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Asymptotic behavior of inexact infinite products of nonexpansive mappings

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Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th anniversary

Abstract. We analyze the asymptotic behavior of inexact infinite products of nonexpansive mappings, which take a nonempty closed subset of a complete metric space into the space, in the case where the errors are sufficiently small.

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1. Introduction

For nearly sixty years now, there has been an intensive research activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 4, 5, 3, 12, 13, 14, 15, 16, 17, 18, 19, 21, 23, 24, 27, 28, 29, 30, 32, 34, 38, 39, 40, 41] and references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) orbits of a nonexpansive mapping to one of its fixed points. Since that seminal result, numerous developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [6, 7, 11, 35, 36, 37, 40, 41]. In this connection, see also the results regarding the asymptotic behavior of infinite products of nonexpansive mappings which have been established in, for instance, [8, 9, 10, 20, 22, 25, 26, 31] and references mentioned therein.

In [30] we collected several results which demonstrate the convergence of inexact iterates of a nonexpansive self-mapping of a complete metric space to one of its fixed points. In the present paper we establish three variants of these results for inexact infinite products of nonexpansive mappings, which take a nonempty closed subset of a complete metric space into the space, in the case where the errors are sufficiently small. Prototypes of these results for inexact orbits of nonexpansive mappings have recently been obtained in [33].

2. Main results

Let (Z, d) be a complete metric space. For each point $z \in Z$ and each positive number M, define

$$B(z, M) := \{y \in Z : \ d(z, y) \leq M\}.$$

For every point $z \in Z$ and every nonempty set $D \subset Z$, put

$$d(z, D) := \inf\{d(z, y) : \ y \in D\}.$$

Let $K \subset Z$ be a nonempty closed set, let the mappings $A_j : K \to Z$, $j = 1, 2, \ldots$, satisfy

$$d(A_j(z), A_j(y)) \leq d(z, y) \text{ for all } z, y \in K \tag{2.1}$$

for each integer $j \geq 1$, and let $\mathcal{R}$ be a nonempty collection of mappings $r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$. Fix a point $\theta \in K$. Assume that a point $z_* \in K$ satisfies

$$A_j(z_*) = z_*, \ j = 1, 2, \ldots, \tag{2.2}$$

and that the following two properties hold:

(P1) if $r \in \mathcal{R}$ and $q \geq 1$ is an integer, then the mapping $n \to r(n+q)$, $n = 1, 2 \ldots$, belongs to the collection $\mathcal{R}$;

(P2) for every positive number ε and every positive number M, there exists an integer $n(M, \varepsilon) \geq 1$ such that if $z \in B(\theta, M)$, $r \in \mathcal{R}$ and

$$\prod_{i=1}^{n(M,\varepsilon)} A_{r(i)}(z)$$

exists, then

$$d\left(\prod_{i=1}^{n(M,\varepsilon)} A_{r(i)}(z), z_*\right) \leq \varepsilon.$$

Note that property (P2) holds for Banach contractions and for many nonexpansive mappings of contractive type [30]. In [30] we consider a large class of sequences of nonexpansive mappings $\{A_i\}_{i=1}^{\infty}$ and show that a generic (typical) sequence of mappings possesses (P2).

In the present paper we establish the following three theorems.

Theorem 2.1. *Let a pair of positive numbers ε, M be given. Then there exists an integer $n_0 \geq 1$ such that for every $\delta \in (0, \varepsilon/2)$, every natural number $n \geq n_0$, every mapping $r \in \mathcal{R}$ and every sequence $\{z_i\}_{i=0}^{n} \subset K$ satisfying*

$$d(z_0, \theta) \leq M,$$

$$d(z_{j+1}, A_{r(j+1)}(z_j)) \leq (4n_0)^{-1}\delta, \ j = 0, \ldots, n-1,$$

and

$$B(z_j, \delta) \subset K, \ j = 0, \ldots, n,$$

the inequality $d(z_j, z_*) \leq \varepsilon$ is true for all integers $j = n_0, \ldots, n$.

Theorem 2.2. *Let $r_* \in (0,1)$ satisfy*

$$B(z_*, r_*) \subset K, \tag{2.3}$$

and let a pair of positive numbers M and $\varepsilon \in (0, r_*/2)$ be given. Then there exists an integer $n_0 \geq 1$ such that for every $\delta \in (0, \varepsilon/2)$, every natural number $n \geq n_0$, every mapping $r \in \mathcal{R}$ and every sequence $\{z_j\}_{j=0}^{n} \subset K$ satisfying

$$d(z_0, \theta) \leq M,$$

$$d(z_{j+1}, A_{r(j+1)}(z_j)) \leq (4n_0)^{-1}\delta,\ j = 0, \dots, n-1,$$

and

$$B(z_j, \delta) \subset K,\ j = 0, \dots, n_0,$$

the inequality $d(z_j, z_*) \leq \varepsilon$ is true for all integers $j = n_0, \dots, n$.

Note that in Theorem 2.1 the sequence $\{z_j\}_{j=0}^{n} \subset K$ satisfies the inclusion $B(z_j, \delta) \subset K$ for all $j = 0, \dots, n$, while in Theorem 2.2 the inclusion holds only for $j = 0, \dots, n_0$. On the other hand, in Theorem 2.2 we assume that z_* is an interior point of K. We do not need this assumption for Theorem 2.1.

Theorem 2.3. *Let $r_* > 0$,*

$$B(z_*, r_*) \subset K, \tag{2.4}$$

$r \in \mathcal{R}$ and let a sequence $\{z_j\}_{j=0}^{\infty} \subset K$ satisfy

$$\lim_{j\to\infty} d(z_{j+1}, A_{r(j+1)}(z_j)) = 0 \tag{2.5}$$

and have a bounded subsequence $\{z_{j_p}\}_{p=1}^{\infty}$. Assume that there exists a positive number Δ such that

$$B(z_j, \Delta) \subset K$$

for all sufficiently large natural numbers j. Then

$$\lim_{j\to\infty} z_j = z_*.$$

The proofs of these three theorems are presented in Sections 4–6 below. We begin, however, with an auxiliary result which is stated and proved in the next section.

3. An auxiliary result

Proposition 3.1. *Let $\varepsilon, M > 0$ be given. Then there exists an integer $n_0 \geq 1$ such that for every number $\delta \in (0, \varepsilon)$, every mapping $r \in \mathcal{R}$ and every sequence $\{z_j\}_{j=0}^{n_0} \subset K$ satisfying*

$$d(z_0, \theta) \leq M, \tag{3.1}$$

$$d(z_{j+1}, A_{r(j+1)}(z_j)) \leq (4n_0)^{-1}\delta,\ j = 0, \dots, n_0 - 1, \tag{3.2}$$

and

$$B(z_j, \delta) \subset K,\ j = 0, \dots, n_0, \tag{3.3}$$

the inequality $d(z_{n_0}, z_*) \leq \varepsilon$ holds true.

Proof. In view of property (P2), there exists an integer $n_0 \geq 1$ such that the following property holds:

(P3) for every point $z \in B(\theta, M)$ and every mapping $r \in \mathcal{R}$ for which $\prod_{j=1}^{n_0} A_{r(j)}(z)$ exists, we have

$$d\left(\prod_{j=1}^{n_0} A_{r(j)}(z), z_*\right) \leq \varepsilon/4.$$

Fix $\delta \in (0, \varepsilon)$ and put

$$\delta_0 := (4n_0)^{-1}\delta. \tag{3.4}$$

Assume that $r \in \mathcal{R}$ and that a sequence of points $\{z_j\}_{j=0}^{n_0} \subset K$ satisfies relations (3.1)–(3.3). Define

$$y_0 := z_0,\ y_1 := A_{r(1)}(z_0). \tag{3.5}$$

By (3.2), (3.4) and (3.5), we have

$$d(y_1, z_1) \leq \delta_0. \tag{3.6}$$

It follows from (3.3), (3.4) and (3.6) that

$$B(z_1, \delta) \subset K,\ y_1 \in K \tag{3.7}$$

and

$$B(y_1, \delta - \delta_0) \subset B(z_1, \delta) \subset K. \tag{3.8}$$

Assume that $1 \leq p < n_0$ is an integer and that a sequence of points $\{y_i\}_{i=0}^{p} \subset K$ satisfies

$$y_0 = z_0, \tag{3.9}$$

$$y_{j+1} = A_{r(j+1)}(y_j),\ j = 0, \ldots, p-1, \tag{3.10}$$

and

$$d(y_j, z_j) \leq j\delta_0,\ j = 0, \ldots, p. \tag{3.11}$$

(It is clear that by relations (3.5)–(3.8), our assumption is valid for $p = 1$.) We claim that our assumption is true for $p+1$ too. Indeed, in view of (3.3), (3.4) and (3.11), we have $y_p \in K$ and

$$d(z_p, y_p) \leq p\delta_0. \tag{3.12}$$

Relations (3.4), (3.11) and (3.12) imply that

$$B(y_p, \delta - p\delta_0) \subset B(z_p, \delta) \subset K. \tag{3.13}$$

By (3.13),

$$y_{p+1} = A_{r(p+1)}(y_p) \tag{3.14}$$

is well defined. It now follows from (2.1), (3.2), (3.4), (3.12) and (3.14) that

$$\begin{aligned} d(y_{p+1}, z_{p+1}) &\leq d(y_{p+1}, A_{r(p+1)}(z_p)) + d(A_{r(p+1)}(z_p), z_{p+1}) \\ &\leq d(A_{r(p+1)}(y_p), A_{r(p+1)}(z_p)) + d(A_{r(p+1)}(z_p), z_{p+1}) \\ &\leq d(y_p, z_p) + \delta_0 \leq (p+1)\delta_0. \end{aligned}$$

Thus the assumption made regarding p also holds for $p+1$, as claimed (see (3.9)-(3.11)). This means that we have shown by using induction that our assumption holds for $p = n_0$. Hence there exists a sequence of points $\{y_j\}_{j=0}^{n_0} \subset K$ which satisfies

$$y_0 = z_0, \tag{3.15}$$

$$y_{j+1} = A_{r(j+1)}(y_j),\ j = 0, \dots, n_0 - 1, \tag{3.16}$$

and

$$d(y_j, z_j) \le j\delta_0,\ j = 0, \dots, n_0. \tag{3.17}$$

By (3.15) and (3.16),

$$y_{n_0} = \prod_{j=1}^{n_0} A_{r(j)}(x_0).$$

It follows from property (P3), (3.1) and (3.15)–(3.17) that

$$d(y_{n_0}, z_*) \le \varepsilon/4.$$

When combined with (3.4) and (3.17), this implies that

$$d(z_{n_0}, z_*) \le d(z_{n_0}, y_{n_0}) + d(y_{n_0}, z_*) \le n_0\delta_0 + \varepsilon/4 < \varepsilon.$$

This completes the proof of Proposition 3.1. □

4. Proof of Theorem 2.1

We may assume that

$$M > d(\theta, z_*) + 1 + \varepsilon. \tag{4.1}$$

In view of Proposition 3.1, there exists an integer $n_0 \ge 1$ such that the following property holds:

(P4) for every number $\delta \in (0, \varepsilon/2)$, every mapping $r \in \mathcal{R}$ and every sequence $\{x_j\}_{j=0}^{n_0} \subset K$ satisfying

$$d(x_0, \theta) \le M,$$

$$d(x_{j+1}, A_{r(j+1)}(x_j)) \le (4n_0)^{-1}\delta,\ j = 0, \dots, n_0 - 1,$$

and

$$B(x_j, \delta) \subset K,\ j = 0, \dots, n_0,$$

the inequality

$$d(x_{n_0}, z_*) \le \varepsilon/2$$

holds true.

Fix $\delta \in (0, \varepsilon/2)$. Assume that $n \ge n_0$ is an integer, $r \in \mathcal{R}$ and that a sequence $\{z_j\}_{j=0}^{n} \subset K$ satisfies

$$d(z_0, \theta) \le M, \tag{4.2}$$

$$d(z_{j+1}, A_{r(j+1)}(z_j)) \le (4n_0)^{-1}\delta,\ j = 0, \dots, n - 1, \tag{4.3}$$

and

$$B(z_j, \delta) \subset K,\ j = 0, \dots, n. \tag{4.4}$$

By (P4), the choice of n_0 and relations (4.2)–(4.4), we have

$$d(z_{n_0}, z_*) \le \varepsilon/2. \tag{4.5}$$

In order to complete the proof of the theorem, it suffices to show that

$$d(z_j, z_*) \le \varepsilon,\ j = n_0, \dots, n.$$

To this end, it is sufficient to consider the case where $n > n_0$. Relations (2.1), (2.2), (4.3) and (4.5) imply that

$$\begin{aligned} d(z_{n_0+1}, z_*) &\le d(z_{n_0+1}, A_{r(n_0+1)}(z_{n_0})) + d(A_{r(n_0+1)}(z_{n_0}), z_*) \\ &\le \delta(4n_0)^{-1} + \varepsilon/2. \end{aligned} \tag{4.6}$$

We claim that for each integer $j \in \{n_0 + 1, \dots, n\}$, we have

$$d(z_j, z_*) \le \varepsilon/2 + \delta(j - n_0)(4n_0)^{-1}. \tag{4.7}$$

By (4.6), relation (4.7) is indeed true for $i = n_0 + 1$.
Assume that $p \in \{n_0 + 1, \dots, n\} \setminus \{n\}$ and that

$$d(z_p, z_*) \le \varepsilon/2 + \delta(p - n_0)(4n_0)^{-1}. \tag{4.8}$$

Relations (2.1), (2.2), (4.3) and (4.8) imply that

$$\begin{aligned} d(z_{p+1}, z_*) &\le d(z_{p+1}, A_{r(p+1)}(z_p)) + d(A_{r(p+1)}(z_p), z_*) \\ &\le \delta(4n_0)^{-1} + d(z_p, z_*) \\ &\le \varepsilon/2 + \delta(p - n_0)(4n_0)^{-1} + \delta(4n_0)^{-1} \\ &\le \varepsilon/2 + \delta(p + 1 - n_0)(4n_0)^{-1}. \end{aligned}$$

Thus the assumption made regarding p also holds for $p + 1$, as claimed.
This means that we have shown by using induction that (4.7) is indeed true for all integers $j = n_0 + 1, \dots, n$.
Suppose now that there exists an integer $q \in \{n_0, \dots, n\}$ for which

$$d(z_q, z_*) > \varepsilon. \tag{4.9}$$

By (4.5) and (4.9), we have

$$q > n_0.$$

In view of (4.7) and (4.9),

$$\varepsilon < d(z_q, z_*) \le \varepsilon/2 + \delta(q - n_0)(4n_0)^{-1},$$

$$\varepsilon/2 < \delta(q - n_0)(4n_0)^{-1} < (\varepsilon/2)(q - n_0)(4n_0)^{-1},$$

$$q - n_0 > 4n_0$$

and

$$q > 5n_0. \tag{4.10}$$

By (4.5) and (4.10), we may assume without any loss of generality that

$$d(z_j, z_*) \le \varepsilon,\ j = n_0, \dots, q - 1. \tag{4.11}$$

Define

$$x_j := z_{j+q-n_0},\ j = 0, 1, \dots, n_0, \tag{4.12}$$

$$\tilde{r}(j) := r(j + q - n_0),\ j = 1, 2, \dots. \tag{4.13}$$

Property (P1) implies that $\tilde{r} \in \mathcal{R}$. It follows from (4.3), (4.12) and (4.13) that for all integers $j = 0, \dots, n_0 - 1$, we have

$$d(x_{j+1}, A_{\tilde{r}(j+1)}(x_j)) = d(z_{j+1+q-n_0}, A_{r(j+1+q-n_0)}(z_{j+q-n_0})) \leq (4n_0)^{-1}\delta.$$

Property (P4), applied to the sequence x_j, $j = 0, \dots, n_0$, (4.1), (4.11), (4.12) and (4.14) imply that

$$d(z_*, z_q) = d(z_*, x_{n_0}) \leq \varepsilon/2.$$

This, however, contradicts (4.9). The contradiction we have reached completes the proof of Theorem 2.1.

5. Proof of Theorem 2.2

We may assume that

$$M > d(\theta, z_*) + 1 + \varepsilon. \tag{5.1}$$

Recall that $\varepsilon < r_*/2$ (see (2.3)). Proposition 3.1 implies that there exists a natural number n_0 such that property (P4), which was introduced in the previous section, holds. We recall it at this point for the convenience of the reader:

(P4) for every number $\delta \in (0, \varepsilon/2)$, every $r \in \mathcal{R}$ and every sequence of points $\{x_j\}_{j=0}^{n_0} \subset K$ which satisfies

$$d(x_0, \theta) \leq M,$$
$$d(x_{j+1}, A_{r(j+1)}(x_j)) \leq (4n_0)^{-1}\delta, \ j = 0, \dots, n_0 - 1,$$

and

$$B(x_j, \delta) \subset K, \ j = 0, \dots, n_0,$$

the inequality

$$d(x_{n_0}, z_*) \leq \varepsilon/2$$

is true.

Let $\delta \in (0, \varepsilon/2)$ be given. Assume that $r \in \mathcal{R}$, $n \geq n_0$ is an integer and that a sequence $\{z_j\}_{j=0}^{n} \subset K$ satisfies

$$d(z_0, \theta) \leq M, \tag{5.2}$$
$$d(z_{j+1}, A_{r(j+1)}(z_j)) \leq (4n_0)^{-1}\delta, \ j = 0, \dots, n - 1, \tag{5.3}$$

and

$$B(z_j, \delta) \subset K, \ j = 0, \dots, n_0. \tag{5.4}$$

Property (P4) and relations (5.2)–(5.4) imply that

$$d(z_{n_0}, z_*) \leq \varepsilon/2. \tag{5.5}$$

In order to complete the proof of the theorem, it is sufficient to show that

$$d(z_j, z_*) \leq \varepsilon, \ j = n_0, \dots, n.$$

Suppose to the contrary that these inequalities are not valid. Then there exists a natural number $q \in \{n_0, \dots, n\}$ for which

$$d(z_q, z_*) > \varepsilon. \tag{5.6}$$

Inequalities (5.5) and (5.6) imply that

$$q > n_0. \tag{5.7}$$

In view of (5.7), we may assume without loss of generality that

$$d(z_j, z_*) \leq \varepsilon,\ j = n_0, \ldots, q-1. \tag{5.8}$$

Using induction and arguing as in the proof of Theorem 2.1, we can show that for all natural numbers $i = n_0 + 1, \ldots, n$, we have

$$d(z_i, z_*) \leq \varepsilon/2 + \delta(i - n_0)(4n_0)^{-1}. \tag{5.9}$$

In view of (5.6) and (5.9),

$$\varepsilon < d(z_q, z_*) \leq \varepsilon/2 + \delta(q - n_0)(4n_0)^{-1},$$

$$\varepsilon/2 < \delta(q - n_0)(4n_0)^{-1} < (\varepsilon/2)(q - n_0)(4n_0)^{-1},$$

$$q - n_0 > 4n_0$$

and

$$q > 5n_0. \tag{5.10}$$

Put

$$x_j := z_{j+q-n_0},\ j = 0, \ldots, n_0, \tag{5.11}$$

$$\tilde{r}(j) := r(j + q - n_0),\ j = 1, 2, \ldots. \tag{5.12}$$

It follows from property (P1) that $\tilde{r} \in \mathcal{R}$. In view of (5.3), (5.11) and (5.12), we have, for every integer $j \in \{0, 1, \ldots, n_0 - 1\}$,

$$d(x_{j+1}, A_{\tilde{r}(j+1)}(x_j)) = d(z_{j+1+q-n_0}, A_{r(j+1+q-n_0)}(z_{j+q-n_0})) \leq (4n_0)^{-1}\delta. \tag{5.13}$$

It follows from property (P4), (5.1), (5.8) and (5.10)–(5.13) that

$$d(z_*, z_q) = d(z_*, x_{n_0}) \leq \varepsilon/2.$$

This, however, contradicts (5.6). The contradiction we have reached completes the proof of Theorem 2.2.

6. Proof of Theorem 2.3

We may assume without any loss of generality that

$$B(z_j, \Delta) \subset K,\ j = 0, 1, \ldots. \tag{6.1}$$

There exists a number

$$M > d(\theta, z_*) + 1$$

for which

$$d(z_{i_p}, \theta) \leq M,\ p = 1, 2, \ldots. \tag{6.2}$$

Fix

$$\varepsilon \in (0, r_*/4) \tag{6.3}$$

(see (2.4)). Proposition 3.1 implies that there exists an integer $n_0 \geq 1$ for which property (P4), which was introduced in Section 4, holds. We recall it now for the convenience of the reader:

(P4) for every number $\delta \in (0, \varepsilon/2)$, every mapping $r \in \mathcal{R}$ and every sequence $\{x_j\}_{j=0}^{n_0} \subset K$ which satisfies

$$d(x_0, \theta) \leq M,$$

$$d(x_{j+1}, A_{r(j+1)}(x_j)) \leq (4n_0)^{-1}\delta,\ j = 0, \ldots, n_0 - 1,$$

and

$$B(x_j, \delta) \subset K,\ j = 0, \dots, n_0,$$

the inequality

$$d(x_{n_0}, x_*) \le \varepsilon/2$$

is true.

Now let

$$\delta := \min\{\varepsilon/4,\ \Delta\}. \tag{6.4}$$

By (6.3) and (6.4),

$$\delta \le 16^{-1} r_*.$$

Relations (2.5) and (6.2) imply that there exists an integer $p_0 \ge 1$ such that

$$d(z_{i_{p_0}}, \theta) \le M \tag{6.5}$$

and

$$d(z_{i+1}, A_{r(i+1)}(z_i)) \le (4n_0)^{-1}\delta \text{ for all integers } i \ge i_{p_0}. \tag{6.6}$$

For all integers $j = 0, \dots, n_0$, define

$$x_j := z_{j+p_0}, \tag{6.7}$$

$$\tilde{r}(i) := r(i + i_{p_0}),\ i = 1, 2, \dots. \tag{6.8}$$

It follows from relations (6.6)–(6.8) that for each integer $j \in \{0, 1, \dots, n_0 - 1\}$, we have

$$d(x_{j+1}, A_{\tilde{r}(j+1)}(x_j)) = d(z_{j+1+i_{p_0}}, A_{r(j+1+i_{p_0})}(z_{j+i_{p_0}})) \le (4n_0)^{-1}\delta. \tag{6.9}$$

Property (P4) and relations (6.5), (6.7) and (6.9) imply that

$$\varepsilon/2 \ge d(x_{n_0}, z_*) = d(z_{i_{p_0}+n_0}, z_*).$$

In order to complete the proof of the theorem, it is sufficient to show that

$$d(z_j, z_*) \le \varepsilon \text{ for all integers } j \ge i_{p_0} + n_0.$$

Suppose to the contrary that this is not true. Then there exists a natural number $q > i_{p_0} + n_0$ such that

$$d(z_q, z_*) > \varepsilon. \tag{6.10}$$

We may assume without any loss of generality that

$$d(z_j, z_*) \le \varepsilon,\ j = i_{p_0} + n_0, \dots, q - 1. \tag{6.11}$$

Using induction and arguing as in the proof of Theorem 2.1, we can show that for each integer $i \ge i_{p_0} + n_0$, we have

$$d(z_i, z_*) \le \varepsilon/2 + \delta(i - i_{p_0} - n_0)(4n_0)^{-1}. \tag{6.12}$$

Inequalities (6.10) and (6.11) imply that

$$\varepsilon < d(z_q, z_*) \le \varepsilon/2 + \delta(q - i_{p_0} - n_0)(4n_0)^{-1},$$

$$\varepsilon/2 \le (\varepsilon/2)(q - i_{p_0} - n_0)(4n_0)^{-1},$$

$$q - i_{p_0} - n_0 \ge 4n_0$$

and

$$q \ge i_{p_0} + 5n_0. \tag{6.13}$$

In view of (6.11) and (6.13), we have

$$d(z_{q-n_0}, z_*) \le \varepsilon. \tag{6.14}$$

By (6.14), we have

$$d(z_{q-n_0}, \theta) \le M. \tag{6.15}$$

For all integers $j = 0, \dots, n_0$, put

$$x_j := z_{q-n_0+j}, \tag{6.16}$$

$$\tilde{r}(j) := r(q - n_0 + j),\ j = 1, 2, \dots. \tag{6.17}$$

It follows from property (P4), (6.1), (6.6), (6.13) and (6.15)–(6.17) that

$$\varepsilon/2 \ge d(z_*, x_{n_0}) = d(z_*, z_q).$$

This, however, contradicts (6.10). The contradiction we have reached completes the proof of Theorem 2.3.

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