S^0_{12}f = 0, \quad S^0_{22}f = 0.$$

Therefore, univalence criterion given by inequality (4.1) becomes the classical criterion of univalence in the unit disk due to Nehari: if such function f satisfies that

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}$$

for all $|z| < 1$, then f is univalent in the unit disk.

Corollary 4.3. *Let f be as in previous theorem have constant (non-zero) Jacobian J_f. If*

$$(1-\|z\|^2)\|S_f(z)\langle z,\cdot\rangle\| \leq 1,$$

then f is univalent in $\mathbb{B}^n$.

Proof. Since J_f is a constant, $\nabla \log J_f = 0$. Furthermore, in this case, the correspondig solution u_0 in the proof of Theorem 4.1 is a constant too. Then the system (2.7) asserts that $S^0_{ij}f$ are identically zero for all i, j, and k. Thus, the inequality (4.1) equals

$$(1-\|z\|^2)\|S_f(z)\langle z,\cdot\rangle\| \leq 1.$$

A direct application of Theorem 4.1 ends the proof. $\square$

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