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Datko criteria for uniform instability in Banach spaces

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Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th anniversary.

Abstract. The main objective of this paper is to present some necessary and sufficient conditions of Datko type for the uniform exponential and uniform polynomial instability concepts for evolution operators in Banach spaces.

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1. Introduction

The instability behavior of evolution operators is a topic that has witnessed a significant progress in recent years. The importance of the role played by this concept in the theory of dynamical systems is illustrated by the appearance of various papers in this domain for the exponential case ([3], [8], [11], [12], [14]) as well as for the polynomial case ([1], [2], [13]), which appeared due to the fact that in some situation the exponential behavior is too restrictive for the dynamics.

Another direction for the study of the instability behavior is given by M. Megan, A.L. Sasu, B. Sasu in [9] where the authors express the uniform exponential instability of evolution families in terms of Banach function spaces. The property of exponential instability is generalized by M. Megan, C. Stoica [10] for skew-evolution semiflows defined by means of evolution semiflows and evolution cocycles. Recently, P.V. Hai [6] in his paper obtains results from the same perspective of using Banach spaces of sequences for the polynomial instability concept.

In this work we deal with both exponential and polynomial instability behavior for the uniform case of evolution operators in Banach spaces. In this sense, we give some necessary and sufficient conditions due to Datko [5], firstly for the uniform exponential instability concept and then we extend the theory to the polynomial case, our theorems being proved by using different techniques from those known so far.

2. Notations and definitions

We consider X a real or complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X and I the identity operator on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|.\|$. We also denote by

$$\Delta = \{(t,s) \in \mathbb{R}^2_+ : t \geq s\}, \qquad T = \{(t,s,t_0) \in \mathbb{R}^3_+ : t \geq s \geq t_0\}.$$

Definition 2.1. An application $U : \Delta \to \mathcal{B}(X)$ is said to be an evolution operator on X if the following relations are satisfied:

(e_1) $U(t,t) = I$ for all $t \geq 0$

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s,t_0) \in T$.

Definition 2.2. An evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be *strongly measurable* if for all $(s,x) \in \mathbb{R}_+ \times X$, the mapping $t \mapsto \|U(t,s)x\|$ is measurable on $[s,\infty)$.

Definition 2.3. The evolution operator $U : \Delta \to \mathcal{B}(X)$ *has uniform exponential decay (u.e.d.)* if there exist the constants $M \geq 1$ and $\omega > 0$ such that:

$$\|U(s,t_0)x_0\| \leq Me^{\omega(t-s)}\|U(t,t_0)x_0\|, \text{ for all } (t,s,t_0,x_0) \in T \times X.$$

Remark 2.4. The evolution operator $U : \Delta \to \mathcal{B}(X)$ has uniform exponential decay if and only if there exist the constants $M \geq 1$ and $\omega > 0$ such that:

$$\|x\| \leq Me^{\omega(t-s)}\|U(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Definition 2.5. The evolution operator U is said to be *uniformly exponentially instable (u.e.is.)* if there exist $N \geq 1$ and $\nu > 0$ such that:

$$\|U(s,t_0)x_0\| \leq Ne^{-\nu(t-s)}\|U(t,t_0)x_0\|, \text{ for all } (t,s,t_0,x_0) \in T \times X.$$

Remark 2.6. The evolution operator U is uniformly exponentially instable if and only if there exist $N \geq 1$ and $\nu > 0$ such that:

$$\|x\| \leq Ne^{-\nu(t-s)}\|U(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Definition 2.7. The evolution operator U has *uniform polynomial decay (u.p.d.)* if there exist the constants $M \geq 1$ and $\omega > 0$ such that:

$$\|U(s,t_0)x_0\| \leq M\left(\frac{t+1}{s+1}\right)^{\omega}\|U(t,t_0)x_0\|, \text{ for all } (t,s,t_0,x_0) \in T \times X.$$

Remark 2.8. The evolution operator U has uniform polynomial decay if and only if there exist the constants $M \geq 1$ and $\omega > 0$ such that:

$$\|x\| \leq M\left(\frac{t+1}{s+1}\right)^{\omega}\|U(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Definition 2.9. The evolution operator U is said to be *uniformly polynomially instable (u.p.is.)* if there exist $N \geq 1$ and $\nu > 0$ such that:

$$\|U(s,t_0)x_0\| \leq N\left(\frac{s+1}{t+1}\right)^{\nu}\|U(t,t_0)x_0\|, \text{ for all } (t,s,t_0,x_0) \in T \times X.$$

Remark 2.10. The evolution operator U is uniformly polynomially instable if and only if there exist $N \geq 1$ and $\nu > 0$ such that:

$$\|x\| \leq N \left(\frac{s+1}{t+1}\right)^{\nu} \|U(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Remark 2.11. The connections between the instability concepts and the decay properties mentioned above are given by the following diagram:

$$\begin{array}{ccc} u.e.is. & \Rightarrow & u.p.is. \\ \Downarrow & & \Downarrow \\ u.e.d. & \Leftarrow & u.p.d. \end{array}$$

The converse implications are not true (see [7], [13]).

We define $U_1 : \Delta \to B(X),\ U_1(t,s) = U(e^t - 1, e^s - 1)$ the evolution operator associated to U.

Proposition 2.12. *The evolution operator $U : \Delta \to \mathcal{B}(X)$ has uniform polynomial decay if and only if the evolution operator $U_1 : \Delta \to \mathcal{B}(X)$ has uniform exponential decay.*

Proof. Necessity. We suppose that U has u.p.d. which implies that there exist the constants $M \geq 1$, $\omega > 0$ such that

$$\left(\frac{s+1}{t+1}\right)^{\omega} \|x\| \leq M\|U(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X,$$

which implies

$$\left(\frac{e^s}{e^t}\right)^{\omega} \|x\| \leq M\|U(e^t - 1, e^s - 1)x\|,$$

which is equivalent to

$$e^{-\omega(t-s)}\|x\| \leq M\|U(e^t - 1, e^s - 1)x\| = M\|U_1(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Then U_1 has u.e.d.
Sufficiency. We suppose that U_1 has u.e.d., which implies that there exist $M \geq 1$, $\omega > 0$ such that

$$e^{-\omega(t-s)}\|x\| \leq M\|U_1(t,s)x\| = M\|U(e^t - 1, e^s - 1)x\|, \tag{2.1}$$

for all $(t,s,x) \in \Delta \times X$.
We denote by $e^t - 1 = u,\ e^s - 1 = v$, which implies $t = \ln(1+u), s = \ln(1+v)$. Then, from relation (2.1) we obtain

$$\left(\frac{1+u}{1+v}\right)^{-\omega} \|x\| \leq M\|U(u,v)x\|, \text{ for all } (u,v,x) \in \Delta \times X,$$

which implies that U has u.p.d. □

Proposition 2.13. *The evolution operator $U : \Delta \to \mathcal{B}(X)$ is uniformly polynomially instable if and only if $U_1 : \Delta \to \mathcal{B}(X)$ is uniformly exponentially instable.*

Proof. For the necessity, we suppose that U u.p.is. Then,

$$\begin{aligned}N\|U_1(t,s)x\| &= N\|U(e^t-1,e^s-1)x\|\\ &\geq \left(\frac{e^t}{e^s}\right)^\nu \|x\|\\ &= e^{\nu(t-s)}\|x\|,\end{aligned}$$

for all $(t,s,x)\in\Delta\times X$, which implies U_1 is u.e.is.
Conversely, we suppose that U_1 is u.e.is. Then,

$$\begin{aligned}N_1\|U_1(t,s)x\| &= N_1\|U(e^t-1,e^s-1)x\|\\ &= N_1\|U(u,v)x\|\\ &\geq e^{\nu(\ln(1+u)-\ln(1+v))}\|x\|\\ &= e^{\nu\ln\frac{1+u}{1+v}}\|x\|\\ &= \left(\frac{1+u}{1+v}\right)^\nu \|x\|,\end{aligned}$$

for all $(t,s,x)\in\Delta\times X$, which implies that U is u.p.is. □

3. The main results

In this section we give some characterization theorems of Datko type for the uniform exponential instability and uniform polynomial instability for evolution operators in Banach spaces.

Theorem 3.1. *Let U be a strongly measurable evolution operator with uniform exponential decay. Then U is uniformly exponentially instable if and only if there exist the constants $D>1$ and $d\in[0,1)$ such that*

$$\int_s^\infty \frac{e^{dt}}{\|U(t,t_0)x_0\|}dt \leq D\frac{e^{ds}}{\|U(s,t_0)x_0\|},$$

for all $(s,t_0,x_0)\in\Delta\times X,\ U(s,t_0)x_0\neq 0$.

Proof. Necessity. Let $d\in(0,\nu)$. We suppose that U is u.e.is. Then,

$$\begin{aligned}\int_s^\infty \frac{e^{dt}}{\|U(t,t_0)x_0\|}dt &\leq N\int_s^\infty \frac{e^{dt}e^{-\nu(t-s)}}{\|U(s,t_0)x_0\|}dt = \frac{Ne^{\nu s}}{\|U(s,t_0)x_0\|}\int_s^\infty e^{(d-\nu)t}dt\\ &= \frac{N}{\nu-d}\cdot e^{ds}\|U(s,t_0)x_0\| \leq De^{ds}\|U(s,t_0)x_0\|,\end{aligned}$$

where $D=1=\dfrac{N}{\nu-d}$.

Sufficiency. Case 1. Let $d \in (0,1)$. For $t \geq s+1$ we obtain

$$\begin{aligned}\frac{e^{dt}}{\|U(t,t_0)x_0\|} &= \int_{t-1}^{t} \frac{e^{dt}}{\|U(t,t_0)x_0\|} d\tau = \int_{t-1}^{t} \frac{e^{dt}}{\|U(t,\tau)U(\tau,t_0)x_0\|} d\tau \\ &\leq M \int_{t-1}^{t} \frac{e^{dt}e^{\omega(t-\tau)}}{\|U(\tau,t_0)x_0\|} d\tau = M \int_{t-1}^{t} \frac{e^{d\tau} \cdot e^{(d+\omega)(t-\tau)}}{\|U(\tau,t_0)x_0\|} d\tau \\ &\leq Me^{d+\omega} \int_{s}^{\infty} \frac{e^{d\tau}}{\|U(\tau,t_0)x_0\|} d\tau \leq \frac{Ne^{ds}}{\|U(s,t_0)x_0\|},\end{aligned}$$

which is equivalent to

$$e^{d(t-s)}\|U(s,t_0)x_0\| \leq N\|U(t,t_0)x_0\|, \forall t \geq s+1, s \geq 0. \tag{3.1}$$

Let $t \in [s, s+1]$.

$$\begin{aligned}e^{d(t-s)}\|U(s,t_0)x_0\| &\leq Me^{(d+\omega)(t-s)}\|U(t,t_0)x_0\| \leq Me^{d+\omega}\|U(t,t_0)x_0\| \\ &\leq N\|U(t,t_0)x_0\|,\end{aligned}$$

which is equivalent to

$$e^{d(t-s)}\|U(s,t_0)x_0\| \leq N\|U(t,t_0)x_0\|, \forall t \in [s, s+1], s \geq 0. \tag{3.2}$$

From (3.1) and (3.2) we obtain that

$$\|U(s,t_0)x_0\| \leq Ne^{-d(t-s)}\|U(t,t_0)x_0\|, \forall (t,s,x_0) \in \Delta \times X,$$

where $N = 1 + DMe^{d+\omega}$, so U is u.p.is.
Case 2. For $d = 0$, see [3]. □

Theorem 3.2. *Let U be a strongly measurable evolution operator with uniform exponential decay. Then U is uniformly exponentially instable if and only if there exist the constants $D > 1$ and $d \in [0,1)$ such that*

$$\int_{t_0}^{t} \frac{\|U(s,t_0)x_0\|}{e^{ds}} ds \leq D\frac{\|U(t,t_0)x_0\|}{e^{dt}},$$

for all $(t,t_0,x_0) \in \Delta \times X$.

Proof. Case 1. For $d \in (0,1)$ see[12].
Case 2. For $d = 0$ see [14] . □

Theorem 3.3. *Let U be a strongly measurable evolution operator with uniform polynomial decay. Then U is uniformly polynomially instable if and only if there exist $D > 1$ and $d \in [0,1)$ such that*

$$\int_{s}^{\infty} \frac{(t+1)^{d-1}}{\|U(t,t_0)x_0\|} dt \leq \frac{D(s+1)^d}{\|U(s,t_0)x_0\|},$$

for all $(s,t_0,x_0) \in \Delta \times X$, $U(s,t_0)x_0 \neq 0$.

Proof. Case 1. Let $d \in (0,1)$. From Proposition 2.12 we have that U has u.p.d. is equivalent to U_1 has u.e.d. and from Proposition 2.13 we have that U u.p.is. is equivalent to U_1 u.e.is. which means from Theorem 3.1 that there exist $D > 1$ and $d \in [0,1)$ such that

$$\int_s^\infty \frac{e^{dt}}{\|U_1(t,t_0)x_0\|} dt \leq \frac{De^{ds}}{\|U_1(s,t_0)x_0\|},$$

that is equivalent to

$$\int_s^\infty \frac{e^{dt}}{\|U(e^t-1,e^{t_0}-1)x_0\|} dt \leq \frac{De^{ds}}{\|U(e^s-1,e^{t_0}-1)x_0\|}. \tag{3.3}$$

Using the change of variables $e^t - 1 = u$ and denoting by $v_0 = e^{t_0} - 1$, $u_0 = e^s - 1$, relation (3.3) becomes

$$\int_{u_0}^\infty \frac{e^{d\ln(u+1)}}{\|U(u,v_0)x_0\|} \cdot \frac{du}{u+1} \leq \frac{De^{d\ln(u_0+1)}}{\|U(u_0,v_0)x_0\|},$$

that is equivalent to

$$\int_{u_0}^\infty \frac{(u+1)^{d-1}}{\|U(u,v_0)x_0\|} du \leq \frac{D(u_0+1)^d}{\|U(u_0,v_0)x_0\|},$$

so the theorem is proved.
Case 2. For $d = 0$ see [4]. $\square$

Theorem 3.4. *Let U be a strongly measurable evolution operator with uniform polynomial decay. Then U is uniformly polynomially instable if and only if there exist $D > 1$ and $d \in [0,1)$ such that*

$$\int_{t_0}^t \frac{\|U(s,t_0)x_0\|}{(s+1)^{d+1}} ds \leq \frac{D\|U(t,t_0)x_0\|}{(t+1)^d},$$

for all $(t,t_0,x_0) \in \Delta \times X$.

Proof. Using Proposition 2.12 and Proposition 2.13 we have that U_1 is u.e.is. with u.e.d. and from Theorem 3.1 we obtain that there exist $D > 1$ and $d \in [0,1)$ such that

$$\int_{t_0}^t \frac{\|U_1(s,t_0)x_0\|}{e^{ds}} ds \leq \frac{D\|U(t,t_0)x_0\|}{e^{dt}},$$

which is equivalent to

$$\int_{t_0}^t \frac{\|U(e^s-1,e^{t_0}-1)x_0\|}{e^{ds}} ds \leq \frac{D\|U(e^t-1,e^{t_0}-1)x_0\|}{e^{dt}}. \tag{3.4}$$

Using the change of variables $e^s - 1 = u$ and denoting by $v = e^t - 1,\ v_0 = e^{t_0} - 1$, relation (3.4) becomes

$$\int_{u_0}^{v} \frac{\|U(u, v_0)x_0\|}{e^{d\ln(u+1)}} \cdot \frac{du}{u+1} \leq \frac{D\|U(v, v_0)x_0\|}{(v+1)^d},$$

that is equivalent to

$$\int_{u_0}^{v} \frac{\|U(u, v_0)x_0\|}{(u+1)^{d+1}} du \leq \frac{D\|U(v, v_0)x_0\|}{(v+1)^d},$$

so the theorem is proved. □

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