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Fuzzy differential subordinations connected with convolution

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Abstract. The object of the present paper is to obtain several fuzzy differential subordinations associated with Linear operator

$$\mathcal{D}^{m}_{n,\delta,g}f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)\, c^{n}(\delta)\right]^{m} a_j b_j z^j .$$

Using the operator $\mathcal{D}^{m}_{n,\delta,g}$, we also introduce a class $\mathcal{H}^{F}_{n,m,\delta}(\eta, g)$ of univalent analytic functions for which we give some properties.

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1. Introduction

Let $\Omega \subset \mathbb{C}$, $H(\Omega)$ the class of holomorphic functions on Ω and denote by $H_d(\Omega)$ the class of holomorphic and univalent functions on Ω. In this paper, we denote by $H(\Delta)$ the class of holomorphic functions in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with $B_\Delta = \{z \in \mathbb{C} : |z| = 1\}$ the boundary of the unit disk. For $\beta \in \mathbb{C}$ and $d \in \mathbb{N}$, we denote

$$H[\beta, d] = \left\{ f \in H(\Delta) : \; f(z) = \beta + \sum_{j=d+1}^{\infty} a_j z^j, \quad z \in \Delta \right\},$$

$$\mathbb{A}_d = \left\{ f \in H(\Delta) : \; f(z) = z + \sum_{j=d+1}^{\infty} a_j z^j, \quad z \in \Delta \right\} \quad \text{with} \quad \mathbb{A}_1 = \mathbb{A},$$

and

$$\mathcal{S} = \{ f \in \mathbb{A} : \; f \text{ is a univalent function in } \Delta \} .$$

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We denote by

$$\mathcal{C}=\left\{f\in\mathbb{A}:\ \Re\left(1+\frac{zf^{''}(z)}{f^{'}(z)}\right)>0,\ z\in\Delta\right\},$$

the set of convex functions in Δ.

Definition 1.1. [4, 11] Let f_1 and f_2 are analytic function in Δ, then f_1 is subordinate to f_2, written $f_1 \prec f_2$ if there exists a Schwarz function w, which is analytic in Δ with $w(0)=0$ and $|w(z)|<1$ for all $z\in\Delta$, such that $f_1(z)=f_2(w(z))$. Furthermore, if the function f_2 is univalent in Δ, then we have the following equivalence:

$$f_1(z)\prec f_2(z)\Leftrightarrow f_1(0)=f_2(0)\text{ and }f_1(\Delta)\subset f_2(\Delta).$$

In order to introduce the notion of fuzzy differential subordination, we use the following definitions and propositions:

Definition 1.2. [10] Fuzzy subset of $\mathcal{Y}$ is a pair $(\mathcal{B},\mathcal{F}_{\mathcal{B}})$, with $\mathcal{F}_{\mathcal{B}}:\mathcal{Y}\to[0,1]$ and

$$\mathcal{B}=\{x\in\mathcal{Y}:0<\mathcal{F}_{\mathcal{B}}(x)\leq 1\}. \tag{1.1}$$

The support of the fuzzy set $(\mathcal{B},\mathcal{F}_{\mathcal{B}})$ is the set $\mathcal{B}$ and the membership function of $(\mathcal{B},\mathcal{F}_{\mathcal{B}})$ is $\mathcal{F}_{\mathcal{B}}$.

Proposition 1.3. [12] *(i) If $(\mathcal{B},\mathcal{F}_{\mathcal{B}})=(\mathcal{U},\mathcal{F}_{\mathcal{U}})$, then we have $\mathcal{B}=\mathcal{U}$, where*

$$\mathcal{B}=\sup(\mathcal{B},\mathcal{F}_{\mathcal{B}})\ \text{ and } \mathcal{U}=\sup(\mathcal{U},\mathcal{F}_{\mathcal{U}});$$

(ii) If $(\mathcal{B},\mathcal{F}_{\mathcal{B}})\subseteq(\mathcal{U},\mathcal{F}_{\mathcal{U}})$, then we have $\mathcal{B}\subseteq\mathcal{U}$, where

$$\mathcal{B}=\sup(\mathcal{B},\mathcal{F}_{\mathcal{B}})\ \text{ and } \mathcal{U}=\sup(\mathcal{U},\mathcal{F}_{\mathcal{U}}).$$

Let $f,g\in H(\Omega)$, we denote by

$$f(\Omega)=\left\{f(z):\ 0<\mathcal{F}_{f(\Omega)}f(z)\leq 1,\ z\in\Omega\right\}=\sup\left(f(\Omega),\mathcal{F}_{f(\Omega)}\right), \tag{1.2}$$

and

$$g(\Omega)=\left\{g(z):\ 0<\mathcal{F}_{g(\Omega)}g(z)\leq 1,\ z\in\Omega\right\}=\sup\left(g(\Omega),\mathcal{F}_{g(\Omega)}\right). \tag{1.3}$$

Definition 1.4. [12] Let $z_0\in\Omega$ be a fixed point and let the functions $f,g\in H(\Omega)$. The function f is said to be fuzzy subordinate to g and write $f\prec_{\mathcal{F}} g$ or $f(z)\prec_{\mathcal{F}} g(z)$, if are satisfied the following conditions:

(*i*) $f(z_0)=g(z_0)$

(*ii*) $\mathcal{F}_{f(\Omega)}f(z)\leq\mathcal{F}_{g(\Omega)}g(z),\quad z\in\Omega.$

Proposition 1.5. [12] *Assume that $z_0\in\Omega$ is a fixed point and the functions $f,\mathrm{g}\in H(\Omega)$. If $f(z)\prec_{\mathcal{F}} g(z)$, $z\in\Omega$, then*

(*i*) $f(z_0)=g(z_0)$

(*ii*) $f(\Omega)\subseteq g(\Omega),\quad \mathcal{F}_{f(\Omega)}f(z)\leq\mathcal{F}_{\mathrm{g}(\Omega)}g(z),\ z\in\Omega,$

where $f(\Omega)$ and $\mathrm{g}(\Omega)$ are defined by (1.2) and (1.3), respectively.

Definition 1.6. [13] Assume that $\Phi:\mathbb{C}^3\times\Delta\to\mathbb{C}$ and $h\in\mathcal{S}$, with $\Phi(\alpha,0,0;0)=h(0)=\alpha$. If p is analytic in Δ, with $p(0)=\alpha$ and satisfies the second order fuzzy differential subordination

$$\mathcal{F}_{\Phi(\mathbb{C}^3\times\Delta)}\Phi\left(p(z),zp^{'}(z),z^2p^{''}(z);z\right)\leq\mathcal{F}_{h(\Delta)}h(z),$$

$$\text{i.e.} \quad \Phi\left(p(z), zp'(z), z^2p''(z); z\right) \prec_{\mathcal{F}} h(z), \quad z \in \Delta, \tag{1.4}$$

then p is said to be a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions for the fuzzy differential subordination if

$$\mathcal{F}_{p(\Delta)}p(z) \leq \mathcal{F}_{q(\Delta)}q(z), \quad \text{i.e.} \quad p(z) \prec_{\mathcal{F}} q(z), \quad z \in \Delta$$

for all p satisfying (1.4).
A fuzzy dominant $\widetilde{q}$ that satisfies

$$\mathcal{F}_{\widetilde{q}(\Delta)}\widetilde{q}(z) \leq \mathcal{F}_{q(\Delta)}q(z), \quad \text{i.e.} \quad \widetilde{q}(z) \prec_{\mathcal{F}} q(z), \quad z \in \Delta$$

for all fuzzy dominants q of (1.4) is called the fuzzy best dominant of (1.4).

Making use the binomial series

$$(1-\delta)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \ \delta^i \qquad (n \in \mathbb{N} = \{1, 2, ...\}),$$

for $f \in \mathbb{A}$, we introduced the linear differential operator as follows:

$$\mathcal{D}^0_{n,\delta,g}f(z) = (f * g)(z),$$

$$\begin{aligned}
\mathcal{D}^1_{n,\delta,g}f(z) &= \mathcal{D}_{n,\delta,g}f(z) = (1-\delta)^n (f*g)(z) + [1-(1-\delta)^n]\, z\, (f*g)'(z) \\
&= z + \sum_{j=2}^{\infty} [1 + (j-1)\, c^n(\delta)]\, a_j b_j z^j \\
&\ \vdots \\
\mathcal{D}^m_{n,\delta,g}f(z) &= \mathcal{D}_{n,\delta,g}\left(\mathcal{D}^{m-1}_{n,\delta,g}f(z)\right) \\
&= (1-\delta)^n \mathcal{D}^{m-1}_{n,\delta,g}f(z) + [1-(1-\delta)^n]\, z \left(\mathcal{D}^{m-1}_{n,\delta,g}f(z)\right)'
\end{aligned}$$

$$= z + \sum_{j=2}^{\infty} [1 + (j-1)\, c^n(\delta)]^m\, a_j b_j z^j \tag{1.5}$$

$$(\delta > 0,\ n \in \mathbb{N},\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

where

$$c^n(\delta) = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \ \delta^i \qquad (n \in \mathbb{N}).$$

From (1.5), we obtain that

$$c^n(\delta)\ z\ \left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' = \mathcal{D}^{m+1}_{n,\delta,g}f(z) - [1 - c^n(\delta)]\, \mathcal{D}^m_{n,\delta,g}f(z).$$

By specializing the parameters n, δ and b_j, we note that
(i) Putting $b_j = 1$ (or $g(z) = \frac{z}{1-z}$), then $\mathcal{D}^m_{n,\delta,\frac{z}{1-z}} = \mathcal{D}^m_{n,\delta}$ defined by Yousef et al. [17].
(ii) Putting $b_j = 1$ (or $g(z) = \frac{z}{1-z}$) and $n = 1$, then $\mathcal{D}^m_{1,\delta,\frac{z}{1-z}} = \mathcal{D}^m_{\delta}$ defined by Al-Oboudi [3].

(iii) Putting $b_j = 1$ (or $g(z) = \frac{z}{1-z}$) and $n = \delta = 1$, then $\mathcal{D}^m_{1,1,\frac{z}{1-z}} = \mathcal{D}^m$ defined by Sălăgean.[15].

(iv) Putting $b_j = \left(\frac{\ell+1}{\ell+j}\right)^{\alpha}$ $(\alpha > 0,\ \ell > -1)$ and $n = 1$, then $\mathcal{D}^m_{1,\delta,g} = \mathcal{I}^{m,\alpha}_{\ell,\delta} f(z)$ defined by El-Deeb and Lupaş [6].

(v) Putting $b_j = \left(\frac{\alpha+1}{\alpha+j}\right)^{n} \frac{m^{j-1}}{(j-1)!} e^{-m}$ $(m, \alpha \geq 0,\ n \in \mathbb{N}_0)$ and $m = 0$, then $\mathcal{D}^0_{n,\delta,g} = \mathcal{H}^n_{\alpha,m} f(z)$ defined by El-Deeb and Oros [9].

(vi) Putting $b_j = \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}}$, $(\upsilon > 0,\ \lambda > -1,\ 0 < q < 1)$ studied by El-Deeb and Bulboacă [7] and El-Deeb [5], we obtain the operator $\mathcal{N}^{m,\lambda,q}_{\upsilon,n,\delta}$, defined as follows:

$$\mathcal{N}^{m,\lambda,q}_{\upsilon,n,\delta} f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)\, c^n(\delta)\right]^m \frac{(-1)^{j-1}\Gamma(\upsilon+1)}{4^{j-1}(j-1)!\Gamma(j+\upsilon)} a_j z^j$$
$$(\lambda > -1;\ 0 < q < 1;\ \delta, \upsilon > 0;\ n \in \mathbb{N};\ m \in \mathbb{N}_0);$$

(vi) Putting $b_j = \left(\frac{\ell+1}{\ell+j}\right)^{\alpha} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}}$, $(\alpha > 0,\ n \geq 0,\ \lambda > -1,\ 0 < q < 1)$ studied by El-Deeb and Bulboacă [8] and Srivastava and El-Deeb [16], we obtain the operator $\mathcal{M}^{m,\lambda,q}_{\ell,n,\delta,\alpha}$, defined as follows:

$$\mathcal{M}^{m,\lambda,q}_{\ell,n,\delta,\alpha} f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)\, c^n(\delta)\right]^m \left(\frac{n+1}{n+k}\right)^{\alpha} \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_j z^j$$
$$(\alpha > 0;\ \lambda > -1;\ \ell \geq 0;\ 0 < q < 1;\ \delta > 0;\ n \in \mathbb{N};\ m \in \mathbb{N}_0).$$

2. Preliminary

To prove our results, we need the following lemmas.

Lemma 2.1. [11] *Let* $\psi \in \mathbb{A}$ *and*

$$\mathcal{G}(z) = \frac{1}{z} \int_0^z \psi(t)dt,\ z \in \Delta.$$

If $\Re\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > \frac{-1}{2},\ z \in \Delta$, *then* $\mathcal{G} \in \mathcal{K}$.

Lemma 2.2. [14, Theorem 2.6] *Let* ψ *be a convex function with* $\psi(0) = \beta$ *and* $\nu \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ *with* $\Re(\nu) \geq 0$. *If* $p \in H[\beta, d]$ *with* $p(0) = \beta$, $\Phi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$,

$$\Phi\left(p(z), zp'(z); z\right) = p(z) + \frac{1}{\nu} zp'(z)$$

is analytic function in Δ *and*

$$\mathcal{F}_{\Phi(\mathbb{C}^2\times\Delta)}\left(p(z) + \frac{1}{\nu} zp'(z)\right) \leq \mathcal{F}_{h(\Delta)} h(z) \quad \to p(z) + \frac{1}{\nu} zp'(z) \prec_{\mathcal{F}} h(z), \quad z \in \Delta,$$

then

$$\mathcal{F}_{p(\Delta)} p(z) \leq \mathcal{F}_{q(\Delta)} q(z) \leq \mathcal{F}_{h(\Delta)} h(z) \ \to\ p(z) \prec_{\mathcal{F}} q(z),\ z \in \Delta,$$

where

$$q(z)=\frac{\nu}{dz^{\frac{\nu}{d}}}\int\limits_0^z\psi(t)t^{\frac{\nu}{d}-1}dt,\ z\in\Delta.$$

The function q is convex and it is the fuzzy best dominant.

Lemma 2.3. [14, Theorem 2.7] *Let g be a convex function in Δ and*

$$\psi(z)=g(z)+d\gamma zg^{'}(z),$$

where $z\in\Delta,\ d\in\mathbb{N}$ and $\gamma>0$. If

$$p(z)=g(0)+p_d\ z^d+p_{d+1}\ z^{d+1}+...$$

belongs to $H(\Delta)$, and

$$\mathcal{F}_{p(\Delta)}\left(p(z)+\gamma zp^{'}(z)\right)\leq\mathcal{F}_{\psi(\Delta)}\psi(z)\quad\rightarrow\quad p(z)+\gamma zp^{'}(z)\prec_{\mathcal{F}}\psi(z),\ z\in\Delta,$$

then

$$\mathcal{F}_{p(\Delta)}\left(p(z)\right)\leq\mathcal{F}_{g(\Delta)}g(z)\quad\rightarrow\quad p(z)\prec_{\mathcal{F}}g(z),\quad z\in\Delta.$$

This result is sharp.

For the general theory of fuzzy differential subordination and its applications, we refer the reader to [1, 2].

In the next section, we obtain several fuzzy differential subordinations associated with the diferential operator $\mathcal{D}^m_{n,\delta,g}f(z)$ by using the method of fuzzy differential subordination.

3. Main results

Assume that $\eta\in[0,1),\ \delta>0,\ n\in\mathbb{N},\ m\in\mathbb{N}_0,\ \lambda>0$ and $z\in\Delta$ are mentioned through this paper.

By using the integral operator $\mathcal{D}^m_{n,\delta,g}$, we define a class of analytic functions and we derive several fuzzy differential subordinations for this class.

Definition 3.1. Let the function $f\in\mathbb{A}$ belongs to the class $\mathcal{H}^F_{n,m,\delta}(\eta,g)$ for all $\eta\in[0,1),\ n\in\mathbb{N}_0,\ m>0$ and $\alpha\geq0$ if it satisfies the inequality:

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)^{'}(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)^{'}>\eta,\qquad(z\in\Delta).$$

Theorem 3.2. *Let k belongs to $\mathcal{C}$ in Δ and suppose that $h(z)=k(z)+\frac{1}{\lambda+2}zk^{'}(z)$. If $f\in\mathcal{H}^F_{n,m,\delta}(\eta,g)$ and*

$$G(z)=I^{\lambda}f(z)=\frac{\lambda+2}{z^{\lambda+1}}\int\limits_0^z t^{\lambda}f(t)dt,\tag{3.1}$$

then

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)^{'}(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)^{'}\leq F_{h(\Delta)}h(z)\rightarrow\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)^{'}\prec_{\mathcal{F}}h(z),\tag{3.2}$$

implies

$$F_{\left(\mathcal{D}^m_{n,\delta,g}G\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \le F_{k(\Delta)}k(z) \to \left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \prec_{\mathcal{F}} k(z),$$

and this result is sharp.

Proof. Since

$$z^{\lambda+1}G(z) = (\lambda+2)\int_0^z t^{\lambda} f(t)dt,$$

by differentiating, it obtain

$$(\lambda+1)\,G(z) + zG'(z) = (\lambda+2)\,f(z),$$

and

$$(\lambda+1)\,\mathcal{D}^m_{n,\delta,g}G(z) + z\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' = (\lambda+2)\,\mathcal{D}^m_{n,\delta,g}f(z), \tag{3.3}$$

and also, by differentiating (3.3) we obtain

$$\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' + \frac{1}{(\lambda+2)}z\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)'' = \left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \tag{3.4}$$

By using (3.4), the fuzzy differential subordination (3.2) is

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' + \frac{1}{(\lambda+2)}z\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)''\right)$$

$$\le F_{h(\Delta)}\left(k(z) + \frac{1}{(\lambda+2)}zk'(z)\right). \tag{3.5}$$

We denote

$$q(z) = \left(\mathcal{D}^m_{n,\delta,g}G(z)\right)', \quad \text{so} \quad q \in \mathcal{H}[1,n]. \tag{3.6}$$

Putting (3.6) in (3.5), we have

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(q(z) + \frac{1}{(\lambda+2)}zq'(z)\right) \le F_{h(\Delta)}\left(k(z) + \frac{1}{(\lambda+2)}zk'(z)\right), \tag{3.7}$$

and applying Lemma (2.3), we have

$$F_{q(\Delta)}q(z) \le F_{k(\Delta)}k(z), \quad \text{i.e} \quad F_{\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \le F_{k(\Delta)}k(z),$$

therefore $\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \prec_{\mathcal{F}} k(z)$, and k is the fuzzy best dominant. □

Theorem 3.3. *Assume that* $h(z) = \frac{1+(2\eta-1)z}{1+z}$, $\eta \in [0,1)$, $\lambda > 0$ *and* $\mathcal{I}^{\lambda}$ *is given by (*3.1*), then*

$$\mathcal{I}^{\lambda}\left[\mathcal{H}^F_{n,m,\delta}(\eta, g)\right] \subset \mathcal{H}^F_{n,m,\delta}(\eta^*, g), \tag{3.8}$$

where

$$\eta^* = 2\eta - 1 + (\lambda+2)(2-2\eta)\int_0^1 \frac{t^{\lambda+2}}{t+1}dt. \tag{3.9}$$

Proof. A function h belongs to $\mathcal{C}$ and using the same technique in the proof of Theorem 3.2, we obtain from the hypothesis of Theorem 3.3 that

$$F_{q(\Delta)}\left(q(z)+\frac{1}{(\lambda+2)}zq^{'}(z)\right)\leq F_{h(\Delta)}h(z),$$

where $q(z)$ is defined in (3.6). By using Lemma 2.2, we obtain

$$F_{q(\Delta)}q(z)\leq F_{k(\Delta)}k(z)\leq F_{h(\Delta)}h(z),$$

which implies

$$F_{\left(\mathcal{D}^m_{n,\delta,g}G\right)^{'}(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)^{'}\leq F_{k(\Delta)}k(z)\leq F_{h(\Delta)}h(z),$$

where

$$\begin{aligned} k(z) &= \frac{\lambda+2}{z^{\lambda+2}}\int\limits_0^z t^{\lambda+1}\frac{1+(2\eta-1)\,t}{1+t}dt \\ &= (2\eta-1)+\frac{(\lambda+2)\,(2-2\eta)}{z^{\lambda+2}}\int\limits_0^z\frac{t^{\lambda+1}}{1+t}dt. \end{aligned}$$

k belongs to $\mathcal{C}$ and $k\,(\Delta)$ is symmetric with respect to the real axis, so we conclude

$$F_{\left(\mathcal{D}^m_{n,\delta,g}G\right)^{'}(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)^{'}\geq\ \min_{|z|=1}F_{k(\Delta)}k(z)=F_{k(\Delta)}k(1), \tag{3.10}$$

and

$$\eta^*=k(1)=2\eta-1+(\lambda+2)\,(2-2\eta)\int\limits_0^1\frac{t^{\lambda+2}}{t+1}dt.$$ □

Theorem 3.4. *Let k belongs to $\mathcal{C}$ in Δ, $k(0)=1$, and $h(z)=k(z)+zk^{'}(z)$. If $f\in\mathbb{A}$ and satisfies the fuzzy differential subordination*

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)^{'}(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)^{'}\leq F_{h(\Delta)}h(z)\rightarrow\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)^{'}\prec_{\mathcal{F}}h(z), \tag{3.11}$$

then

$$F_{\mathcal{D}^m_{n,\delta,g}f(\Delta)}\frac{\mathcal{D}^m_{n,\delta,g}f\,(z)}{z}\leq F_{k(\Delta)}k(z)\rightarrow\frac{\mathcal{D}^m_{n,\delta,g}f\,(z)}{z}\prec_{\mathcal{F}}k(z). \tag{3.12}$$

The result is sharp.

Proof. For

$$\begin{aligned} q(z) &= \frac{\mathcal{D}^m_{n,\delta,g}f\,(z)}{z}=\frac{z+\sum\limits_{j=2}^{\infty}\left[1+(j-1)\,c^n(\delta)\right]^m a_jb_jz^j}{z} \\ &= 1+\sum_{j=2}^{\infty}\left[1+(j-1)\,c^n(\delta)\right]^m a_jb_jz^{j-1}, \end{aligned}$$

we obtain that

$$q(z)+zq^{'}(z)=\left(\mathcal{D}^m_{n,\delta,g}f\,(z)\right)^{'},$$

so

$$F_{\left(\mathcal{D}_{n,\delta,g}^{m}f\right)'(\Delta)}\left(\mathcal{D}_{n,\delta,g}^{m}f(z)\right)' \leq F_{h(\Delta)}h(z)$$

implies

$$F_{q(\Delta)}\left(q(z)+zq'(z)\right) \leq F_{h(\Delta)}h(z) = F_{k(\Delta)}\left(k(z)+zk'(z)\right).$$

Applying Lemma 2.3, we have

$$F_{q(\Delta)}q(z) \leq F_{k(\Delta)}k(z) \rightarrow F_{\mathcal{D}_{n,\delta,g}^{m}f(\Delta)}\frac{\mathcal{D}_{n,\delta,g}^{m}f(z)}{z} \leq F_{k(\Delta)}k(z),$$

and we get

$$\frac{\mathcal{D}_{n,\delta,g}^{m}f(z)}{z} \prec_{\mathcal{F}} k(z).$$

The result is sharp. ☐

Theorem 3.5. *Consider $h \in \mathcal{H}(\Delta)$ with $h(0)=1$, which satisfies*

$$\Re\left(1+\frac{zh''(z)}{h'(z)}\right) > \frac{-1}{2}.$$

If $f \in \mathbb{A}$ and the fuzzy differential subordination

$$F_{\left(\mathcal{D}_{n,\delta,g}^{m}f\right)'(\Delta)}\left(\mathcal{D}_{n,\delta,g}^{m}f(z)\right)' \leq F_{h(\Delta)}h(z) \rightarrow \left(\mathcal{D}_{n,\delta,g}^{m}f(z)\right)' \prec_{\mathcal{F}} h(z), \tag{3.13}$$

holds, then

$$F_{\mathcal{D}_{n,\delta,g}^{m}f(\Delta)}\frac{\mathcal{D}_{n,\delta,g}^{m}f(z)}{z} \leq F_{k(\Delta)}k(z) \qquad i.e \quad \frac{\mathcal{D}_{n,\delta,g}^{m}f(z)}{z} \prec_{\mathcal{F}} k(z), \tag{3.14}$$

where

$$k(z) = \frac{1}{z}\int_{0}^{z} h(t)dt,$$

the function k is convex and it is the fuzzy best dominant.

Proof. Let

$$q(z) = \frac{\mathcal{D}_{n,\delta,g}^{m}f(z)}{z} = 1+\sum_{j=2}^{\infty}\left[1+(j-1)c^{n}(\delta)\right]^{m}a_{j}b_{j}z^{j-1}, \quad q \in \mathcal{H}[1,1],$$

where $\Re\left(1+\frac{zh''(z)}{h'(z)}\right) > \frac{-1}{2}$. From Lemma 2.1, we have

$$k(z) = \frac{1}{z}\int_{0}^{z} h(t)dt$$

belongs to the class $\mathcal{C}$, which satisfies the fuzzy differential subordination (3.13). Since

$$k(z)+zk'(z) = h(z),$$

it is the fuzzy best dominant.

We have

$$q(z) + zq^{'}(z) = \left(\mathcal{D}^m_{n,\delta,g} f(z)\right)^{'},$$

then (3.13) becomes

$$F_{q(\Delta)}\left(q(z) + zq^{'}(z)\right) \leq F_{h(\Delta)} h(z).$$

Applying Lemma 2.3, we have

$$F_{q(\Delta)} q(z) \leq F_{k(\Delta)} k(z), \quad \text{i.e.} \quad F_{\mathcal{D}^m_{n,\delta,g} f(\Delta)} \frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} \leq F_{k(\Delta)} k(z),$$

then

$$\frac{\mathcal{D}^m_{n,\delta,g} f(z)}{z} \prec_{\mathcal{F}} k(z).$$

□

Putting $h(z) = \frac{1+(2\beta-1)z}{1+z}$ in Theorem 3.5, we obtain the following corollary:

Corollary 3.6. *Let* $h = \frac{1+(2\beta-1)z}{1+z}$ *a convex function in* Δ, *with* $h(0) = 1$, $0 \leq \beta < 1$. *If* $f \in \mathbb{A}$ *and verifies the fuzzy differential subordination*

$$F_{\left(\mathcal{D}^m_{n,\delta,g} f\right)^{'}(\Delta)} \left(\mathcal{D}^m_{n,\delta,g} f(z)\right)^{'} \leq F_{h(\Delta)} h(z), \quad i.e \quad \left(\mathcal{D}^m_{n,\delta,g} f(z)\right)^{'} \prec_{\mathcal{F}} h(z),$$

then

$$k(z) = 2\beta - 1 + \frac{2(1-\beta)}{z} \ln(1+z),$$

the function k *is convex and it is the fuzzy best dominant.*

Concluding, all the above results give us information about fuzzy differential subordinations for the operator $\mathcal{D}^m_{n,\delta,g}$, we give some properties for the class $\mathcal{H}^F_{\alpha,m}(n,\eta)$ of univalent analytic functions.

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