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On the order of convolution consistence of certain classes of harmonic functions with varying arguments

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Abstract. Making use of a modified Hadamard product or convolution of harmonic functions with varying arguments, combined with an integral operator, we study when these functions belong to a given class. Following an idea of U. Bednarz and J. Sokol we define the order of convolution consistence of three classes of functions and determine it for certain classes of harmonic functions with varying arguments defined using a convolution operator.

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A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [5]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f'_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the functions h and g analytic in $\mathcal{U}$ can be expressed in the following

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forms:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m \quad (0 \le b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m} \quad (0 \le b_1 < 1). \tag{2}$$

For functions $f \in \mathcal{H}$ given by (2) and $F \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{m=2}^{\infty} A_m z^m + \overline{\sum_{m=1}^{\infty} B_m z^m}, \tag{3}$$

we recall the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{m=2}^{\infty} a_m A_m z^m + \overline{\sum_{m=1}^{\infty} b_m B_m z^m} \quad (z \in \mathcal{U}). \tag{4}$$

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathcal{H}$ such that $(f * F)(z)$ exists for any $f \in \mathcal{H}$, and for various choices of F we get different linear operators which have been studied in recent past.
In [10] it is defined and studied a subclass of $\mathcal{H}$ denoted by $S_{\mathcal{H}}(F;\gamma)$, for $0 \le \gamma < 1$, which involves the convolution (4) and consist of functions of the form (1) satisfying the inequality:

$$\frac{\partial}{\partial \theta} \left(arg \left[(f * F)(z) \right] \right) > \gamma \tag{5}$$

$0 \le \theta < 2\pi$ and $z = re^{i\theta}$. Equivalently

$$Re \left\{ \frac{z \left(h(z) * H(z) \right)' - \overline{z \left(g(z) * G(z) \right)'}}{h(z) * H(z) + \overline{g(z) * G(z)}} \right\} \ge \gamma \tag{6}$$

where $z \in \mathcal{U}$. We also let $\mathcal{V}_{\mathcal{H}}(F;\gamma) = S_{\mathcal{H}}(F;\gamma) \bigcap V_{\mathcal{H}}$ where $V_{\mathcal{H}}$ is the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [8], consisting of functions f of the form (1) in $\mathcal{H}$ for which there exists a real number ϕ such that

$$\eta_m + (m-1)\phi \equiv \pi \,(mod 2\pi), \quad \delta_m + (m+1)\phi \equiv 0 \,(mod 2\pi), \quad m \ge 2, \tag{7}$$

where $\eta_m = arg\,(a_m)$ and $\delta_m = arg\,(b_m)$.
Some of the function classes emerge from the function class $S_{\mathcal{H}}(F;\gamma)$ defined above. Indeed, if we specialize the function $F(z)$ we can obtain, respectively, (see [10]) the class of functions defined using: the Wright's generalized operator on harmonic functions ([11], [16]), the Dziok-Srivastava operator on harmonic functions ([1]), the Carlson-Shaffer operator ([4]), the Ruscheweyh derivative operator on harmonic functions ([7], [9], [12]), the Srivastava-Owa fractional derivative operator ([15]), the Sălăgean derivative operator for harmonic functions ([6], [13]).

In the following we suppose that $F(z)$ is of the form

$$F(z) = H(z) + \overline{G(z)} = z + \overline{z} + \sum_{m=2}^{\infty} C_m \left(z^m + \overline{z^m} \right), \tag{8}$$

where $C_m \geq 0\,(m \geq 2)$.
In [10] the following characterization theorem is proved

Theorem 1. *Let $f = h + \overline{g}$ be given by* (2) *with restrictions* (7) *and* $0 \leq b_1 < \dfrac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$. *Then $f \in \mathcal{V}_{\mathcal{H}}(F, \gamma)$ if and only if the inequality*

$$\sum_{m=2}^{\infty} \left(\frac{m-\gamma}{1-\gamma} |a_m| + \frac{m+\gamma}{1-\gamma} |b_m| \right) C_m \leq 1 - \frac{1+\gamma}{1-\gamma} b_1 \tag{9}$$

holds true.

Let consider the integral operator (for the analytic case see [3], [2], [13])

$$\mathcal{I}^s : f \in \mathcal{V}_{\mathcal{H}}(F, \gamma) \to \mathcal{V}_{\mathcal{H}}(F, \gamma), \ s \in \mathbb{R},$$

such that

$$\mathcal{I}^s f(z) = \mathcal{I}^s \left(z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m} \right) = z + \sum_{m=2}^{\infty} \frac{a_m}{m^s} z^m + \overline{\sum_{m=1}^{\infty} \frac{b_m}{m^s} z^m}. \tag{10}$$

Definition 1. The modified Hadamard product or $\circledast$-convolution of two functions f_1 and f_2 in $\mathcal{V}_{\mathcal{H}}$ of the form

$$f_1(z) = z + \sum_{m=2}^{\infty} a_{1,m} z^m + \overline{\sum_{m=1}^{\infty} b_{1,m} z^m} \text{ and } f_2(z) = z + \sum_{m=2}^{\infty} a_{2,m} z^m + \overline{\sum_{m=1}^{\infty} b_{2,m} z^m} \tag{11}$$

is the function $(f \circledast g)$ defined as

$$(f_1 \circledast f_2)(z) = z - \sum_{m=2}^{\infty} a_{1,m} a_{2,m} z^m + \overline{\sum_{m=1}^{\infty} b_{1,m} b_{2,m} z^m}.$$

We note that $(f \circledast g)$ also belongs to $\mathcal{V}_{\mathcal{H}}$.

Definition 2. ([3], [14]) Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be subsets of $\mathcal{V}_{\mathcal{H}}(F; \gamma)$. We say that the three $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is $S_{\circledast}$-closed under the convolution if there exists a number $S_{\circledast}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that

$$S_{\circledast}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = min \left\{ s \in \mathbb{R} : \mathcal{I}^s (f \circledast g) \in \mathcal{Z}, \forall f \in \mathcal{X}, \forall g \in \mathcal{Y} \right\} \tag{12}$$

The number $S_{\circledast}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is called the order of $\circledast$-convolution consistence of the three $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$

U. Bednarz and J. Sokol in [3] obtained the order of convolution consistence concerning certain classes of univalent functions (starlike, convex, uniform-starlike or uniform-convex functions) and in [14] it is obtained the order of $\circledast$-convolution consistence for certain classes of analytic functions with negative coefficients. In this paper we obtain similar results, but concerning the class $\mathcal{V}_{\mathcal{H}}(F; \gamma)$ and for $\circledast$-convolution.

Let denote by $\mathcal{V}^1_{\mathcal{H}}(F; \gamma)$ the subset of $\mathcal{V}_{\mathcal{H}}(F; \gamma)$ consisting of functions of the form (2) which satisfy $|a_m| \leq 1, |b_m| \leq 1, \forall m \geq 2$.

Main results

Theorem 2. *Let f_1, f_2 be two functions in $\mathcal{V}^1_{\mathcal{H}}(F;\gamma)$ of the form (1); then $(f_1 \circledast f_2)$also belongs to $\mathcal{V}^1_{\mathcal{H}}(F;\gamma)$.*

Proof. Since $f_1, f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma)$, from *Theorem 1* we have

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}\left|a_{1,m}\right|+\frac{m+\gamma}{1-\gamma}\left|b_{1,m}\right|\right)C_m \le 1-\frac{1+\gamma}{1-\gamma}b_{1,1}$$

and

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}\left|a_{2,m}\right|+\frac{m+\gamma}{1-\gamma}\left|b_{2,m}\right|\right)C_m \le 1-\frac{1+\gamma}{1-\gamma}b_{2,1}$$

and by the Cauchy-Schwarz inequality we deduce

$$\sum_{m=2}^{\infty}\sqrt{\left(\frac{m-\gamma}{1-\gamma}\left|a_{1,m}\right|+\frac{m+\gamma}{1-\gamma}\left|b_{1,m}\right|\right)\left(\frac{m-\gamma}{1-\gamma}\left|a_{2,m}\right|+\frac{m+\gamma}{1-\gamma}\left|b_{2,m}\right|\right)}C_m$$
$$\le\sqrt{\left(1-\frac{1+\gamma}{1-\gamma}b_{1,1}\right)\left(1-\frac{1+\gamma}{1-\gamma}b_{2,1}\right)}.$$

In order to prove that $f_1 \circledast f_2 \in \mathcal{V}^1_{\mathcal{H}}(F,\gamma)$ we need to show that

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma}{1-\gamma}\left|b_{1,m}\right|\left|b_{2,m}\right|\right)C_m \le 1-\frac{1+\gamma}{1-\gamma}b_{1,1}b_{2,1}.$$

But by using again Cauchy-Schwarz inequality we have

$$\left(\frac{m-\gamma}{1-\gamma}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma}{1-\gamma}\left|b_{1,m}\right|\left|b_{2,m}\right|\right)^2$$
$$\le\left[\left(\frac{m-\gamma}{1-\gamma}\left(\left|a_{1,m}\right|\right)^2+\frac{m+\gamma}{1-\gamma}\left(\left|b_{1,m}\right|\right)^2\right)\right]\left[\left(\frac{m-\gamma}{1-\gamma}\left(\left|a_{2,m}\right|\right)^2+\frac{m+\gamma}{1-\gamma}\left(\left|b_{2,m}\right|\right)^2\right)\right], \forall m\ge 2$$
$$\left[\left(\frac{m-\gamma}{1-\gamma}\left(\left|a_{1,m}\right|\right)^2+\frac{m+\gamma}{1-\gamma}\left(\left|b_{1,m}\right|\right)^2\right)\right]\left[\left(\frac{m-\gamma}{1-\gamma}\left(\left|a_{2,m}\right|\right)^2+\frac{m+\gamma}{1-\gamma}\left(\left|b_{2,m}\right|\right)^2\right)\right]$$
$$\le\left(\frac{m-\gamma}{1-\gamma}\left|a_{1,m}\right|+\frac{m+\gamma}{1-\gamma}\left|b_{1,m}\right|\right)\left(\frac{m-\gamma}{1-\gamma}\left|a_{2,m}\right|+\frac{m+\gamma}{1-\gamma}\left|b_{2,m}\right|\right).$$

On the other hand

$$\sqrt{\left(1-\frac{1+\gamma}{1-\gamma}b_{1,1}\right)\left(1-\frac{1+\gamma}{1-\gamma}b_{2,1}\right)}\le\sqrt{\left(1-\frac{1+\gamma}{1-\gamma}b_{1,1}b_{2,1}\right)\left(1-\frac{1+\gamma}{1-\gamma}b_{1,1}b_{2,1}\right)}$$
$$\le 1-\frac{1+\gamma}{1-\gamma}b_{1,1}b_{2,1},\quad 0\le b_{1,1}<1,\quad 0\le b_{2,1}<1.$$

□

Remark. Let the function $F = F_{m_0}, (m_0 \ge 2)$ be of the form (8) with $C_{m_0} = \dfrac{1-\gamma}{m_0-\gamma}$.
Then if

$$f_1(z)=f_2(z)=z-\frac{z^{m_0}}{C_0\dfrac{m_0-\gamma}{1-\gamma}} \tag{13}$$

then the condition (9) for f_1 becomes $\frac{m_0-\gamma}{1-\gamma}\left(|a_{1,m_0}|+|b_{1,m_0}|\right)C_{m_0}=1$ and similar for f_2 and this shows that f_1, f_2 belong to $\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$. For the function $(f_1 \circledast f_2)$ we have

$$\frac{m_0-\gamma}{1-\gamma}\left(|a_{1,m_0}|\,|a_{2,m_0}|+|b_{1,m_0}|\,|b_{2,m_0}|\right)C_{m_0}=\frac{m_0-\gamma}{1-\gamma}\frac{1}{C^2_{m_0}}\left(\frac{1-\gamma}{m_0-\gamma}\right)^2 C_{m_0}=1$$

and this imply that also $(f_1 \circledast f_2) \in \mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$. This shows that the result is Theorem 2 is sharp when $F=F_{m_0}, (m_0 \geq 2)$.

Corollary 1. *The order of* $\circledast$*-convolution consistence for the classes* $\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$ *is*

$$S_{\circledast}\left(\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)\right)=0 \tag{14}$$

Proof. From Theorem 2 we know that if $f_1, f_2 \in \mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$, then

$$\mathcal{I}^0(f_1 \circledast f_2)=(f_1 \circledast f_2) \in \mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma).$$

This means that

$$S_{\circledast}\left(\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)\right)\leq 0$$

But the functions f_1, f_2 given by (13) for which the coefficients of $(f_1 \circledast f_2)$ satisfy the inequalities with equality show that

$$S_{\circledast}\left(\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)\right)\geq 0. \qquad \square$$

Theorem 3. *Let* $f_1 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_1)$, $f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_2)$ *be two functions of the form* (1) *then* $(f_1 \circledast f_2)$ *belongs to* $\mathcal{V}^1_{\mathcal{H}}(F;\gamma^*)$, *where*

$$\gamma^*=\frac{(2+\gamma_1)(2+\gamma_2)(1-b_{1,1}b_{2,1})-2\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2+\gamma_1)(2+\gamma_2)(1+b_{1,1}b_{2,1})+\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]},$$

if

$$1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)>0$$

or

$$\gamma^*=\frac{(2-\gamma_1)(2-\gamma_2)(1-b_{1,1}b_{2,1})-2\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2-\gamma_1)(2-\gamma_2)(1+b_{1,1}b_{2,1})-\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]},$$

if

$$1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)<0.$$

Proof. Since $f_1 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_1)$ and $f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_2)$, from *Theorem 1* we have

$$\frac{\sum_{m=2}^{\infty}\left(\frac{m-\gamma_1}{1-\gamma_1}|a_{1,m}|+\frac{m+\gamma_1}{1-\gamma_1}|b_{1,m}|\right)C_m}{1-\frac{1+\gamma_1}{1-\gamma_1}b_{1,1}}\leq 1$$

and

$$\frac{\sum_{m=2}^{\infty}\left(\frac{m-\gamma_2}{1-\gamma_2}|a_{2,m}|+\frac{m+\gamma_2}{1-\gamma_2}|b_{2,m}|\right)C_m}{1-\frac{1+\gamma_2}{1-\gamma_2}b_{2,1}}\leq 1$$

and by the Cauchy-Schwarz inequality we deduce

$$\frac{\sum_{m=2}^{\infty}\sqrt{\left(\frac{m-\gamma_1}{1-\gamma_1}\left|a_{1,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\left|b_{1,m}\right|\right)\left(\frac{m-\gamma_2}{1-\gamma_2}\left|a_{2,m}\right|+\frac{m+\gamma_2}{1-\gamma_2}\left|b_{2,m}\right|\right)}C_m}{\sqrt{\left(1-\frac{1+\gamma_1}{1-\gamma_1}b_{1,1}\right)\left(1-\frac{1+\gamma_2}{1-\gamma_2}b_{2,1}\right)}}\leq 1. \tag{15}$$

In order to prove that $f_1 \circledast f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma^*)$ we need to show that

$$\frac{\sum_{m=2}^{\infty}\left(\frac{m-\gamma^*}{1-\gamma^*}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma^*}{1-\gamma^*}\left|b_{1,m}\right|\left|b_{2,m}\right|\right)C_m}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}}\leq 1. \tag{16}$$

We note that

$$\frac{\sum_{m=2}^{\infty}\left(\frac{m-\gamma^*}{1-\gamma^*}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma^*}{1-\gamma^*}\left|b_{1,m}\right|\left|b_{2,m}\right|\right)C_m}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}}$$

$$\leq\frac{\sum_{m=2}^{\infty}\sqrt{\left(\frac{m-\gamma_1}{1-\gamma_1}\left|a_{1,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\left|b_{1,m}\right|\right)\left(\frac{m-\gamma_2}{1-\gamma_2}\left|a_{2,m}\right|+\frac{m+\gamma_2}{1-\gamma_2}\left|b_{2,m}\right|\right)}C_m}{\sqrt{\left(1-\frac{1+\gamma_1}{1-\gamma_1}b_{1,1}\right)\left(1-\frac{1+\gamma_2}{1-\gamma_2}b_{2,1}\right)}} \tag{17}$$

implies (16).
But by using again Cauchy-Schwarz inequality we have

$$\left(\frac{m-\gamma_1}{1-\gamma_1}\frac{m-\gamma_2}{1-\gamma_2}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\frac{m+\gamma_2}{1-\gamma_2}\left|b_{1,m}\right|\left|b_{2,m}\right|\right)^2$$
$$\leq\left[\frac{m-\gamma_1}{1-\gamma_1}\left(\left|a_{1,m}\right|\right)^2+\frac{m+\gamma_1}{1-\gamma_1}\left(\left|b_{1,m}\right|\right)^2\right]\left[\frac{m-\gamma_2}{1-\gamma_2}\left(\left|a_{2,m}\right|\right)^2+\frac{m+\gamma_2}{1-\gamma_2}\left(\left|b_{2,m}\right|\right)^2\right]$$
$$\leq\left(\frac{m-\gamma_1}{1-\gamma_1}\left|a_{1,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\left|b_{1,m}\right|\right)\left(\frac{m-\gamma_2}{1-\gamma_2}\left|a_{2,m}\right|+\frac{m+\gamma_2}{1-\gamma_2}\left|b_{2,m}\right|\right)$$

and using in (17):

$$\frac{\frac{m-\gamma^*}{1-\gamma^*}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma^*}{1-\gamma^*}\left|b_{1,m}\right|\left|b_{2,m}\right|}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}}$$
$$\leq\frac{\frac{m-\gamma_1}{1-\gamma_1}\frac{m-\gamma_2}{1-\gamma_2}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\frac{m+\gamma_2}{1-\gamma_2}\left|b_{1,m}\right|\left|b_{2,m}\right|}{1-\frac{1+\gamma_1}{1-\gamma_1}\frac{1+\gamma_2}{1-\gamma_2}b_{1,1}b_{2,1}}.$$

It is sufficient to determine γ^* such that

$$\frac{\frac{m-\gamma^*}{1-\gamma^*}}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}} \leq \frac{\frac{m-\gamma_1}{1-\gamma_1}\frac{m-\gamma_2}{1-\gamma_2}}{1-\frac{1+\gamma_1}{1-\gamma_1}\frac{1+\gamma_2}{1-\gamma_2}b_{1,1}b_{2,1}}$$

or equivalently

$$\gamma_1^* = \gamma^* \leq \frac{(m-\gamma_1)(m-\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m-\gamma_1)(m-\gamma_2)(1+b_{1,1}b_{2,1}) - \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]} \tag{18}$$

and

$$\frac{\frac{m+\gamma^*}{1-\gamma^*}}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}} \leq \frac{\frac{m+\gamma_1}{1-\gamma_1}\frac{m+\gamma_2}{1-\gamma_2}}{1-\frac{1+\gamma_1}{1-\gamma_1}\frac{1+\gamma_2}{1-\gamma_2}b_{1,1}b_{2,1}}$$

or equivalently

$$\gamma_2^* = \gamma^* \leq \frac{(m+\gamma_1)(m+\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m+\gamma_1)(m+\gamma_2)(1+b_{1,1}b_{2,1}) + \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}. \tag{19}$$

$$\begin{aligned} b_{1,1} < \frac{1-\gamma_1}{1+\gamma_1}, b_{2,1} < \frac{1-\gamma_2}{1+\gamma_2} &\Leftrightarrow b_{1,1}b_{2,1} < \frac{1-\gamma_1}{1+\gamma_1}\frac{1-\gamma_2}{1+\gamma_2} \\ &\Leftrightarrow (1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1} > 0 \\ &\Leftrightarrow (1-b_{1,1}b_{2,1})(1+\gamma_1\gamma_2)-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) > 0 \end{aligned} \tag{20}$$

From (18) and (19) we choose the smaller one:
1. If

$$1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) > 0 \tag{21}$$

then $\gamma_1^* > \gamma_2^*$ or

$$\frac{(m-\gamma_1)(m-\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m-\gamma_1)(m-\gamma_2)(1+b_{1,1}b_{2,1}) - \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$$

$$> \frac{(m+\gamma_1)(m+\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m+\gamma_1)(m+\gamma_2)(1+b_{1,1}b_{2,1}) + \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$$

or equivalently

$$m^2\left[1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)\right] - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 > 0.$$

We substitute $m = 2$, the smallest value and we make the calculations, we get:

$$(1-b_{1,1}b_{2,1})(2-\gamma_1\gamma_2) - 2(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) > 0 \tag{22}$$

which is true, because if we add (20) with (22) and we divide with 3, we get the (21) condition.

Let us consider the function $E:[2;\infty)\to\mathbb{R}$

$$E(x)=\frac{(x+\gamma_1)(x+\gamma_2)(1-b_{1,1}b_{2,1})-x\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(x+\gamma_1)(x+\gamma_2)(1+b_{1,1}b_{2,1})+\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}.$$

Then its derivative is:

$$E'(x)=\frac{\Delta\left[(1+b_{1,1}b_{2,1})x^2+(1-b_{1,1}b_{2,1})(2x-1)+(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2+\gamma_1\gamma_2)\right]}{\left\{(x+\gamma_1)(x+\gamma_2)(1+b_{1,1}b_{2,1})+\Delta\right\}^2}>0,$$

where $\Delta=\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]$.
$E(x)$ is an increasing function. In our case we need $\gamma^*\le E(m),\forall m\ge 2$ and for this reason we choose

$$\gamma^*=E(2)$$
$$=\frac{(2+\gamma_1)(2+\gamma_2)(1-b_{1,1}b_{2,1})-2\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2+\gamma_1)(2+\gamma_2)(1+b_{1,1}b_{2,1})+\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}.$$

2. If

$$1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)<0 \tag{23}$$

then $\gamma_1^*<\gamma_2^*$ or

$$\frac{(m-\gamma_1)(m-\gamma_2)(1-b_{1,1}b_{2,1})-m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m-\gamma_1)(m-\gamma_2)(1+b_{1,1}b_{2,1})-\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$$
$$<\frac{(m+\gamma_1)(m+\gamma_2)(1-b_{1,1}b_{2,1})-m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m+\gamma_1)(m+\gamma_2)(1+b_{1,1}b_{2,1})+\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$$

or equivalently

$$m^2\left[1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)\right]-m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]$$
$$+(1-b_{1,1}b_{2,1})\gamma_1\gamma_2<0.$$

We substitute $m=2$, the smallest value and we make the calculations, we get:

$$(1-b_{1,1}b_{2,1})(2-\gamma_1\gamma_2)-2(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)<0 \tag{24}$$

which is true, because if we multiply (23) with 2 and add with $-\gamma_1\gamma_2(1-b_{1,1}b_{2,1})<0$, we get the (24).

Let us consider the function $E:[2;\infty)\to\mathbb{R}$

$$E(x)=\frac{(x-\gamma_1)(x-\gamma_2)(1-b_{1,1}b_{2,1})-x\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(x-\gamma_1)(x-\gamma_2)(1+b_{1,1}b_{2,1})-\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}.$$

Then its derivative is:

$$E'(x)=\frac{\Delta\left[(1+b_{1,1}b_{2,1})(x-1)^2+2b_{1,1}b_{2,1}(x-\gamma_1\gamma_2-\gamma_1-\gamma_2)+2b_{1,1}b_{2,1}(x-1)\right]}{\left\{(x-\gamma_1)(x-\gamma_2)(1+b_{1,1}b_{2,1})-\Delta\right\}^2}>0,$$

where

$$\Delta=\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right].$$

$E(x)$ is an increasing function. In our case we need $\gamma^*\le E(m),\forall m\ge 2$ and for this reason we choose

$$\gamma^*=E(2)=\frac{(2-\gamma_1)(2-\gamma_2)(1-b_{1,1}b_{2,1})-2\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2-\gamma_1)(2-\gamma_2)(1+b_{1,1}b_{2,1})-\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}.$$

3. If

$$1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)=0$$

or

$$1 - b_{1,1}b_{2,1} = (1 + b_{1,1}b_{2,1})(\gamma_1 + \gamma_2) \tag{25}$$

then $\gamma_1^* = \gamma_2^*$ or

$$\frac{(m-\gamma_1)(m-\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m-\gamma_1)(m-\gamma_2)(1+b_{1,1}b_{2,1}) - \left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$$

$$= \frac{(m+\gamma_1)(m+\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m+\gamma_1)(m+\gamma_2)(1+b_{1,1}b_{2,1}) + \left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$$

or equivalently

$$m^2\left[1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)\right] - m\left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0.$$

If we use (25) we get

$$-m\left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0.$$

$$-m\left[(1-b_{1,1}b_{2,1})(1+\gamma_1\gamma_2) - (1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0$$

$$\Leftrightarrow -m\left[(1-b_{1,1}b_{2,1})(1+\gamma_1\gamma_2) - (1-b_{1,1}b_{2,1})\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0$$

$$\Leftrightarrow -m\left[(1-b_{1,1}b_{2,1})\gamma_1\gamma_2\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0$$

$$\Leftrightarrow (1-m)\left[(1-b_{1,1}b_{2,1})\gamma_1\gamma_2\right] = 0$$

$$\Leftrightarrow m = 1(\text{false}) \text{ or } b_{1,1} = \frac{1}{b_{2,1}} \text{ or } \gamma_1 = 0 \text{ or } \gamma_2 = 0.$$

If we put $b_{1,1} = \dfrac{1}{b_{2,1}}$ in (25) we get $\gamma_2 = -\gamma_1$. If we substitute this in (18) and (19) we get $\gamma_1^* = \gamma_2^* = 0$. □

References

[1] Al-Kharsani, H.A., Al-Khai, R.A., *Univalent harmonic functions,* J. Inequal. Pure and Appl. Maths., **8**, no. 2, Article 59.

[2] Bălăeţi, C.M., *An integral operator associated with differential superordinations,* An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat., **17**(2009), no. 3, 37-44.

[3] Bednarz, U., Sokol, J., *On order of convolution consistence of the analytic functions,* Stud. Univ. Babeş-Bolyai Math., **55**(2010), no. 3.

[4] Carlson, B.C., Shaffer, S.B., *Starlike and prestarlike hypergeometric functions,* SIAM J. Math. Anal., **15**(2002), 737-745.

[5] Clunie, J., Sheil-Small, T., *Harmonic univalent functions,* Ann. Acad. Sci. Fenn. Ser. A.I. Math., **9**(1984) 3-25.

[6] Jahangiri, J.M., Murugusundaramoorthy, G., Vijaya, K., *Sălăgean-type harmonic univalent functions,* Southwest. J. Pure and Appl., Math., **2**(2002), 77-82.

[7] Jahangiri, J.M., Murugusundaramoorthy, G., Vijaya, K., *Starlikeness of Rucheweyh type harmonic univalent functions,* J. Indian Acad. Math., **26**(2004), 191-200.

[8] Jahangiri, J.M., Silverman, H., *Harmonic univalent functions with varying arguments,* Int. J. Appl. Math., **8**(2002), 267-275.

[9] Murugusundaramoorthy, G., *A class of Ruscheweyh-type harmonic univalent functions with varying arguments,* Southwest J. Pure Appl. Math., **2**(2003), 90-95.

[10] Murugusundaramoorthy, G., Sălăgean, G.S., *On a certain class of harmonic functions associated with a convolution structure,* Mathematica, **54**(77)(2012), 131-142.

[11] Murugusundaramoorthy, G., Vijaya, K., *On certain subclasses of harmonic functions associated with Wright hypergeometric functions,* Adv. Stud. in Contemp. Math., **1**(2009).

[12] Ruscheweyh, S., *New criteria for univalent functions,* Proc. Amer. Math. Soc., **49**(1975), 109-115.

[13] Sălăgean, G.S., *Subclasses of univalent functions,* Complex Analysis – Fifth Romanian – Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., 1013, Springer, Berlin, (1983), 362-372.

[14] Sălăgean, G.S., Venter, A., *On the order of convolution consistence of the analytic functions with negative coefficients,* Math. Bohem., **142**(2017), no. 4, 381-386.

[15] Srivastava, H.M., Owa, S., *Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions,* Nagoya Math. J., **106**(1987), 1-28.

[16] Wright, E.M., *The asymptotic expansion of the generalized hypergeometric function,* Proc. Lond. Math. Soc., **46**(1946), 389-408.

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