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A maximum theorem for generalized convex functions

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Dedicated to the memory of Professors Gábor Kassay and Csaba Varga.

Abstract. Motivated by the Maximum Theorem for convex functions (in the setting of linear spaces) and for subadditive functions (in the setting of Abelian semigroups), we establish a Maximum Theorem for the class of generalized convex functions, i.e., for functions $f : X \to \mathbb{R}$ that satisfy the inequality $f(x \circ y) \le pf(x) + qf(y)$, where $\circ$ is a binary operation on X and p, q are positive constants. As an application, we also obtain an extension of the Karush–Kuhn–Tucker theorem for this class of functions.

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1. Introduction

In what follows, a linear space X always means a vector space over the field of real numbers. If X is a topological linear space, then its (topological) dual space is denoted by X^*. The Maximum Theorem for convex functions, which is due to Dubovitskii and Milyutin (cf. [9]), can be stated as follows.

Theorem 1.1. *Let X be a linear space, let $D \subseteq X$ be a convex set and let $f_1, \dots, f_n : D \to \mathbb{R}$ be convex functions such that*

$$0 \le \max(f_1(x), \dots, f_n(x)) \qquad (x \in D).$$

Then there exist $\lambda_1, \dots, \lambda_n \ge 0$ with $\lambda_1 + \dots + \lambda_n = 1$ such that

$$0 \le \lambda_1 f_1(x) + \dots + \lambda_n f_n(x) \qquad (x \in D).$$

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A standard application of the Maximum Theorem is to prove the subdifferential formula for the pointwise maximum of convex functions, which was established by Dubovitskii and Milyutin (see [9]). For the standard terminologies and notations, we refer to the list of monographs in the list of references, where the reader can find many more details and applications.

Theorem 1.2. *Let X be a topological vector space, $D \subseteq X$ be an open convex set, $p \in D$ and $f_1, \dots, f_n : D \to \mathbb{R}$ be continuous convex functions with $f_1(p) = \dots = f_n(p)$ and define $f := \max(f_1, \dots, f_n)$. Then*

$$\partial f(p) = \operatorname{conv}\big(\partial f_1(p) \cup \dots \cup \partial f_n(p)\big).$$

Proof. Using that $f(p) = f_1(p) = \dots = f_n(p)$, for all $h \in X$, we obtain

$$\begin{aligned}
f'(p,h) :&= \lim_{t\to 0^+} \frac{f(p+th) - f(p)}{t} \\
&= \lim_{t\to 0^+} \frac{\max(f_1(p+th), \dots, f_n(p+th)) - f(p)}{t} \\
&= \lim_{t\to 0^+} \max\left(\frac{f_1(p+th) - f(p)}{t}, \dots, \frac{f_n(p+th) - f(p)}{t}\right) \\
&= \lim_{t\to 0^+} \max\left(\frac{f_1(p+th) - f_1(p)}{t}, \dots, \frac{f_n(p+th) - f_n(p)}{t}\right) \\
&= \max\left(\lim_{t\to 0^+} \frac{f_1(p+th) - f_1(p)}{t}, \dots, \lim_{t\to 0^+} \frac{f_n(p+th) - f_n(p)}{t}\right) \\
&= \max(f_1'(p,h), \dots, f_n'(p,h)).
\end{aligned}$$

First assume that a continuous linear functional $\varphi \in X^*$ belongs to $\partial f(p)$. Then, in view of the above formula for directional derivatives, we get

$$\varphi(h) \le f'(p,h) = \max(f_1'(p,h), \dots, f_n'(p,h)) \qquad (h \in X).$$

This relation implies that

$$0 \le \max(f_1'(p,h) - \varphi(h), \dots, f_n'(p,h) - \varphi(h)) \qquad (h \in X).$$

This inequality states that the maximum of the convex functions $h \mapsto f_i'(p,h) - \varphi(h)$ is nonnegative. Thus, by the Maximum Theorem, there exist $\lambda_1, \dots, \lambda_n \ge 0$ with $\lambda_1 + \dots + \lambda_n = 1$ such that

$$0 \le \lambda_1(f_1'(p,h) - \varphi(h)) + \dots + \lambda_n(f_n'(p,h) - \varphi(h)) \qquad (h \in X),$$

equivalently,

$$\varphi(h) \le \lambda_1 f_1'(p,h) + \cdots \lambda_n f_n'(p,h) = (\lambda_1 f_1 + \cdots \lambda_n f_n)'(p,h) \qquad (h \in X).$$

Using the so-called Sum Rule, we get

$$\begin{aligned}
\varphi \in \partial(\lambda_1 f_1 + \cdots \lambda_n f_n)(p) &= \lambda_1 \partial f_1(p) + \dots + \lambda_n \partial f_n(p) \\
&\subseteq \operatorname{conv}\big(\partial f_1(p) \cup \dots \cup \partial f_n(p)\big).
\end{aligned}$$

The proof of the reversed inclusion is simpler, thus it is left to the reader. $\square$

Another motivation for this paper comes from the theory of subadditive functions defined on Abelian semigroups. The following result was stated in the monograph [7] of Fuchssteiner and Lusky.

Theorem 1.3. *Let $(X,+)$ be an Abelian semigroup and let $f_1,\dots,f_n : X \to \mathbb{R}$ be subadditive functions such that*

$$0 \le \max(f_1(x),\dots,f_n(x)) \qquad (x \in X).$$

Then there exist $\lambda_1,\dots,\lambda_n \ge 0$ with $\lambda_1+\dots+\lambda_n = 1$ such that

$$0 \le \lambda_1 f_1(x)+\dots+\lambda_n f_n(x) \qquad (x \in X).$$

This result has beautiful applications in the book [7], for instance, the Phragmen–Lindelöf Principle and the Hadamard Three Circle Theorem (both results belong to the theory of complex functions) can elegantly be verified in terms of them.

2. The general maximum problem

The two Maximum Theorems described in the Introduction motivate the following definition.

Definition 2.1. Let X be a nonempty set. A family $\mathcal{F} \subseteq \{f : X \to \mathbb{R}\}$ is said to have the *discrete maximum property* if

$$f_1,\dots,f_n \in \mathcal{F}, \qquad 0 \le \max(f_1(x),\dots,f_n(x)) \qquad (x \in X)$$

implies that there exist $(\lambda_1,\dots,\lambda_n) \in S_n$ such that

$$0 \le \lambda_1 f_1(x)+\dots+\lambda_n f_n(x) \qquad (x \in X).$$

Here, for convenience, S_n denotes the $(n-1)$-dimensional simplex

$$\{(\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n \mid \lambda_1,\dots,\lambda_n \ge 0,\ \lambda_1+\dots+\lambda_n = 1\}.$$

If X has at least two elements, then the set of all functions $\mathcal{F} := \{f : X \to \mathbb{R}\}$ does not have the discrete maximum property. Indeed, Let $\{A_1, A_2\}$ be a partition of X and $f_i(x) := 0$ if $x \in A_i$, $f_i(x) := -1$ if $x \notin A_i$. Then $\max(f_1,f_2) = 0$, but $\lambda f_1 + (1-\lambda) f_2 < 0$ for all $\lambda \in [0,1]$. This example shows that, in order to possess the discrete maximum property, the family $\mathcal{F} \subseteq \{f : X \to \mathbb{R}\}$ must satisfy some additional nontrivial conditions.

In the next result we characterize the situation when a finite family of given functions possess a nonnegative convex combination.

Theorem 2.2. *Let X be nonempty and $f_1,\dots,f_n : X \to \mathbb{R}$. Then there exists $(\lambda_1,\dots,\lambda_n) \in S^n$ such that*

$$0 \le \lambda_1 f_1(x)+\dots+\lambda_n f_n(x) \qquad (x \in X) \tag{2.1}$$

if and only if

$$0 \le \max_{i\in\{1,\dots,n\}} \big(t_1 f_i(x_1)+\dots+t_n f_i(x_n)\big) \quad (x_1,\dots,x_n \in X,\ (t_1,\dots,t_n) \in S_n). \tag{2.2}$$

Proof. Assume first that (2.1) holds for some $\lambda \in S_n$. To verify the necessity of (2.2), let $x_1, \dots, x_n \in X$ and $(t_1, \dots, t_n) \in S_n$ be arbitrary. Then, using (2.1) for $x \in \{x_1, \dots, x_n\}$, we get

$$\begin{aligned} 0 &\le \sum_{j=1}^{n} t_j \big(\lambda_1 f_1(x_j) + \dots + \lambda_n f_n(x_j)\big) \\ &= \sum_{i=1}^{n} \lambda_i \big(t_1 f_i(x_1) + \dots + t_n f_i(x_n)\big) \\ &\le \max_{i \in \{1,\dots,n\}} \big(t_1 f_i(x_1) + \dots + t_n f_i(x_n)\big). \end{aligned}$$

This shows the necessity of condition (2.2).
Now assume that (2.2) holds and, for $x \in X$, define the set $\Lambda_x \subseteq S_n$ by

$$\Lambda_x := \big\{(\lambda_1, \dots, \lambda_n) \in S_n \mid 0 \le \lambda_1 f_1(x) + \dots + \lambda_n f_n(x)\big\}. \tag{2.3}$$

The inequality (2.1) is now equivalent to the condition

$$\bigcap_{x \in X} \Lambda_x \ne \emptyset, \tag{2.4}$$

because every element λ of the above intersection will satisfy (2.1). It easily follows from the definition that Λ_x is a compact convex subset of the $(n-1)$-dimensional affine space

$$\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 + \dots + \lambda_n = 1\}.$$

Therefore, according to Helly's Theorem, the condition (2.4) is satisfied if and only every n-member subfamily of $\{\Lambda_x \mid x \in X\}$ has a nonempty intersection. To verify this, let $x_1, \dots . x_n \in X$ be fixed arbitrarily. According to inequality (2.2), the pointwise maximum of the convex functions

$$S_n \ni (t_1, \dots, t_n) \mapsto t_1 f_i(x_1) + \dots + t_n f_i(x_n)$$

is nonnegative over S_n.
Therefore, in view of Theorem 1.1, there exists $(\lambda_1, \dots, \lambda_n) \in S_n$ such that

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} \lambda_i \big(t_1 f_i(x_1) + \dots + t_n f_i(x_n)\big) \\ &= \sum_{j=1}^{n} t_j \big(\lambda_1 f_1(x_j) + \dots + \lambda_n f_n(x_j)\big) \qquad ((t_1, \dots, t_n) \in S_n). \end{aligned}$$

If $i \in \{1, \dots, n\}$, then substituting $(t_1, \dots, t_n) := (\delta_{i,j})_{j=1}^n$ into the above inequality, we get that

$$\lambda_1 f_1(x_i) + \dots + \lambda_n f_n(x_i) \qquad (i \in \{1, \dots, n\}).$$

This shows that $\lambda \in \Lambda_{x_1} \cap \dots \cap \Lambda_{x_n}$, proving that this intersection is nonempty, as it was desired. $\square$

In the case $n = 2$, the above theorem immediately implies the following statement.

Corollary 2.3. *Let X be a nonempty set and $f, g : X \to \mathbb{R}$. Then there exists $\lambda \in [0,1]$ such that*

$$0 \le \lambda f(x) + (1-\lambda)g(x) \qquad (x \in X) \tag{2.5}$$

if and only if

$$0 \le \max\big(tf(x) + (1-t)f(y), tg(x) + (1-t)g(y)\big) \qquad (x, y \in X,\ t \in [0,1]). \tag{2.6}$$

3. Generalized convexity

The general convexity property that we introduce below is going to play an important role in the sequel.

Definition 3.1. Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, $p, q > 0$ be constants. A function $f : X \to \mathbb{R}$ is called $(\circ, p, q)$*-convex* if

$$f(x \circ y) \le pf(x) + qf(y) \qquad (x, y \in X).$$

Trivially, if X is a convex subset of a linear space, $p = q = \frac{1}{2}$, and $x \circ y = \frac{x+y}{2}$, then f is $(\circ, p, q)$-convex if and only if f is Jensen convex. On the other hand, if X is an Abelian semigroup, $p = q = 1$, and $x \circ y = x + y$, then f is $(\circ, p, q)$-convex if and only if f is subadditive.

The proof of the following assertion is elementary, therefore it is omitted.

Theorem 3.2. *The family of $(\circ, p, q)$-convex functions is closed with respect to addition, multiplication by positive scalars and pointwise maximum.*

The main result of this paper is stated in the following theorem.

Theorem 3.3. *Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, and $p, q > 0$ be constants. Let $f_1, \dots, f_n : X \to \mathbb{R}$ be $(\circ, p, q)$-convex functions such that*

$$0 \le \max(f_1(x), \dots, f_n(x)) \qquad (x \in X).$$

Then there exist $\lambda_1, \dots, \lambda_n \ge 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ such that

$$0 \le \lambda_1 f_1(x) + \cdots + \lambda_n f_n(x) \qquad (x \in X).$$

The following auxiliary result establishes the key tool for the proof of Theorem 3.3.

Lemma 3.4. *Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, and $p, q > 0$ be constants. Let*

$$S := \left\{ \frac{a}{a+b} \;\middle|\; \begin{array}{l} \text{There is an operation } * : X \times X \to X \text{ such that} \\ \text{every } (\circ, p, q)\text{-convex function is } (*, a, b)\text{-convex.} \end{array} \right\}$$

Then $1 - S \subseteq S$ and S is dense multiplicative subsemigroup of $[0,1]$.

Proof. If $s \in S$, then there exists an operation $* : X \times X \to X$ and $a, b > 0$ such that $s = \frac{a}{a+b}$ and f is $(*, a, b)$-convex, i.e.,

$$f(x * y) \le af(x) + bf(y) \qquad (x, y \in X).$$

Thus, interchanging the roles of x and y, we get

$$f(y * x) \le bf(x) + af(y) \qquad (x, y \in X),$$

which means that f is $(*', b, a)$-convex, where $x*'y := y*x$. Therefore $1-s = \frac{b}{a+b} \in S$, which shows that $1 - S \subseteq S$.
Additionally, let $t \in S$ be arbitrary.
Then there exists a binary operation $\cdot : X \times X \to X$ and $c, d > 0$ such that $t = \frac{c}{c+d}$ and f is also $(\cdot, c, d)$-convex, i.e.,

$$f(x \cdot y) \le cf(x) + df(y) \qquad (x, y \in X).$$

Using the $(\cdot, c, d)$- and the $(*, a, b)$-convexity of f (twice), for all $x, y \in X$, we obtain

$$\begin{aligned} f((x*y)\cdot(y*y)) &\le cf(x*y) + df(y*y) \\ &\le c(af(x) + bf(y)) + d(af(y) + bf(y)) \\ &= acf(x) + (bc + ad + bd)f(y). \end{aligned}$$

This implies that f is $(\diamond, ac, bc + ad + bd)$-convex, where $x \diamond y := (x*y)\cdot(y*y)$. Therefore,

$$st = \frac{ac}{ac + bc + ad + bd} \in S,$$

which proves that S is closed with respect to multiplication.
By induction, it follows that

$$s^n \in S \qquad (s \in S,\ n \in \mathbb{N}). \tag{3.7}$$

The assumption that f is $(\circ, p, q)$-convex implies that $S \cap]0, 1[\neq \emptyset$. Therefore, (3.7) yields that $\inf S = 0$. Using the inclusion $1 - S \subseteq S$, we can see that $\sup S = 1$.
Finally, to prove the density of S in $[0, 1]$, let $0 < a < b < 1$ be arbitrary. By $\sup S = 1$, we can choose $s \in S$ so that $\dfrac{a}{b} < s < 1$. Then, for some $n \in \mathbb{N}$, (in particular, with $n := \lfloor \frac{\log(a)}{\log(s)} \rfloor$), we have $s^n \in [a, b]$, which implies that $S \cap [a, b]$ is nonempty. $\square$

In the next result, we verify the Maximum Theorem for two functions.

Theorem 3.5. *Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, and $p, q > 0$ be constants. If $f, g : X \to \mathbb{R}$ are $(\circ, p, q)$-convex functions satisfying*

$$0 \le \max(f(x), g(x)) \qquad (x \in X), \tag{3.8}$$

then there exists $\lambda \in [0, 1]$ such that (2.5) *holds true.*

Proof. First we show that f and g satisfy the inequality (2.6). To verify this, let $x, y \in X$ and let $s \in S$ (where the set S was defined in Lemma 3.4.) Then there exist a binary operation $* : X \times X \to X$ and constans $a, b > 0$ such that the $(\circ, p, q)$-convexity of f and g implies the $(*, a, b)$-convexity of them. Thus, by the maximum inequality (3.8) at $x * y$, we get

$$0 \le \max(f(x*y), g(x*y)) \le \max(af(x) + bf(y), ag(x) + bg(y)).$$

Therefore

$$0 \le \max\Big(\frac{a}{a+b}f(x) + \frac{b}{a+b}f(y), \frac{a}{a+b}g(x) + \frac{b}{a+b}g(y)\Big),$$

and hence

$$0 \le \max\big(sf(x) + (1-s)f(y), sg(x) + (1-s)g(y)\big).$$

Because $s \in S$ was arbitrary and S is dense in $[0,1]$ (according to Lemma 3.4), we can conclude that (2.6) is satisfied for all $t \in [0,1]$.
Having proved that (2.6) is valid, in view of Corollary 2.3, it follows that there exists $\lambda \in [0,1]$ such that (2.5) holds. □

Proof of the discrete Maximum Theorem. The statement is trivial for $n = 1$ and it has been proved for $n = 2$. Assume its validity for some $n \geq 2$. Let $f_0, f_1, \dots, f_n$ be $(\circ, p, q)$-convex functions such that

$$0 \leq \max(f_0(x), f_1(x), \dots, f_n(x)) \qquad (x \in X).$$

Let $g(x) := \max(f_1(x), \dots, f_n(x))$. Then, by Theorem 3.2, we have that g is $(\circ, p, q)$-convex and

$$0 \leq \max(f_0(x), g(x)) \qquad (x \in X).$$

Using now Theorem 3.5, we obtain the existence of $\lambda \in [0,1]$ such that

$$\begin{aligned} 0 &\leq \lambda f_0(x) + (1-\lambda) g(x) \\ &= \max\big(\lambda f_0(x) + (1-\lambda) f_1(x), \dots, \lambda f_0(x) + (1-\lambda) f_n(x)\big) \qquad (x \in X). \end{aligned}$$

By the inductive assumption, there exists $(\lambda_1, \dots, \lambda_n) \in S_n$ such that

$$\begin{aligned} 0 &\leq \lambda_1\big(\lambda f_0(x) + (1-\lambda) f_1(x)\big) + \dots + \lambda_n\big(\lambda f_0(x) + (1-\lambda) f_n(x)\big) \\ &= \lambda f_0(x) + \lambda_1(1-\lambda) f_1(x) + \dots + \lambda_n(1-\lambda) f_n(x) \qquad (x \in X), \end{aligned}$$

which proves the statement for $(n+1)$ functions. □

4. An application

In the subsequent result we establish an extension of the Karush–Kuhn–Tucker Theorem.

Theorem 4.1. *Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, and $p, q > 0$ be constants. Let $f_0, f_1, \dots, f_n : X \to \mathbb{R}$ be $(\circ, p, q)$-convex functions and assume that $f_0(x_0) = 0$ and $x_0 \in X$ is a solution of the constrained optimization problem*

$$\textit{Minimize} \quad f_0(x) \quad \textit{subject to} \quad f_1(x), \dots, f_n(x) \leq 0. \tag{4.9}$$

Then there exist $(\lambda_0, \lambda_1, \dots, \lambda_n) \in S_{n+1}$ such that

$$\lambda_1 f_1(x_0) = \dots = \lambda_1 f_1(x_0) = 0 \tag{4.10}$$

and

$$0 \leq \lambda_0 f_0(x) + \lambda_1 f_1(x) + \dots + \lambda_n f_n(x) \qquad (x \in X). \tag{4.11}$$

Conversely, if conditions (4.10) *and* (4.11) *hold for some $(\lambda_0, \lambda_1, \dots, \lambda_n) \in S_{n+1}$ with $\lambda_0 > 0$, then x_0 is a solution of the optimization problem* (4.9).

Proof. If x_0 is a solution of the optimization problem then, for all $x \in X$, the inequalities

$$f_0(x) < f_0(x_0) = 0 \qquad \text{and} \qquad f_1(x), \dots, f_n(x) \leq 0$$

cannot hold simultaneously. Hence

$$0 \leq \max(f_0(x), f_1(x), \dots, f_n(x)) \qquad (x \in X).$$

Therefore, in view of Theorem 3.3, there exist $(\lambda_0, \lambda_1, \dots, \lambda_n) \in S_{n+1}$ such that (4.11) holds.

Being a solution to (4.9), x_0 is admissible for the optimization problem, that is, we have that $f_1(x_0), \dots, f_n(x_0) \leq 0$. Hence

$$0 \leq \lambda_0 f_0(x_0) + \lambda_1 f_1(x_0) + \cdots + \lambda_n f_n(x_0) = \lambda_1 f_1(x_0) + \cdots + \lambda_n f_n(x_0) \leq 0.$$

The terms in the last sum are nonpositive, therefore, the only way this sum can be zero is that it is zero termwise. Hence the transversality condition (4.10) is also true.

To prove the reversed statement, assume that (4.10) and (4.11) hold for some $(\lambda_0, \lambda_1, \dots, \lambda_n) \in S_{n+1}$ with $\lambda_0 > 0$. Let $x \in X$ be an admissible point with respect to problem (4.9), i.e., assume that $f_1(x_0), \dots, f_n(x_0) \leq 0$. Then, by (4.10) and (4.11), we get

$$\begin{aligned} \lambda_0 f_0(x_0) &= \lambda_0 f_0(x_0) + \lambda_1 f_1(x_0) + \cdots + \lambda_n f_n(x_0) \\ &= 0 \leq \lambda_0 f_0(x) + \lambda_1 f_1(x) + \cdots + \lambda_n f_n(x) \leq \lambda_0 f_0(x), \end{aligned}$$

which, using that $\lambda_0 > 0$, implies $f_0(x_0) \leq f_0(x)$, and proves the minimality of x_0. □

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