Stud. Univ. Babeş-Bolyai Math. 67(2022), No. 1, 31–45
DOI: 10.24193/subbmath.2022.1.03

Continuity and maximal quasimonotonicity of normal cone operators

Monica Bianchi, Nicolas Hadjisavvas and Rita Pini

Dedicated to the memory of Professor Gábor Kassay.

Abstract. In this paper we study some properties of the adjusted normal cone operator of quasiconvex functions. In particular, we introduce a new notion of maximal quasimotonicity for set-valued maps different from similar ones recently appeared in the literature, and we show that it is enjoyed by this operator. Moreover, we prove the $s \times w^*$ cone upper semicontinuity of the normal cone operator in the domain of f in case the set of global minima has non empty interior.

Mathematics Subject Classification (2010): 47H05, 47H04, 49J53, 90C33.

Keywords: Quasimonotone operator, maximal quasimonotone operator, cone upper semicontinuity, upper sign continuity, quasiconvex function.

1. Introduction

The notion of maximal monotone operator dates back to the sixties and, since then, it has been extensively studied in literature (see, for instance, [9] and the references therein). One of the main interests for maximal monotone operators is the strong relationship between convexity of a function and maximal monotonicity of its associated subdifferential operator.

In recent years different generalizations of monotonicity have been proposed, both in the scalar (see [16]) and in the set-valued case, in finite and infinite dimensional spaces. Among them the most studied are, without a doubt, pseudomonotonicity and quasimonotonicity. Many nice properties of these classes of operators have been proved, but little effort has been devoted to the study of a suitable notion of maximality. To fill this gap, Hadjisavvas in [14] introduced and studied maximal pseudomonotone operators $T : X \rightrightarrows X^*$, where X is a Banach space and X^* denotes its dual, while the notion of maximality for quasimonotone operators has been addressed in the recent works by Aussel and Eberhard [6], and by Bueno and Cotrina

Received 16 November 2021; Accepted 20 December 2021.

[11]. In particular, in [11] the authors extend the notion of polarity introduced by Martínez-Legaz and Svaiter in 2005 ([17]), by defining the quasimonotone polar of a set-valued operator in order to characterize maximal quasimonotone operators via graph inclusion.

In this work we define a new notion of maximality for a quasimonotone operator defined on a Banach space, that is based both on the notion of quasimonotone polar of an operator T and on its behaviour at the points in the interior of the effective domain of T. This property is enjoyed, in particular, by the Clarke subdifferential $\partial^o f$, where f is quasiconvex and locally Lipschitz, under suitable restrictions on ∂^o, as well as by the adjusted normal cone operator to the sublevel sets of a quasiconvex, lower semicontinuous and solid function, provided suitable assumptions on the minima are satisfied. The interest in studying the properties of the adjusted normal cone operator is due to the crucial role it plays in characterizing quasiconvexity (see [7]).

The paper is organized as follows: In Section 2 we present some preliminary notions and results. In Section 3 the new definition of maximal quasimonotonicity for operators is introduced; some properties of maximal quasimonotone operators are established, together with a sufficient condition that can be compared with a similar one for maximal monotone operators. Section 4 is devoted to the investigation of the properties of the adjusted normal cone operator of a lower semicontinuous and quasiconvex function in terms of maximal quasimonotonicity and cone upper semicontinuity. In particular, the cone upper semicontinuity is proved in the domain of f in case the set of global minima has non empty interior, thereby extending a result in [7].

2. Preliminaries

Let X be a real Banach space, X^* its topological dual, and $\langle \cdot, \cdot \rangle$ the duality mapping. In the following, $\{x_\alpha\}$ and $\{x^*_\alpha\}$, with $\alpha \in \Gamma$ will denote nets in X and X^*, respectively.

For $x \in X$ and $r > 0$, $B(x, r)$, $\overline{B}(x, r)$ and $S(x, r)$ will denote the open ball, the closed ball and the sphere centered at x with radius r, respectively. Also, given a nonempty set $A \subseteq X$, let $B(A, \epsilon) = \{x \in X : \operatorname{dist}(x, A) < \epsilon\}$ and $\overline{B}(A, \epsilon) = \{x \in X : \operatorname{dist}(x, A) \leq \epsilon\}$, where $\operatorname{dist}(x, A) = \inf_{y \in A} \|x - y\|$ is the distance of x from A. A set L in a topological vector space is said to be a cone if it is closed under multiplication by nonnegative scalars; a set L is said to be an open cone if it is an open set, closed under multiplication by positive scalars. A convex set B is called a base of a cone L if and only if $0 \notin \overline{B}$ and $L = \cup_{t \geq 0} tB$.

The domain and the graph of a set valued map $T : X \rightrightarrows X^*$ will be denoted by $\operatorname{dom}(T)$ and $\operatorname{Gr}(T)$, while the *effective domain* of T is given by

$$\operatorname{edom}(T) = \{x \in \operatorname{dom}(T) : T(x) \neq \{0\}\}.$$

For any $x^* \in X^*$, let $\mathbb{R}_+ x^* = \{tx^* \in X^* : t \geq 0\}$ and for any $B \subseteq X^*$ let $\mathbb{R}_+ B = \cup_{x^* \in B} \mathbb{R}_+ x^*$. The operator $(\mathbb{R}_+ T) : X \rightrightarrows X^*$ is given by

$$(\mathbb{R}_+ T)(x) = \mathbb{R}_+(T(x)) = \cup_{x^* \in T(x)} \mathbb{R}_+ x^*.$$

Given $(x,x^*),(y,y^*) \in X \times X^*$, (x,x^*) is said to be *quasimonotonically related* to (y,y^*), denoted by $(x,x^*) \uparrow (y,y^*)$, if

$$\min\{\langle x^*, y-x\rangle, \langle y^*, x-y\rangle\} \le 0$$

(see for instance [11] and the references therein). Note that $(x,0)$ is quasimonotonically related to any $(y,y^*) \in X \times X^*$. Relation $\uparrow$ is a tolerance relation, i.e., it is reflexive and symmetric but in general not transitive.

The *quasimonotone polar* $T^\nu : X \rightrightarrows X^*$ of T is given by

$$\begin{aligned} T^\nu(x) &= \{x^* \in X^* : (x,x^*) \uparrow (y,y^*)\ \forall y^* \in T(y), y \in \mathrm{dom}(T)\} \\ &= \{x^* \in X^* : (x,x^*) \uparrow (y,y^*)\ \forall y^* \in T(y), y \in \mathrm{edom}(T)\} \end{aligned}$$

Note that $0 \in T^\nu(x)$ and that $T^\nu(x)$ is a cone for all $x \in X$. Moreover, $T^\nu(x)$ is a convex and w^*-closed set (see Corollary 3.8 in [11]), that can be not pointed (see, for instance, the next Example 3.2).

Moreover, the following proposition, related to Lemma 1 in [6] and to Proposition 3.5 in [14] holds :

Proposition 2.1. *Let $T : X \rightrightarrows X^*$ be an operator. If $(x_\alpha, x_\alpha^*) \in \mathrm{Gr}(T^\nu)$, $(x_\alpha, x_\alpha^*) \to (x,x^*)$ in the $w \times w^*$ topology, and $\limsup_\alpha \langle x_\alpha^*, x_\alpha\rangle \le \langle x^*, x\rangle$, then $x^* \in T^\nu(x)$. In particular, $\mathrm{Gr}(T^\nu)$ is sequentially closed in the $s \times w^*$ topology and in the $w \times s$ topology.*

Proof. Take any $(y,y^*) \in \mathrm{Gr}(T)$. Since $(x_\alpha, x_\alpha^*) \uparrow (y,y^*)$, we have

$$\min\{\langle x_\alpha^*, y-x_\alpha\rangle, \langle y^*, x_\alpha - y\rangle\} \le 0.$$

By our assumptions,

$$\liminf_\alpha \langle x_\alpha^*, y - x_\alpha\rangle = \langle x^*, y\rangle - \limsup_\alpha \langle x_\alpha^*, x_\alpha\rangle \ge \langle x^*, y-x\rangle .$$

Thus

$$\min\{\langle x^*, y-x\rangle, \langle y^*, x-y\rangle\} \le 0$$

which says that $(x,x^*) \in \mathrm{Gr}(T^\nu)$.

In particular, $\mathrm{Gr}(T^\nu)$ is sequentially closed in the $s \times w^*$ and in the $w \times s$ topologies, because, in these cases, we have $\lim \langle x_n^*, x_n\rangle = \langle x^*, x\rangle$. □

In the sequel we will introduce the notions of quasimonotonicity, cone upper semicontinuity, upper sign continuity for an operator T. The reader can easily convince himself that all the definitions hold for T if and only if they hold for $\mathbb{R}_+ T$.

A map $T : X \rightrightarrows X^*$ is said to be

(i) *quasimonotone* if $T(x) \subseteq T^\nu(x)$, for all $x \in X$; equivalently, for every $x, y \in X$, $x^* \in T(x)$, $y^* \in T(y)$,

$$\min\{\langle x^*, y-x\rangle, \langle y^*, x-y\rangle\} \le 0;$$

(ii) *$s \times w^*$ cone upper semicontinuous (cone usc)* at $x \in \mathrm{edom}(T)$ if for every w^*-open cone K such that $T(x) \subseteq K \cup \{0\}$, there exists a neighborhood U of x such that $T(y) \subseteq K \cup \{0\}$ for all $y \in U$ (see Definition 5 in [6]);

(iii) *upper sign continuous* at x if for every $v \in X$,

$$\begin{aligned} &\exists \delta > 0 : \forall t \in \,]0, \delta[\,, \exists x^* \in T(x+tv)\backslash\{0\} : \langle x^*, v\rangle \geq 0 \\ &\quad \Rightarrow \exists x^* \in T(x)\backslash\{0\} : \langle x^*, v\rangle \geq 0 \end{aligned} \tag{2.1}$$

In particular, the second definition fits well with operators $T : X \rightrightarrows X^*$ whose values are unbounded convex cones. In this case, if $T(x)$ has a base for every $x \in \mathrm{edom}(T)$, the notion is equivalent to Definition 2.1 in [7]. Moreover, our definition of upper sign continuity is slightly different from Definition 9 in [6].

It is easy to verify that the definition (ii) is stronger than (iii). Indeed, the following result holds:

Proposition 2.2. *If $T : X \rightrightarrows X^*$ is cone upper semicontinuous at $x \in \mathrm{edom}(T)$, then T is upper sign continuous at x.*

Proof. Suppose first that for every $v \in X$, the l.h.s. in (2.1) is never satisfied; in this case, there is nothing to prove. Otherwise, suppose that there exists $v \in X$ such that the l.h.s. holds, but $\langle x^*, v\rangle < 0$ for every $x^* \in T(x) \setminus \{0\}$. The set

$$K_v = \{x^* \in X^* : \langle x^*, v\rangle < 0\}$$

is a w^*-open cone with $T(x) \subseteq K_v \cup \{0\}$. From the cone upper semicontinuity at x, for t small enough, $T(x+tv) \subseteq K_v \cup \{0\}$, a contradiction. □

The cone upper semicontinuity of a conic valued operator, under mild conditions, implies also the closedness of the graph of the operator in the $s \times w^*$ topology as shown in the following result:

Proposition 2.3. *Let $T : X \rightrightarrows X^*$ be such that for all $x \in X$, $T(x)$ is a convex, w^*-closed cone with a w^*-compact base. If* $\mathrm{dom}(T)$ *is closed and T is cone usc, then* $\mathrm{Gr}(T)$ *is closed in the $s \times w^*$ topology.*

Proof. Let (x_α, x_α^*), $\alpha \in \mathcal{A}$ be a net in $\mathrm{Gr}(T)$, converging to (x, x^*) in the $s \times w^*$ topology. Since $\mathrm{dom}(T)$ is closed, $x \in \mathrm{dom}(T)$. We have to show that $x^* \in T(x)$. If $x^* = 0$ this is trivial, so we suppose that $x^* \neq 0$ and $x^* \notin T(x)$. Let $B(x)$ be a w^*-compact base of $T(x)$. Then $B(x) \cap \mathbb{R}_+ x^* = \emptyset$.

By Lemma 3.3 of [14], there exists $b \in X$ such that $\langle x^*, b\rangle > 0 > \langle y^*, b\rangle$ for all $y^* \in B(x)$, so $\langle x^*, b\rangle > 0 > \langle y^*, b\rangle$ for all $y^* \in T(x)\backslash\{0\}$. The set $V := \{y^* \in X^* : \langle y^*, b\rangle < 0\}$ is an open cone and $T(x) \subseteq V \cup \{0\}$. By cone upper semicontinuity, there exists $\alpha_0 \in \mathcal{A}$ such that $T(x_\alpha) \subseteq V \cup \{0\}$ for $\alpha \succcurlyeq \alpha_0$. Thus, $\langle x_\alpha^*, b\rangle \leq 0$ for $\alpha \succcurlyeq \alpha_0$. This contradicts $\langle x^*, b\rangle > 0$ and $x_\alpha^* \overset{w^*}{\to} x^*$. □

Remark 2.4. In the Euclidean setting, a conic-valued map with closed graph is always cone usc. Indeed, one can consider the operator $T'(x) = T(x) \cap S(0,1)$; T' has compact range and closed graph, and therefore it is upper semicontinuous. This is equivalent to say that T is cone usc (see for instance [1], [8]). This is no longer true in infinite dimensional settings, as the following example shows. Let $X = X^* = \ell^2$, $\{x_n\}_n \subset \ell^2$ be a sequence of points different from 0 and strongly convergent to 0, and consider the set-valued map $T : \ell^2 \rightrightarrows \ell^2$ with domain $\{x_n\}_n \cup \{0\}$ and defined as follows:

$$T(0) = \{0\}, \quad T(x_n) = \mathbb{R}_+ e_n,$$

where e_n denotes the sequence $\{e_n^i\}_i$ such that $e_n^i = 1$ if $i = n$, and $e_n^i = 0$, otherwise. This operator is not cone usc at $x = 0$; indeed, taking $V = \emptyset$, $T(0) \subset V \cup \{0\}$, but $T(x_n) \notin V \cup \{0\}$, for any n. Let us show that $\mathrm{Gr}(T)$ is in fact $s \times w^*$ closed. Suppose that $(x_n, x_n^*) \in \mathrm{Gr}(T)$, and $x_n^* \xrightarrow{w^*} x^*$. From the definition of T, $x_n^* = t_n e_n$, for some $t_n \geq 0$. In addition, the sequence $\{t_n e_n\}$ is bounded. This implies that, for every $x \in \ell^2$, $\langle t_n e_n, x\rangle \to \langle x^*, x\rangle$ if and only if $x^* = 0$, thereby showing the closedness of $\mathrm{Gr}(T)$.

In order to define the notion of the operator we are interested in, i.e., the adjusted normal cone operator, we need first to recall some necessary preliminary definitions.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function and $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ its domain, which is always assumed nonempty.

For every $\lambda \in \mathbb{R}$ define the sublevel set $S_{f,\lambda} = \{x \in X : f(x) \leq \lambda\}$ and the strict sublevel set $S^<_{f,\lambda} = \{x \in X : f(x) < \lambda\}$. In particular, in order to simplify the notation, for every $x \in \operatorname{dom} f$, we set

$$S_f(x) = S_{f,f(x)}, \qquad S_f^<(x) = S^<_{f,f(x)}.$$

The function f is said to be *lower semicontinuous (lsc)* if $S_{f,\lambda}$ is a closed set for every $\lambda \in \mathbb{R}$, and *solid* if $\operatorname{int} S_{f,\lambda} \neq \emptyset$ for every $\lambda > \inf_X f$.

Moreover, let $\rho_x^f = \operatorname{dist}(x, S_f^<(x))$ and for any $x \in \operatorname{dom} f$ define the adjusted sublevel set $S_f^a(x)$ by

$$S_f^a(x) = \begin{cases} S_f(x) \cap \overline{B}(S_f^<(x), \rho_x^f) & x \in \operatorname{dom} f \setminus \operatorname{argmin} f \\ S_f(x) & x \in \operatorname{argmin} f. \end{cases}$$

In particular, $S_f^a(x) = S_f(x)$ for every $x \in \operatorname{dom} f$ whenever every minimum of f is global.

In general, $S_f^<(x) \subset S_f^a(x) \subseteq S_f(x)$ for any $x \in \operatorname{dom} f$.

The function f is said to be *quasiconvex* if for every $x, y \in \operatorname{dom} f$ and $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}$$

It is well known that the convexity of the sublevel sets $S_f(x)$, of the strict sublevel sets $S_f^<(x)$ as well as of the adjusted sublevel sets $S_f^a(x)$ for every $x \in X$, characterizes the quasiconvexity of the function f (see [7]).

Let us recall that a map $T : X \rightrightarrows X$ is said to be *lower semicontinuous* at x if for every $x_n \xrightarrow{s} x$ with $x \in \operatorname{dom}(T)$, and for every $y \in T(x)$, there exists $y_n \in T(x_n)$ such that $y_n \xrightarrow{s} y$ (see for instance [3], p. 39-40).

The following result, whose proof is very similar to the proof in the finite dimensional case in [1, Th. 3.1], holds:

Theorem 2.5. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be quasiconvex. If $S_f(x)$ is closed for all $x \in \operatorname{dom} f$, then the map $x \rightrightarrows S_f^a(x)$ is lower semicontinuous on* $\operatorname{dom} f$.

For any function f let us define the normal cone operator $N_f : X \rightrightarrows X^*$ and the adjusted normal cone operator $N_f^a : X \rightrightarrows X^*$ as follows: if $x \in \operatorname{dom} f$,

$$N_f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \ \forall y \in S_f(x)\}$$
$$N_f^a(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \ \forall y \in S_f^a(x)\};$$

if $x \notin \operatorname{dom} f$, we set $N_f(x) = N_f^a(x) = \emptyset$. Obviously, $N_f(x) \subseteq N_f^a(x)$.

These operators are always quasimonotone, indeed they satisfy a stronger property known as cyclic quasimonotonicity (see [7] and the references therein).

3. A new notion of maximal quasimonotone map

The study of a suitable definition of maximal quasimonotone set-valued map was recently addressed by Aussel and Eberhard [6] and also by Bueno and Cotrina [11]. The new notion of maximal quasimonotonicity we introduce in this section is enjoyed, in particular, by the Clarke subdifferential operator of a locally Lipschitz and quasiconvex function, and by the adjusted normal cone operator of a quasiconvex function.

Definition 3.1. Let $T : X \rightrightarrows X^*$ be a quasimonotone operator with $\operatorname{int} \operatorname{edom}(T) \neq \emptyset$. T is maximal quasimonotone if for every $x^* \in T^\nu(x)$ with $x \in \operatorname{int} \operatorname{edom}(T)$, we have $x^* \in \mathbb{R}_+ T(x)$, i.e. $T^\nu(x) = \mathbb{R}_+ T(x)$ for every $x \in \operatorname{int} \operatorname{edom}(T)$.

As a consequence of [11, Th. 4.7(4)], our notion of maximal quasimonotone operator is weaker than the notion introduced in [6].

The following trivial example exhibits a maximal quasimonotone map according to Definition 3.1 which is not maximal quasimonotone neither according to [6] or [11].

Example 3.2. Define $T : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$T(x) = \begin{cases} 0, & \text{if } x < 0 \\ [0, +\infty) & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

Then $\operatorname{edom}(T) = [0, +\infty)$. It is straightforward to verify that T is maximal quasimonotone according to Definition 3.1. Indeed, for $x \in (0, +\infty)$, $(x, x^*) \uparrow (y, y^*)$ for every $y^* \in T(y)$ if and only if $x^* \in \mathbb{R}_+ T(x)$.

On the other hand, a quasimonotone extension of T on $[0, +\infty)$ can be provided by setting $T(0) = (-\infty, +\infty)$. Thus T is not maximal quasimonotone either according to Definition 1 in [6] or according to the definition in [11] given in terms of inclusion. In addition, note that T is not even pre-maximal quasimonotone as defined in [11] since

$$T^\nu(x) = \begin{cases} (-\infty, +\infty) & \text{if } x \leq 0 \\ [0, +\infty) & \text{if } x > 0 \end{cases}$$

is not quasimonotone.

The following example shows a quasimonotone operator which is not maximal quasimonotone according to Definition 3.1.

Example 3.3. Define $T : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$T(x,y) = \begin{cases} \mathbb{R}_+(1,1) & \text{if } x \geq 0, y \geq 0,\ (x,y) \neq (0,0) \\ \mathbb{R}_+(1,-1) & \text{if } x > 0, y < 0 \\ \mathbb{R}_+(-1,1) & \text{if } x < 0, y > 0 \\ \mathbb{R}_+(-1,-1) & \text{if } x \leq 0, y \leq 0 \end{cases}$$

It is straighforward to verify that this operator is quasimonotone with $\operatorname{edom} T = \mathbb{R}^2$ but it is not maximal quasimonotone; indeed, $T^\nu(0,0) = \mathbb{R}^2$, but $T(0,0) = \mathbb{R}_+(-1,-1)$.

In the next proposition some properties of maximal quasimonotone operators are summarized. Some of them extend to maximal quasimonotone operators results similar to those involving maximal monotone ones (see, for instance, [15], Ch. 3).

Proposition 3.4. *Let $T : X \rightrightarrows X^*$ be a maximal quasimonotone operator. Then,*

i) $\mathbb{R}_+T : X \rightrightarrows X^*$ *is maximal quasimonotone.*

ii) $\mathbb{R}_+T(x)$ *is convex for all* $x \in \operatorname{int}\operatorname{edom}(T)$.

iii) *If* $x \in \operatorname{int}\operatorname{edom}(T)$, $x_n \xrightarrow{s} x$, $x_n^* \xrightarrow{w^*} x^*$ *with* $x_n^* \in T(x_n)$, *then* $x^* \in \mathbb{R}_+T(x)$. *In particular,* $\mathbb{R}_+T(x)$ *is sequentially w^*-closed, for every* $x \in \operatorname{int}\operatorname{edom}(T)$.

iv) *If* $x \in \operatorname{int}\operatorname{edom}(T)$, $x_n \xrightarrow{w} x$, $x_n^* \xrightarrow{s} x^*$ *with* $x_n^* \in T(x_n)$, *then* $x^* \in \mathbb{R}_+T(x)$.

Proof. Recall that by definition of maximal quasimonotone operators,

$$T^\nu(x) = \mathbb{R}_+T(x) \text{ for all } x \in \operatorname{int}\operatorname{dom}(T).$$

i) Trivial, noting that $(\mathbb{R}_+T)^\nu(x) = T^\nu(x) = \mathbb{R}_+T(x)$ for all

$$x \in \operatorname{int}\operatorname{edom}(\mathbb{R}_+T) = \operatorname{int}\operatorname{edom}(T).$$

ii) follows from Corollary 3.8 in [11].

iii) and iv) follows from Proposition 2.1 observing that for quasimonotone operators $x_n^* \in T(x_n) \subseteq T^\nu(x_n)$.

□

Remark 3.5. Note that $\mathbb{R}_+T(x)$ is not necessarily convex or w^*-closed at the boundary of $\operatorname{edom}(T)$. For example, take $X = \mathbb{R}^2$ and define T by

$$T(x) = \begin{cases} \mathbb{R}_+ \times \{0\} & \text{if } x > 0, y \geq 0 \\ \mathbb{R}_- \times \{0\} & \text{if } x < 0, y \geq 0 \\ \mathbb{R} \times \mathbb{R}_+ & \text{if } x = 0, y > 0 \\ \{(x,y) : -2|x| < y < -|x|\} & \text{if } x = y = 0 \\ \emptyset & \text{if } y < 0 \end{cases}$$

Then T is maximal quasimonotone according to Definition 3.1, but $\mathbb{R}_+T(0,0)$ is neither closed, nor convex.

The next two results try to adapt known properties of maximal monotone operators to the case of maximal quasimonotone ones.

It is well known that any maximal monotone operator is upper semicontinuous in the interior of its domain (see Theorem 1.28, Section 3 in [15]). In case of maximal quasimonotone operators a similar result holds in a finite dimensional setting.

Proposition 3.6. *If $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal quasimonotone, then T is cone usc at every $x \in \operatorname{int}\operatorname{edom}(T)$.*

Proof. Without loss of generality we will suppose that $T = \mathbb{R}_+T$. Let $x \in \operatorname{int}\operatorname{edom}(T)$ be a point where T is not cone usc. Then, there exist an open cone K, and sequences $\{x_n\}$ and $\{x_n^*\}$ such that: $T(x) \subset K \cup \{0\}$, $x_n \to x$ and $x_n^* \in T(x_n) \setminus (K \cup \{0\})$. Without loss of generality, suppose that $\|x_n^*\| = 1$ and $x_n^* \to x^*$, with $\|x^*\| = 1$. From Proposition 3.4-iii), $x^* \in T(x)$. On the other hand, $x_n^* \in (K \cup \{0\})^c \subset K^c$, which is a closed set, so $x^* \in K^c$, $x^* \neq 0$, a contradiction. □

The next result provides a sufficient condition for maximal quasimonotonicity, that can be compared with a similar one for maximal monotone operators (see Theorem 1.33, Section 3 in [15]; see also Lemma 9.i-ii. in [6]):

Proposition 3.7. *Let $T : X \rightrightarrows X^*$ be upper sign-continuous, with convex, w^*-compact values. If $\operatorname{int}\operatorname{edom}(T) \neq \emptyset$ and $0 \notin T(x)$ for every $x \in \operatorname{int}\operatorname{edom}(T)$, then $T^\nu(x) \subseteq \mathbb{R}_+T(x)$, for every $x \in \operatorname{int}\operatorname{edom}(T)$. In particular, if T is quasimonotone, then it is maximal quasimonotone.*

Proof. Let us assume that there exists $x \in \operatorname{int}\operatorname{edom}(T)$ and $x_0^* \neq 0$, such that $x_0^* \in T^\nu(x)\setminus\mathbb{R}_+T(x)$. From the assumption $0 \notin T(x)$, and thus $\mathbb{R}_+x_0^* \cap T(x) = \emptyset$. Therefore, we can apply Lemma 3.3. in [14] and find $b \in X$ such that

$$\langle x_0^*, b\rangle > 0 > \langle x^*, b\rangle, \quad \forall x^* \in T(x). \tag{3.1}$$

Set $x_t = x + tb \in \operatorname{int}\operatorname{edom}(T)$ for $t > 0$ sufficiently small. Since $\langle x_0^*, x_t - x\rangle > 0$, from the definition of quasimonotone polar it follows that $\langle x^*, b\rangle \geq 0$ for all $x^* \in T(x_t)$. By upper sign-continuity, there exists $x^* \in T(x) \setminus \{0\}$ such that $\langle x^*, b\rangle \geq 0$, contradicting (3.1).

In case T is quasimonotone, from the inclusion $T^\nu(x) \supseteq \mathbb{R}_+T(x)$ the maximal quasimonotonicity easily follows. □

The example below shows that the assumption $0 \notin T(x)$ cannot be dropped, even in case we strengthen the continuity of T by imposing its cone upper semicontinuity:

Example 3.8. Define $T : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$T(x,y) = \begin{cases} \mathbb{R} \times (-\infty, 0] & \text{if } x = 0, y = 0 \\ [0, +\infty) \times \{0\} & \text{if } x > 0, y \geq 0 \\ (-\infty, 0] \times \{0\} & \text{if } x < 0, y \geq 0 \\ \mathbb{R} \times \{0\} & \text{if } x = 0, y > 0 \\ \mathbb{R}_+(x,y) & \text{if } x \in \mathbb{R}, y < 0 \end{cases}$$

It is straighforward to verify that $\operatorname{edom}(T) = \mathbb{R}^2$, T is quasimonotone, cone usc with closed, conic and convex values, but it is not maximal quasimonotone. As a matter of fact, $T^\nu(0,0) = \mathbb{R}^2$, while $T(0,0) = \mathbb{R} \times (-\infty, 0]$.

In the last result of this section we apply Proposition 3.7 to show the maximal quasimonotonicity of the Clarke subdifferential.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz function and denote by $\partial^o f : X \rightrightarrows X^*$ its Clarke subdifferential. It is well known that $\operatorname{dom}(\partial^o f) = \operatorname{dom} f$, $\partial^o f(x)$

is w^*-compact and convex for all $x \in \operatorname{dom}(\partial^o f)$, and $\partial^o f$ is upper semicontinuous in the $s \times w^*$ topology (see [12], and [18] Prop. 7.3.8). Thus, Proposition 3.7 and Theorem 4.1 in [5] imply

Corollary 3.9. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz quasiconvex function. Assume that $0 \notin \partial^o f(x)$ for all $x \in \operatorname{int} \operatorname{edom}(\partial^o f)$. Then, $\partial^o f$ is maximal quasimonotone.*

Note that a function satisfying the assumptions of the corollary above is necessarily pseudoconvex (see [4], Theorem 4.1). This means that $\partial^o f$ is D-maximal pseudomonotone (see [14], Corollary 3.2). However, this does not automatically imply maximal quasimonotonicity, as shown by the next example. The example also shows that the assumption $0 \notin \partial^o f(x)$, $\forall x \in \operatorname{dom} f$, cannot be omitted from Corollary 3.9.

Example 3.10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = \frac{1}{2}x_1^2 + |x_2|$. Then f is convex, thus quasiconvex. Its subdifferential $\partial f = \partial^o f$ is given by

$$\partial f\,(x_1, x_2) = \begin{cases} \{(x_1, 1)\} & \text{if } x_2 > 0 \\ \{(x_1, -1)\} & \text{if } x_2 < 0 \\ \{x_1\} \times [-1, 1] & \text{if } x_2 = 0 \end{cases}$$

Note that $\partial^o f$ is usc with compact convex values and $\operatorname{edom}(\partial^o f) = \mathbb{R}^2$. The operator $\partial^o f$ is maximal monotone and D-maximal pseudomonotone. It is not maximal quasimonotone, because $(1, 0) \in (\partial^o f)^\nu\,(0, 0)$, but $(1, 0) \notin \mathbb{R}_+ \partial^o f(0, 0)$.

Finally, note that the function $f(x) = |x|$ does not satisfy the assumptions of Corollary 3.9, but $\partial^o f$ is maximal quasimonotone.

4. Maximal quasimonotonicity and continuity properties of the adjusted normal cone operator

We start by proving the maximal quasimonotonicity of the normal operator N_f^a. To this purpose, it is necessary to describe the interior of the effective domain of this operator.

Let us first introduce some preliminary useful notions. Given a convex set $K \subseteq X$, a point $x_0 \in K$ is called a *support point* of K if there exists $x^* \in X^* \backslash \{0\}$ such that

$$\langle x^*, x_0 \rangle = \sup_{x \in K} \langle x^*, x \rangle\,,$$

or equivalently, if $x_0 \in \operatorname{edom}(N_K)$, where $N_K : K \rightrightarrows X^*$ is defined as follows

$$N_K(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \le 0,\ \forall y \in K\}.$$

The set of support points of K is denoted by $\operatorname{supp}(K)$; this definition is consistent with the one in [2], Ch. 7, but is different from the one in [10], that corresponds in fact to the notion of proper support points given in [2]. The set of nonsupport points (or quasi-interior points, see [13] Prop. 2.2) is the set

$$\operatorname{nsupp}(K) := K \backslash \operatorname{supp}(K).$$

Note that, if K is a nonempty, convex and closed set with nonempty interior, then every boundary point x of K is a support point for K (see Lemma 7.7 in [2]). Therefore, $\operatorname{nsupp}(K) = \operatorname{int} K$. In infinite dimensional spaces we may have nonsupport points even if $\operatorname{int} K$ is empty (see Example 7.8 in [2]).

If $\operatorname{nsupp}(K) \neq \emptyset$, then $\operatorname{nsupp}(K)$ is dense in K. In fact, we have the easy property:

Proposition 4.1. *Let $K \subseteq X$ be convex. If $x_1 \in K$ and $x_2 \in \operatorname{nsupp}(K)$, then*

$$]x_1, x_2] \subseteq \operatorname{nsupp}(K).$$

In particular, $\operatorname{nsupp}(K)$ *is dense in K.*

Proof. Assume that there exists $x_3 = tx_1 + (1-t)x_2$, $t \in]0,1[$, such that

$$x_3 \notin \operatorname{nsupp}(K).$$

Then there exists $x^* \in X^* \setminus \{0\}$ such that

$$\langle x^*, tx_1 + (1-t)x_2 \rangle = \sup_{x \in K} \langle x^*, x \rangle \geq \langle x^*, x_1 \rangle \tag{4.1}$$

$$\langle x^*, tx_1 + (1-t)x_2 \rangle = \sup_{x \in K} \langle x^*, x \rangle > \langle x^*, x_2 \rangle \tag{4.2}$$

The strict inequality in (4.2) is due to the fact that

$$\langle x^*, tx_1 + (1-t)x_2 \rangle = \langle x^*, x_2 \rangle$$

would imply that $x_2 \in \operatorname{supp}(K)$, contrary to our assumption.

Combining (4.1) and (4.2) we get a contradiction. Hence, $x_3 \in \operatorname{nsupp}(K)$. □

Let now $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc, solid and quasiconvex function and set

$$C = \operatorname{argmin} f.$$

Under the assumptions on f, C is closed and convex, and $\operatorname{int} \operatorname{dom} f \neq \emptyset$.

Proposition 4.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex, lsc and solid function. Then*

$$\operatorname{int} \operatorname{edom}(N_f^a) = \begin{cases} \operatorname{int} \operatorname{dom} f & \text{if } \operatorname{nsupp}(C) = \emptyset \\ (\operatorname{int} \operatorname{dom} f) \setminus C & \text{if } \operatorname{nsupp}(C) \neq \emptyset \end{cases}$$

Proof. By Proposition 3.4 in [7] we have $\operatorname{dom} f \setminus C \subseteq \operatorname{edom}(N_f^a)$, so

$$(\operatorname{int} \operatorname{dom} f) \setminus C \subseteq \operatorname{edom}(N_f^a).$$

Since $(\operatorname{int} \operatorname{dom} f) \setminus C$ is open, we obtain

$$(\operatorname{int} \operatorname{dom} f) \setminus C \subseteq \operatorname{int} \operatorname{edom}(N_f^a). \tag{4.3}$$

We consider two cases:

(i) Let $\operatorname{nsupp}(C) = \emptyset$. Then $C = \operatorname{supp}(C) = \operatorname{edom}(N_C) \subseteq \operatorname{edom}(N_f^a)$. Combining with $(\operatorname{int} \operatorname{dom} f) \setminus C \subseteq \operatorname{edom}(N_f^a)$ we obtain $\operatorname{int} \operatorname{dom} f \subseteq \operatorname{edom}(N_f^a)$. Hence $\operatorname{int} \operatorname{dom} f \subseteq \operatorname{int} \operatorname{edom}(N_f^a)$. The reverse implication is obvious, since $\operatorname{edom}(N_f^a) \subseteq \operatorname{dom} f$, so $\operatorname{int} \operatorname{edom}(N_f^a) = \operatorname{int} \operatorname{dom} f$.

(ii) Let $\operatorname{nsupp}(C) \neq \emptyset$. Take $x_0 \in \operatorname{int} \operatorname{edom}(N_f^a)$. There exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq \operatorname{int} \operatorname{edom}(N_f^a)$. Then $B(x_0, \varepsilon) \subseteq \operatorname{int} \operatorname{dom} f$. If we had $B(x_0, \varepsilon) \cap C \neq \emptyset$, then we would also have $B(x_0,\varepsilon)\cap \operatorname{nsupp} C\neq\emptyset$, due to Proposition 4.1. But then there would exist a point $y\in B(x_0,\varepsilon)\subseteq \operatorname{edom}(N_f^a)$ such that $y\in \operatorname{nsupp} C$. This is clearly impossible. Hence, $B(x_0,\varepsilon)\subseteq (\operatorname{int}\operatorname{dom} f)\setminus C$, which shows that

$$\operatorname{int}\operatorname{edom}(N_f^a)\subseteq (\operatorname{int}\operatorname{dom} f)\setminus C.$$

The reverse implication was already shown in (4.3). □

An immediate consequence of Proposition 4.2 is the following:

Corollary 4.3. *Let $f: X\to\mathbb{R}\cup\{+\infty\}$ be a quasiconvex, lsc function. Assume that* $\operatorname{int} C\neq\emptyset$. *Then*

$$\operatorname{int}\operatorname{edom}(N_f^a) = (\operatorname{int}\operatorname{dom} f)\setminus C.$$

Proof. If $\operatorname{int} C\neq\emptyset$, then f is solid, and $\operatorname{nsupp}(C)=\operatorname{int} C\neq\emptyset$. Proposition 4.2 yields the result. □

We are now in position to prove maximality of the quasimonotone operator N_f^a. To this aim, it is necessary to provide a description for $(N_f^a)^\nu$.

Theorem 4.4. *Let $f: X\to\mathbb{R}\cup\{+\infty\}$ be a quasiconvex, lsc and solid function. Then*

$$(N_f^a)^\nu(x)=\begin{cases} N_f^a(x), & \text{if } x\in\operatorname{dom} f\setminus C\\ X^*, & \text{if } x\in C\end{cases}$$

Proof. Let $x\in C$. Take any $(y,y^*)\in \operatorname{Gr}(N_f^a)$. If $y\in C$, then $x\in S_f(y)=S_f^a(y)$. If $y\notin C$, then $x\in S_f^<(y)\subseteq S_f^a(y)$. In both cases, $x\in S_f^a(y)$ so $\langle y^*, x-y\rangle\le 0$. It follows that for every $x^*\in X^*$,

$$\min\{\langle x^*, y-x\rangle, \langle y^*, x-y\rangle\}\le 0.$$

Thus, $(x,x^*)\uparrow(y,y^*)$ so $(N_f^a)^\nu(x)=X^*$.
Now let $x\in\operatorname{dom} f\setminus C$. Since N_f^a is quasimonotone, we always have $N_f^a(x)\subseteq (N_f^a)^\nu(x)$, so we have to show that

$$(N_f^a)^\nu(x)\subseteq N_f^a(x). \tag{4.4}$$

Suppose by contradiction that there exists $x_0^*\in (N_f^a)^\nu(x)\setminus N_f^a(x)$. It follows that $\langle x_0^*, y'-x\rangle>0$ for some $y'\in S_f^a(x)$.
Since f is solid, $\operatorname{int} S_f^a(x)\neq\emptyset$ and $S_f^a(x)=\overline{\operatorname{int} S_f^a(x)}$. Thus, there exists some $\overline{y}$ such that

$$\langle x_0^*, \overline{y}-x\rangle>0,\quad \overline{y}\in\operatorname{int} S_f^a(x). \tag{4.5}$$

Set $y_t=x+t(\overline{y}-x)$, $t\in(0,1]$. Then (4.5) implies that for all $t\in(0,1]$,

$$\langle x_0^*, y_t-x\rangle>0,\quad y_t\in\operatorname{int} S_f^a(x).$$

Combining with $x_0^*\in (N_f^a)^\nu(x)$ and $\langle y^*, y_t-x\rangle=t\langle y^*,\overline{y}-x\rangle$, we deduce

$$\langle y^*, \overline{y}-x\rangle\ge 0,\quad \forall y^*\in N_f^a(y_t),\ t\in(0,1]. \tag{4.6}$$

By Proposition 3.4 (ii) in [7], for every quasiconvex, lsc and solid function f and $x\in\operatorname{dom} f\setminus C$, we have $N_f^a(x)\setminus\{0\}\neq\emptyset$. Thus, $x\in\operatorname{edom}(N_f^a)$. Take any $x^*\in N_f^a(x)\setminus\{0\}$. Then

$$\overline{y}\in\operatorname{int} S_f^a(x)\subseteq \operatorname{int}\{y\in X:\langle x^*, y-x\rangle\le 0\}=\{y\in X:\langle x^*, y-x\rangle<0\}.$$

This means that $\langle x^*, \overline{y} - x\rangle < 0$. Hence,

$$N_f^a(x) \subset \{z^* \in X^* : \langle z^*, \overline{y} - x\rangle < 0\} \cup \{0\}. \tag{4.7}$$

Set $K = \{z^* \in X^* : \langle z^*, \overline{y} - x\rangle < 0\}$. This is a w^*-open cone, and $N_f^a(x) \subseteq K \cup \{0\}$. Taking into account the cone upper semicontinuity of the map N_f^a at $x \in \operatorname{dom} f \backslash C$ implied by Proposition 3.5 in [7], we obtain $N_f^a(y_t) \subseteq K \cup \{0\}$ for all $t > 0$ small enough. From (4.6), we get $N_f^a(y_t) = \{0\}$. But for $t > 0$ small enough, we have that $y_t \notin C$ so $y_t \in \operatorname{edom}(N_f^a)$, a contradiction. □

Theorem 4.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex, lsc and solid function. In addition, if $\sharp C \geq 2$, we assume that* $\operatorname{int} C \neq \emptyset$. *If* $\operatorname{int}\operatorname{edom}(N_f^a) \neq \emptyset$, *then N_f^a is maximal quasimonotone.*

Proof. Let $x \in \operatorname{int}\operatorname{edom}(N_f^a)$. In the special case $\sharp C = 1$ and $C = \{x\}$, we have $N_f^a(x) = X^* = (N_f^a)^\nu(x)$ by Theorem 4.4. According to Corollary 4.3, in all other the cases we have $x \notin C$. Applying again Theorem 4.4 we obtain $N_f^a(x) = (N_f^a)^\nu(x)$, so N_f^a is maximal quasimonotone. □

Remark 4.6. The assumption about the set C in the theorem above cannot be relaxed. Take, for instance, the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = |x_1|$.
The set $C = \{0\} \times \mathbb{R}$ has empty interior, $(N_f^a)^\nu(0,0) = \mathbb{R}^2$ from Theorem 4.4, but $(0,1) \notin \mathbb{R}_+ N_f^a(0,0) = N_f^a(0,0)$.

Note that N_f^a can be maximal quasimonotone also in case the function f is not quasiconvex. Take for instance, $f(x) = xe^{-x}$. Indeed, it is easy to verify that

$$N_f^a(x) = \begin{cases} [0, +\infty) & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

is maximal quasimonotone, despite f being trivially not quasiconvex.

In this last part we will investigate some continuity properties of the map N_f^a. Let us first state the following result:

Proposition 4.7. *Let $A : X \rightrightarrows X$ be a map which is lsc on its domain. Define $M :$* $\operatorname{dom}(A) \rightrightarrows X^*$ *by $M(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \forall y \in A(x)\}$. Then the graph of M is $s \times w^*$ sequentially closed on* $\operatorname{dom}(A) \times X^*$.

Proof. Assume that $x_n \xrightarrow{s} x \in \operatorname{dom}(A)$, $x_n^* \in M(x_n)$ and $x_n^* \xrightarrow{w^*} x^*$. Since A is a lsc map, for every $y \in A(x)$ there exists a subnet x_{n_i} of x_n and $y_{n_i} \in A(x_{n_i})$ s.t. $y_{n_i} \xrightarrow{s} y$. Let β be a bound of the sequence $\{x_n^*\}$. Then

$$\begin{aligned} \left|\langle x^*, y - x\rangle - \left\langle x_{n_i}^*, y_{n_i} - x_{n_i}\right\rangle\right| &\leq \left|\left\langle x^* - x_{n_i}^*, y - x\right\rangle\right| \\ &\quad + \left|\left\langle x_{n_i}^*, (y - x) - (y_{n_i} - x_{n_i})\right\rangle\right| \\ &\leq \left|\left\langle x^* - x_{n_i}^*, y - x\right\rangle\right| + \beta \left\|(y - x) - (y_{n_i} - x_{n_i})\right\| \to 0. \end{aligned}$$

We find

$$\langle x^*, y - x\rangle = \lim \left\langle x_{n_i}^*, y_{n_i} - x_{n_i}\right\rangle \leq 0.$$

Hence, $x^* \in M(x)$. □

As an immediate consequence of Theorem 2.5 and Proposition 4.7 we find the following:

Corollary 4.8. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be quasiconvex. If $S_f(x)$ is closed for all $x \in \operatorname{dom} f$, then the graph of the map $x \rightrightarrows N_f^a(x)$ is sequentially closed on $\operatorname{dom} f \times X^*$ in the $s \times w^*$ topology.*

In finite dimensions, the above corollary entails that N_f^a is cone usc (see Corollaries 3.1 and 3.2 in [1]).

In infinite dimensions, by assuming that f is solid, we can show, via the $s \times w^*$ closedness of the graph, the cone upper semicontinuity of the normal cone operator N_f^a in $\operatorname{dom} f$ under a suitable assumption on C. In particular, we recover Proposition 3.5 in [7].

Theorem 4.9. *Let f be quasiconvex, lsc and solid. Then N_f^a is $s \times w^*$ cone upper semicontinuous in $\operatorname{dom} f \setminus C$. If in addition $\#C \leq 1$, or $\#C \geq 2$ and $\operatorname{int} C \neq \emptyset$, then N_f^a is $s \times w^*$ cone upper semicontinuous in $\operatorname{dom} f$.*

Proof. First of all note that if C is a singleton, then N_f^a is $s \times w^*$ cone upper semicontinuous at that point. In the following we will assume that C is not a singleton. Let $x \in \operatorname{dom} f$.

Suppose by contradiction that there exist a w^*-open cone M and a sequence $x_n \in \operatorname{dom} f$, $x_n \xrightarrow{s} x$, such that $N_f^a(x) \subseteq M \cup \{0\}$, but

$$N_f^a(x_n) \nsubseteq M \cup \{0\}. \tag{4.8}$$

Thus, there exists $z_n^* \neq 0$, with $z_n^* \in N_f^a(x_n) \setminus M$. We will show that there exist $n_0 \in \mathbb{N}$, $\varepsilon > 0$ and $y_0 \in X$ such that for all $n \geq n_0$ and $v \in \overline{B}(0,1)$, we have $y_0 + \varepsilon v \in S_f^a(x_n)$. To see this, we consider two cases:

(i) If $x \notin C$, then take λ such that $\inf f < \lambda < f(x)$. Since f is solid, $\operatorname{int} S_{f,\lambda} \neq \emptyset$. By lower semicontinuity of f, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $f(x_n) > \lambda$. Now take $y_0 \in X$ and $\varepsilon > 0$ such that $\overline{B}(y_0, \varepsilon) \subseteq S_{f,\lambda}$. Then for every $v \in \overline{B}(0,1)$ and $n \geq n_0$, we have $y_0 + \varepsilon v \in S_{f,\lambda} \subseteq S_f^<(x_n) \subseteq S_f^a(x_n)$.

(ii) If $x \in C$, then by assumption $\operatorname{int} C \neq 0$; take $y_0 \in \operatorname{int} C$ and $\varepsilon > 0$ such that $\overline{B}(y_0, \varepsilon) \subseteq C$. Then we obtain $y_0 + \varepsilon v \in C \subseteq S_f^a(x_n)$ for all $n \in \mathbb{N}$ and $v \in \overline{B}(0,1)$.

In both cases, $z_n^* \in N_f^a(x_n)$ implies that for $n \geq n_0$,

$$\varepsilon \langle z_n^*, v \rangle \leq \langle z_n^*, x_n - y_0 \rangle \quad \forall v \in \overline{B}(0,1),$$

so

$$\varepsilon \|z_n^*\| \leq \langle z_n^*, x_n - y_0 \rangle.$$

Consequently, taking $n_1 \geq n_0$ such that $\|x_n - x\| \leq \frac{\varepsilon}{2}$ for $n \geq n_1$, we find

$$\varepsilon \|z_n^*\| \leq \langle z_n^*, x_n - x \rangle + \langle z_n^*, x - y_0 \rangle \leq \frac{\varepsilon}{2} \|z_n^*\| + \langle z_n^*, x - y_0 \rangle, \quad n \geq n_1.$$

Thus,

$$0 < \frac{\varepsilon}{2} \|z_n^*\| \leq \langle z_n^*, x - y_0 \rangle, \quad n \geq n_1. \tag{4.9}$$

Since $\langle z_n^*, x - y_0\rangle > 0$, we can choose $t_n > 0$ such that $\langle t_n z_n^*, x - y_0\rangle = 1$. From (4.9) we deduce $\|t_n z_n^*\| \le \frac{2}{\varepsilon}$, $n \ge n_1$. Thus there exists $z^* \in X^*$ and a subsequence $t_{n_k} z_{n_k}^* \xrightarrow{w^*} z^*$. From the $s \times w^*$ sequential closedness of $\mathrm{Gr}(N_f^a)$, it follows that $z^* \in N_f^a(x) \subseteq M \cup \{0\}$. But from (4.8) we obtain that $t_n z_n^*$ belongs to the w^*-closed set $X^* \setminus M$ for all n, so $z^* \notin M$. It follows that $z^* = 0$. Therefore $\langle t_n z_n^*, x - y_0\rangle \to 0$, a contradiction. $\square$

Acknowledgement. The first and third authors were partially supported by GNAMPA-INdAM, Italy.

References

[1] Al-Homidan, S., Hadjisavvas, N., Shaalan, L., *Tranformation of quasiconvex functions to eliminate local minima*, J. Optim. Theory Appl., **177**(2018), 93-105.

[2] Aliprantis, C.D., Border, K., *Infinite Dimensional Analysis*, Springer-Verlag Berlin Heidelberg, 2006.

[3] Aubin, J.P., Frankowska, H., *Set-Valued Analysis*, Birkhäuser, 1990.

[4] Aussel, D., *Subdifferential properties of quasiconvex and pseudoconvex functions: Unified approach*, J. Optim. Theory Appl., **97**(1998), 29-45.

[5] Aussel, D., Corvellec, J.-N., Lassonde, M., *Subdifferential characterization of quasiconvexity and convexity*, J. Convex Anal., **1**(1994), 195-201.

[6] Aussel, D., Eberhard, A., *Maximal quasimonotonicity and dense single-directional properties of quasimonotone operators*, Math. Program. Ser. B, **139**(2013), 27-54.

[7] Aussel, D., Hadjisavvas, N., *Adjusted sublevel sets, normal operator, and quasiconvex programming*, SIAM J. Optim., **16**(2005), 358-367.

[8] Borde, J., Crouzeix, J.-P., *Continuity properties of the normal cone to the level sets of a quasiconvex function*, J. Optim. Theory Appl., **66**(1990), 415-429.

[9] Borwein, J.M, *Fifty years of maximal monotonicity*, Optim. Lett., **4**(2010), 473-490.

[10] Borwein, J.M, Lewis, A.S., *Partially finite convex programming, Part I: Quasi relative interiors and duality theory*, Math. Program., **57**(1992), 15-48.

[11] Bueno, O., Cotrina, J., *On maximality of quasimonotone operators*, Set-Valued Var. Anal., **27**(2019), 87-101.

[12] Clarke, F.H., *Optimization and Nonsmooth Analysis*, Wiley Interscience, New York, 1983.

[13] Csetnek, E.R., *Overcoming the failure of the classical generalized interior-point regularity conditions in convex optimization. Applications of the duality theory to enlargements of maximal monotone operators*, PhD Thesis, Logos Verlag, Berlin, 2010.

[14] Hadjisavvas, N., *Continuity and maximality properties of pseudo-monotone operators*, J. Convex Anal., **178**(2003), 459-469.

[15] Hu, S., Papageorgiou, S., *Handbook of Multivalued Analysis, Vol. I: Theory*, Kluwer Academic Publisher, Norwell, MA, 1997.

[16] Karamardian, S., Schaible, S., *Seven kinds of monotone maps*, J. Optim. Theory Appl., **66**(1990), 37-46.

[17] Martínez-Legaz, J.-E., Svaiter, B.F., *Monotone operators representable by l.s.c. convex functions*, Set-Valued Anal., **13**(2005), 21-46.

[18] Schirotzek, W., *Nonsmooth Analysis*, Springer Verlag, Berlin Heidelberg, 2007.

Monica Bianchi
Università Cattolica del Sacro Cuore di Milano,
Dipartimento di Matematica per le Scienze Economiche, Finanziarie ed Attuariali,
Via Necchi 9, Milano, Italy
e-mail: monica.bianchi@unicatt.it

Nicolas Hadjisavvas
University of the Aegean,
Department of Product and Systems Design Engineering,
Konstantinoupoleos 1, Ermoupoli, Syros, Greece
e-mail: nhad@aegean.gr

Rita Pini
Università degli Studi di Milano-Bicocca,
Dipartimento di Matematica e Applicazioni,
Via R. Cozzi 55, Milano, Italy
e-mail: rita.pini@unimib.it