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Characterizations of hilbertian norms involving the areas of triangles in a smooth space

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. In the previous paper, we have defined together with I. Ionică the heights of a nontrivial triangle with respect to Birkhoff orthogonality in a real smooth space X, $\dim X \geq 2$. In the present paper, we remark that, generally, the area of a nontrivial triangle in X has not the same value for different heights of the triangle. The purpose of this paper is to characterize the norm of X if this space has the property that the area of any triangle is well defined (independent of considered height). In this line we give five equivalent properties using the directional derivative of the norm. If X is strictly convex and $\dim X \geq 3$, then each of these five properties characterizes the hilbertian norms (generated by inner products).

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1. Introduction

Let X be a real normed space and let X^* be its dual space. We recall that two elements $x, y \in X$ are *Birkhoff orthogonal*, $x \perp y$, if

$$\|x\| \leq \|x + ty\|, \text{ for all } t \in \mathbb{R},$$

where we denote by $\mathbb{R}$ the set of real numbers.

If the norm of X is generated by an inner product then this norm is called *hilbertian.* Also, we recall that the space X is smooth if there exists

$$\lim_{t\to 0} \frac{\|x + ty\|^2 - \|x\|^2}{2t} = n'(x; y), \text{for all } x, y \in X.$$

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Since the function $t \mapsto \|x + ty\|, t \in \mathbb{R}$, is convex, it follows that

$$x \perp y \text{ if and only if } n'(x; y) = 0.$$

In a real smooth space X the basic properties of the norm derivative are the following:

(P_1) $n'(x; x) = \|x\|^2$, for all $x \in X$.

(P_2) $n'(x; a_1y_1 + a_2y_2) = a_1n'(x; y_1) + a_2n'(x; y_2)$, for all $a_1, a_2 \in \mathbb{R}$ and $x, y_1, y_2 \in X$.

(P_3) For every $x \in X$, the map $y \mapsto n'(x; y), y \in X$, is a linear continuous functional, that is $n'(x; \cdot) \in X^*$.

(P_4) $n'(x; y) \leq \|x\|\|y\|$, for all $x, y \in X$.

(P_5) $n'(ax; x + by) = a\|x\|^2 + abn'(x; y)$, for all $a, b \in \mathbb{R}, x, y \in X$.

(P_6) The mapping $x \mapsto n'(x; \cdot)$, $x \in X$, is continuous from X, endowed with norm topology, into X^* with the w^*-topology.

Particularly, we have the following homogeneous property:

$$n'(ax; by) = abn'(x; y), \text{ for all } a, b \in \mathbb{R}, x, y \in X.$$

Also, if Y is a finite dimensional subspace of X, then $x \mapsto n'(x; y)$ is continuous on Y for any fixed $y \in X$. (For details concerning these properties see for instance [5-8].)

A simple and useful characterization of the hilbertian norm in a smooth space using norm derivative was established by Joichi [11], Leduc [12], Tapia [17], namely

$$n'(x; y) = n'(y; x), \text{ for all } x, y \in X.$$

Moreover, Leduc [12] proved that if $\dim X \geq 3$, then it is sufficient to have the following weaker property:

$$n'(x; y) = 0 \text{ whenever } n'(y; x) = 0. \tag{1.1}$$

This property was extended using right norm derivatives by Papini [13]. There exists many different characterizations of hilbertian norms using norm derivatives or norm directional derivatives if smoothness is not request (see, for example [1-4,9-11,15-16] and the monography of Amir [5]). Generally, these characterizations are obtained using some known properties of remarkable lines of a triangle.

We recall a characterization of strictly convex spaces established by Tapia [17], namely, a linear normed space X is strictly convex if and only if the equality $|n'(x; y)| = \|x\|\|y\|$ holds only if x, y are linear dependent elements.

In this paper we consider the heights of a triangle defined in [14], using Birkhoff orthogonality. We establish five properties such that the areas of a triangle corresponding to each height of the triangle have the same value, that is the area is well defined. If X is also strictly convex and $\dim X \geq 3$, then every of these five properties characterizes the hilbertian norms.

2. Main result

Let x, y, z be three distinct elements in a real smooth space X. In [14] there are defined the *heights of the triangle*, having the vertices x, y, z, with respect to Birkhoff orthogonality. This concept is different to other concepts considered in [1,2,4]. In our paper, we use the concept of height defined in [14]. Thus, the height corresponding to vertex z is as follows:

$$h_{z;x,y} = \left\{ z + t\left(x - z + \frac{n'(x-y; x-z)}{\|x-y\|^2}(y-x) \right) \mid t \in \mathbb{R} \right\}.$$

Indeed, this straight line has the following property: $x - y \perp h_{z;x,y}$. Consequently, the area of triangle corresponding to this height is

$$A_{z;x,y} = \frac{1}{2}\|x-y\| \left\| x - z + \frac{n'(x-y; x-z)}{\|x-y\|^2}(y-x) \right\|. \tag{2.1}$$

Similarly, we get the areas $A_{x;z,y}$ and $A_{y;z,x}$. We remark that

$$A_{z;x,y} = A_{z;y,x} = A_{z+u,x+u,y+u}, \text{ for any } x, y, z, u \in X.$$

Since the area of a triangle is conserved by translation we can consider only triangles having a vertex in origin. We say that *the area of nontrivial triangle is well defined* if

$$A_{x;y,z} = A_{y;z,x} = A_{z;x,y}. \tag{2.2}$$

Obviously, we can suppose in the paper that $\dim X \geq 2$.

Theorem 2.1. *Let X be a real smooth space with $\dim X \geq 2$. The areas of nontrivial triangles in X are well defined if and only if one of the following equivalent properties is true:*

(i) *The area of any triangle having a vertex in the origin is well defined*;

(ii) $\|x-y\| \cdot \|x - z + \frac{n'(x-y;x-z)}{\|x-y\|^2}(y-x)\| = \|y-z\| \cdot \|y - x + \frac{n'(y-z;y-x)}{\|y-z\|^2}(z-y)\|$ $= \|z-x\| \cdot \|z - y + \frac{n'(z-x;z-y)}{\|z-x\|^2}(x-z)\|$, *for all distinct elements $x, y, z \in X$*;

(iii) $\|x-y\| \cdot \|x - z + \frac{n'(x-y;x-z)}{\|x-y\|^2}(y-x)\| = \|y-z\| \cdot \|y - x + \frac{n'(y-z;y-x)}{\|y-z\|^2}(z-y)\|$, *for all distinct elements $x, y, z \in X$*;

(iv) $\|x\| \cdot \| \|y\|^2 x - n'(y;x) y \| = \|y\| \cdot \| \|x\|^2 y - n'(x;y) x \|$, *for all $x, y \in X$*;

(v) *for any two-dimensional subspace $Y \subset X$ there exists a constant $K > 0$ such that, for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in Y, the following equality holds:*

$$\| \|x\|^2 y - n'(x;y) x \| = K\|x\| |x_1 y_2 - x_2 y_1|.$$

If X is strictly convex and $\dim X \geq 3$, then every of these properties is true if and only if the norm of X is hilbertian.

Proof. According to the equalities (2.1) and (2.2) the area of every triangle with vertices x, y, z is well defined if and only if the equalities (ii) hold. Since we can consider only triangles with a vertex in origin, we have the equivalence (i)⇔(ii). Also, (ii) is equivalent with the equalities obtained if $z = 0$, that is

$$\begin{aligned} \|x-y\| \left\| x + \frac{n'(x-y;x)}{\|x-y\|^2}(y-x) \right\| &= \|y\| \left\| y - x - \frac{n'(y;y-x)}{\|y\|^2} y \right\| \\ &= \|x\| \left\| y - \frac{n'(x;y)}{\|x\|^2} x \right\|, \end{aligned}$$

which proves that equality (iv) holds for all non zero distinct elements $x, y \in X$. Therefore, (ii)⇔(iv). Conversely, if we change x, y with $x - y$ and $z - y$ respectively, we get $(iv)\Rightarrow(iii)$. The implications $(ii)\Rightarrow(iii)$ and $(v)\Rightarrow(iv)$ are obvious. Next we observe that the equality (iii) applied to the elements y, z, x proves that $(iii)\Rightarrow(ii)$.

Now, we prove the equivalence $(iv)\Leftrightarrow(v)$. If Y is the linear subspace generated by the linear independent elements x, y, then there exist two functions $A, B : \mathbb{R}^2 \to \mathbb{R}$ such that

$$n'(u; v) = A(u_1, u_2)v_1 + B(u_1, u_2)v_2, \text{ for all } u = (u_1, u_2), v = (v_1, v_2),$$

where A, B are homogeneous. Thus, we have

$$\begin{aligned}&\|x\|^2 y - n'(x; y)x = ((A(x_1, x_2)x_1 + B(x_1, x_2)x_2)y_1\\ &- (A(x_1, x_2)y_1 + B(x_1, x_2)y_2)x_1,\\ &(A(x_1, x_2)x_1 + B(x_1, x_2)x_2)y_2 - (A(x_1, x_2)y_1 + B(x_1, x_2)y_2)x_2)\\ &= (B(x_1, x_2)(x_2y_1 - x_1y_2), -A(x_1, x_2)(x_2y_1 - x_1x_2))\\ &= (x_2y_1 - x_1y_2)(B(x_1, x_2), -A(x_1, x_2)).\end{aligned}$$

Therefore, the equality (iv) becomes

$$\begin{aligned}&\|y\| \cdot |x_1y_2 - x_2y_2| \cdot \|(B(x_1, x_2), -A(x_1, x_2))\|\\ &= \|x\| \cdot |x_1y_2 - x_2y_1| \cdot \|(B(y_1, y_2), -A(y_1, y_2)\|,\end{aligned}$$

for all $x, y \in Y$, that is the function

$$x \mapsto \|x\|^{-1}\|(B(x_1, x_2), -A(x_1, x_2)\|, \ x = (x_1, x_2) \in Y\backslash\{0\},$$

is a constant function. Since

$$|x_1y_2 - x_2y_1| \cdot \|(B(x_1, x_2), -A(x_1, x_2))\| = \|\|x\|^2 y - n'(x; y\|x\|)\|,$$

it follows that (iv) is equivalent with (v).

Assume that X is strictly convex. If $n'(x; y) = 0$, it follows by (iv) that

$$\|\|y\|^2 x - n'(y; x)y\| = \|x\|\|y\|^2.$$

Since $n'(x; \|y\|^2 x - n'(y; x)y) = \|x\|^2\|y\|^2$, we obtain that

$$n'(x; \|y\|^2 x - n'(y; x)y) = \|x\|\|\|y\|^2 x - n'(y; x)y\|.$$

By Tapia's characterization of strictly convex spaces [17], we get that the elements x and $\|y\|^2 x - n'(y; x)y$ are linear dependent, that is $n'(y; x) = 0$. But $\dim X \geq 3$, and so, by Leduc's result [12], it follows that the norm is necessary hilbertian.

Remark 2.2. The characterization of hilbertian norms by property (iv) was given in [10].

Remark 2.3. By the homogeneity property of the norm it follows that we can consider only elements having equal norms. Consequently, we have the following equivalent properties:

(i′) the area of any isosceles triangle is well defined;

(iv′) $\|y - n'(x; y)x\| = \|x - n'(y; x)y\|$, for all $x, y \in X$, with $\|x\| = \|y\| = 1$.

Also, in equality (v) we can consider only x, y having the same norm.

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References

[1] Alsina, C., Guijarro, P., Tomas, M.S., *On heights in real normed spaces and characterizations of inner product structures,* J. Inst. Math. Comput. Sci. Math. Ser., **6**(1993), 151-159.

[2] Alsina, C., Guijarro, P., Tomas, M.S., *A characterization of inner product space based on a property of height's transformation,* Arch. Mat., **6**(1993), 560-566.

[3] Alsina, C., Guijarro, P., Tomas, M.S., *Some remarkables lines of triangles in real normed spaces and characterizations of inner product structures*, Aequationes Math., **54**(1997), 234-241.

[4] Alsina, C., Sikorska, J., Tomas, M.S., *Norm Derivative Characterizations of Inner Product Spaces,* World Sci. Publish. Co., (2010), 188 pp.

[5] Amir, D., *Characterizations of Inner Product Spaces*, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1986.

[6] Barbu, V., Precupanu, T., *Convexity and Optimization in Banach Spaces,* Fourth Edition, Springer, Dordrecht Heidelberg London New York, 2012.

[7] Diestel, I., *Geometry of Banach Spaces Selected topics,* Lecture Notes in Mathematics, Springer, Berlin, 1978.

[8] Giles, J.R., *On a characterization of differentiability of the norm of a normed linear spaces,* J. Austral. Math. Soc., **12**(1971), 106-114.

[9] Guijarro, P., Tomas, M.S., *Characterizations of inner product spaces by geometrical properties of the heights in a triangle,* Arch. Math., **73**(1999), 64-72.

[10] Ionică, I., *Bisectrices of a triangle in a linear normed space and hilbertian norms,* An. Şt. Univ. Iaşi, **58**(2012), 145-156.

[11] Joichi, J.I., *More characterizations of inner product spaces,* Proc. Amer. Math. Soc., **19**(1968), 1180-1186.

[12] Leduc, M., *Caractérizations des espaces euclidiens,* C.R. Acad. Sci. Paris, Ser. A-B, **268**(1969), 943-946.

[13] Papini, P.L., *Inner products and norm derivatives,* J. Math. Anal. Appl., **91**(1983), 592-598.

[14] Precupanu, T., Ionică, I., *Heights of a triangle in a linear normed space and hilbertian norms,* A. St. Univ. "Al.I. Cuza" Iaşi, Matematica, **55**(2009), 35-43.

[15] Precupanu, T., *Characterizations of hilbertian norms,* Bull. U.M.I., **5**(1978), no. 15-B, 161-169.

[16] Precupanu, T., *On a characterization of hilbertian norms,* An. St. Univ. "Al.I. Cuza" Iaşi, Matematica, **56**(2010).

[17] Tapia, R.A., *A characterizations of inner product spaces,* Proc. AMS, **41**(1973), 569-574.

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