A coupled system of fractional difference equations with anti-periodic boundary conditions

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Abstract. In this article, we give sufficient conditions for the existence, uniqueness and Ulam–Hyers stability of solutions for a coupled system of two-point nabla fractional difference boundary value problems subject to anti-periodic boundary conditions, using the vector approach of Precup [4, 14, 19, 21]. Some examples are included to illustrate the theory.

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1. Introduction

In [21], Precup described the advantage of vector-valued norms in the study of the semilinear operator system

\[
\begin{aligned}
N_1(u_1, u_2) &= u_1, \\
N_2(u_1, u_2) &= u_2,
\end{aligned}
\]

in a Banach space \(X\) with norm \(|\cdot|\), by some methods of nonlinear analysis. Here \(N_1, N_2 : X^2 \to X\) are given nonlinear operators. Obviously, this system can be viewed as a fixed point problem:

\[
Nu = u,
\]
in the space $X^2$, where $u = (u_1, u_2)$ and $N = (N_1, N_2)$. Precup [21] proposed the applications of a few fixed point theorems to the system 1.1 in $X^2$, by using the vector-valued norm
\[
\|u\| = \begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix},
\]
for $u = (u_1, u_2) \in X^2$. Also, Precup [21] demonstrated that the results obtained by using the vector-valued norm are better than those established by means of any scalar norm in $X^2$.

**Theorem 1.1.** [21] Assume that
(H1) for each $i \in \{1, 2\}$, there exist nonnegative numbers $a_i$ and $b_i$ such that
\[
|N_i(u_1, u_2) - N_i(v_1, v_2)| \leq a_i|u_1 - v_1| + b_i|u_2 - v_2|,
\]
for all $(u_1, u_2), (v_1, v_2) \in X^2$;
(H2) The spectral radius of $M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is less than one.

Then, (1.1) has a unique solution $(u_1, u_2) \in X^2$.

**Theorem 1.2.** [21] Assume that
(H3) for each $i \in \{1, 2\}$, the operator $N_i$ is completely continuous and, there exist nonnegative numbers $a_i$, $b_i$ and $c_i$ such that
\[
|N_i(u_1, u_2)| \leq a_i|u_1| + b_i|u_2| + c_i,
\]
for all $(u_1, u_2) \in X^2$.

In addition, assume that condition (H2) is satisfied. Then, (1.1) has at least one solution $(u_1, u_2) \in X^2$ satisfying
\[
\begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix} \leq (I - M)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.
\]

Further, in [25], the author used the following theorem to establish Ulam–Hyers stability of solutions of (1.1):

**Theorem 1.3.** [25] Assume that the hypothesis of Theorem 1.1 holds. Then, the system (1.1) is Ulam–Hyers stable.

Motivated by these results, in this article, we consider the following coupled system of nabla fractional difference equations with anti-periodic boundary conditions
\[
\begin{align*}
\nabla_0^{\alpha_1-1}(\nabla u_1)(t) + f_1(u_1(t), u_2(t)) &= 0, \quad t \in \mathbb{N}_0^T, \\
\nabla_0^{\alpha_2-1}(\nabla u_2)(t) + f_2(u_1(t), u_2(t)) &= 0, \quad t \in \mathbb{N}_0^T,
\end{align*}
\]
\[
\begin{align*}
&\quad u_1(0) + u_1(T) = 0, \quad (\nabla u_1)(1) + (\nabla u_1)(T) = 0, \\
&\quad u_2(0) + u_2(T) = 0, \quad (\nabla u_2)(1) + (\nabla u_2)(T) = 0,
\end{align*}
\]
and apply Theorems 1.1 - 1.3 to establish sufficient conditions on existence, uniqueness, and Ulam–Hyers stability [5, 6, 7, 17, 11, 13, 15, 22, 23, 24] of its solutions. For this purpose, we convert the system (1.6) in the form of (1.1). But the results may not be straightforward because the computation of nonnegative numbers in each theorem
for the system (1.6) is complicated due to the presence of nabla fractional difference operators in it.

Here \( T \in \mathbb{N}_2; 1 < \alpha_1, \alpha_2 < 2; f_1, f_2 : \mathbb{R}^2 \to \mathbb{R} \) are continuous, \( \nabla^\nu_0 \) denotes the \( \nu \)-th order Riemann–Liouville type backward (nabla) difference operator where \( \nu \in \{ \alpha_1 - 1, \alpha_2 - 1 \} \) and \( \nabla \) denotes the first order nabla difference operator.

The present article is organized as follows: Section 2 contains preliminaries. In Section 3, we establish sufficient conditions on existence, uniqueness, and Ulam–Hyers stability of solutions of the system (1.6). We provide two examples in Section 4 to illustrate the applicability of established results.

2. Preliminaries

For our convenience, in this section, we present a few useful definitions and fundamental facts of nabla fractional calculus, which can be found in \([1, 2, 3, 8, 9, 10, 16, 18, 20]\).

Denote by \( \mathbb{N}_a = \{ a, a + 1, a + 2, \ldots \} \) and \( \mathbb{N}_b = \{ a, a + 1, a + 2, \ldots, b \} \) for any \( a, b \in \mathbb{R} \) such that \( b - a \in \mathbb{N}_1 \). The backward jump operator \( \rho : \mathbb{N}_a \to \mathbb{N}_a \) is defined by \( \rho(t) = \max\{a, t - 1\} \), for all \( t \in \mathbb{N}_a \). Define the \( \mu \)-th order nabla fractional Taylor monomial by

\[
H_\mu(t, a) = \frac{(t - a)^\mu}{\Gamma(\mu + 1)} = \frac{\Gamma(t - a + \mu)}{\Gamma(t - a)\Gamma(\mu + 1)}, \quad t \in \mathbb{N}_a, \quad \mu \in \mathbb{R} \setminus \{\ldots, -2, -1\}.
\]

Here \( \Gamma(\cdot) \) denotes the Euler gamma function. Observe that \( H_\mu(a, a) = 0 \) and \( H_\mu(t, a) = 0 \) for all \( \mu \in \{\ldots, -2, -1\} \) and \( t \in \mathbb{N}_a \). The first order backward (nabla) difference of \( u : \mathbb{N}_a \to \mathbb{R} \) is defined by \( (\nabla u)(t) = u(t) - u(t - 1) \), for \( t \in \mathbb{N}_{a+1} \).

**Definition 2.1** (See \([9]\)). Let \( u : \mathbb{N}_{a+1} \to \mathbb{R} \) and \( \nu > 0 \). The \( \nu \)-th order nabla sum of \( u \) based at \( a \) is given by

\[
(\nabla_a^{-\nu} u)(t) = \sum_{s = a + 1}^{t} H_{\nu - 1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,
\]

where by convention \( (\nabla_a^{-\nu} u)(a) = 0 \).

**Definition 2.2** (See \([9]\)). Let \( u : \mathbb{N}_{a+1} \to \mathbb{R} \) and \( 0 < \nu \leq 1 \). The \( \nu \)-th order nabla difference of \( u \) is given by

\[
(\nabla_a^\nu u)(t) = \left(\nabla (\nabla_a^{-(1-\nu)} u)\right)(t), \quad t \in \mathbb{N}_{a+1}.
\]

**Lemma 2.3** (See \([9]\)). We have the following properties of nabla fractional Taylor monomials.

1. \( \nabla H_\mu(t, a) = H_{\mu - 1}(t, a), \quad t \in \mathbb{N}_a \).
2. \( \sum_{s = a + 1}^{t} H_\mu(s, a) = H_{\mu + 1}(t, a), \quad t \in \mathbb{N}_a \).
3. \( \sum_{s = a + 1}^{t} H_\mu(t, \rho(s)) = H_{\mu + 1}(t, a), \quad t \in \mathbb{N}_a \).

**Proposition 2.4** (See \([12]\)). Let \( s \in \mathbb{N}_a \) and \( -1 < \mu \). The following properties hold:

(a) \( H_\mu(t, \rho(s)) \geq 0 \) for \( t \in \mathbb{N}_{\rho(s)} \) and, \( H_\mu(t, \rho(s)) > 0 \) for \( t \in \mathbb{N}_{s} \).
(b) $H_\mu(t, \rho(s))$ is a decreasing function with respect to $s$ for $t \in \mathbb{N}_\rho(s)$ and $\mu \in (0, \infty)$.
(c) If $t \in \mathbb{N}_s$ and $\mu \in (-1, 0)$, then $H_\mu(t, \rho(s))$ is an increasing function of $s$.
(d) $H_\mu(t, \rho(s))$ is a non-decreasing function with respect to $t$ for $t \in \mathbb{N}_\rho(s)$ and $\mu \in [0, \infty)$.
(e) If $t \in \mathbb{N}_s$ and $\mu \in (0, \infty)$, then $H_\mu(t, \rho(s))$ is an increasing function of $t$.
(f) $H_\mu(t, \rho(s))$ is a decreasing function with respect to $t$ for $t \in \mathbb{N}_{s+1}$ and $\mu \in (-1, 0)$.

Proposition 2.5 (See [12]). Let $u$ and $v$ be two nonnegative real-valued functions defined on a set $S$. Further, assume $u$ and $v$ achieve their maximum values in $S$. Then,

$$|u(t) - v(t)| \leq \max\{u(t), v(t)\} \leq \max\{\max_{t \in S} u(t), \max_{t \in S} v(t)\},$$

for every fixed $t$ in $S$.

3. Green’s function and its property

Assume $T \in \mathbb{N}_2$, $1 < \alpha < 2$, and $h : \mathbb{N}^T_2 \to \mathbb{R}$. Consider the boundary value problem

$$\begin{aligned}
\left\{ \begin{array}{l}
\left( \nabla_0^{\alpha-1/2} \nabla u \right)(t) + h(t) = 0, \quad t \in \mathbb{N}^T_2, \\
u(0) + u(T) = 0, \quad (\nabla u)(1) + (\nabla u)(T) = 0.
\end{array} \right.
\end{aligned} \tag{3.1}
$$

First, we construct the Green’s function, $G(t,s)$ corresponding to (3.1), and obtain an expression for its unique solution. Denote by

$$D_1 = \{(t,s) \in \mathbb{N}^T_0 \times \mathbb{N}^T_2 : t \geq s\}, \quad D_2 = \{(t,s) \in \mathbb{N}^T_0 \times \mathbb{N}^T_2 : t \leq \rho(s)\},$$

and

$$\xi_\alpha = 2[1 + H_{\alpha-2}(T,0)]. \tag{3.2}$$

Theorem 3.1. The unique solution of the nabla fractional boundary value problem (3.1) is given by

$$u(t) = \sum_{s=2}^{T} G_\alpha(t,s)h(s), \quad t \in \mathbb{N}^T_2, \tag{3.3}$$

where

$$G_\alpha(t,s) = \begin{cases}
K_\alpha(t,s) - H_{\alpha-1}(t,\rho(s)), & (t,s) \in D_1, \\
K_\alpha(t,s), & (t,s) \in D_2.
\end{cases} \tag{3.4}$$

Here

$$K_\alpha(t,s) = \frac{1}{\xi_\alpha} \left[ H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) \\
+ H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0) - H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s)) \right].$$

Proof. Denote by

$$(\nabla u)(t) = v(t), \quad t \in \mathbb{N}^T_2.$$  

Subsequently, the difference equation in (3.1) takes the form

$$(\nabla_0^{\alpha-1} v)(t) + h(t) = 0, \quad t \in \mathbb{N}^T_2. \tag{3.5}$$
Let $v(1) = c_2$. Then, by Lemma 5.1 of [1], the unique solution of (3.5) is given by
\[ v(t) = H_{\alpha-2}(t,0)c_2 - (\nabla_1^{-(\alpha-1)}h)(t), \quad t \in \mathbb{N}_1^T. \]
That is,
\[ (\nabla u)(t) = H_{\alpha-2}(t,0)c_2 - (\nabla_1^{-(\alpha-1)}h)(t), \quad t \in \mathbb{N}_1^T. \]
Applying the first order nabla sum operator, $\nabla^{-1}$ on both sides of (3.6), we obtain
\[ u(t) = c_1 + H_{\alpha-1}(t,0)c_2 - (\nabla_1^{-\alpha}h)(t), \quad t \in \mathbb{N}_0^T, \]
where $c_1 = u(0)$. We use the pair of anti-periodic boundary conditions considered in (3.1) to eliminate the constants $c_1$ and $c_2$ in (3.7). It follows from the first boundary condition $u(0) + u(T) = 0$ that
\[ 2c_1 + H_{\alpha-1}(T,0)c_2 = (\nabla_1^{-\alpha}h)(T). \]
The second boundary condition $(\nabla u)(1) + (\nabla u)(T) = 0$ yields
\[ [1 + H_{\alpha-2}(T,0)]c_2 = (\nabla_1^{-(\alpha-1)}h)(T). \]
Solving (3.8) and (3.9) for $c_1$ and $c_2$, we obtain
\[ c_1 = \frac{1}{2} \left[ \sum_{s=2}^{T} H_{\alpha-1}(T,\rho(s))h(s) - \frac{2H_{\alpha-1}(T,0)}{\xi_\alpha} \sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s) \right], \quad (3.10) \]
\[ c_2 = \frac{2}{\xi_\alpha} \sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s). \]
Substituting these expressions in (3.7), we achieve (3.4). □

**Lemma 3.2.** Observe that
\[ |K_\alpha(t,s)| \leq \frac{1}{\xi_\alpha} \left[ H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1) \right], \quad (3.12) \]
for all $(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T$.

**Proof.** Denote by
\[ K_\alpha'(t,s) = \frac{1}{\xi_\alpha} \left[ H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) \right. \]
\[ \left. + H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0) \right], \quad (3.13) \]
and
\[ K_\alpha''(t,s) = \frac{1}{\xi_\alpha} \left[ H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s)) \right], \quad (3.14) \]
so that
\[ K_\alpha(t,s) = K_\alpha'(t,s) - K_\alpha''(t,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T. \]
Clearly, from Proposition 2.4,
\[ K_\alpha'(t,s) \geq 0, \quad K_\alpha''(t,s) > 0, \quad \text{for all} \ (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T. \]
From Proposition 2.5, it is obvious that

\[ |K_\alpha(t, s)| \leq \left\{ \max_{(t, s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T} K'_\alpha(t, s), \max_{(t, s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T} K''_\alpha(t, s) \right\}. \]  

(3.15)

First, we evaluate the first backward difference of \( K'_\alpha(t, s) \) with respect to \( t \) for a fixed \( s \). Consider

\[ \nabla K'_\alpha(t, s) = \frac{1}{\xi_\alpha} \left[ 2H_{\alpha-2}(t, 0)H_{\alpha-2}(T, \rho(s)) \right] > 0, \]

for all \((t, s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T\), implying that \( K'_\alpha(t, s) \) is an increasing function of \( t \) for a fixed \( s \). Thus, we have

\[ K'_\alpha(t, s) \leq K'_\alpha(T, s), \quad (t, s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T. \]

(3.16)

It follows from (3.13) - (3.16) that

\[ |K_\alpha(t, s)| \leq \left\{ \max_{(t, s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T} K'_\alpha(t, s), \max_{(t, s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T} K''_\alpha(t, s) \right\} \]

\[ \leq \left\{ \max_{s \in \mathbb{N}_2^T} K'_\alpha(T, s), \max_{s \in \mathbb{N}_2^T} K''_\alpha(t, s) \right\} \]

\[ = \max_{s \in \mathbb{N}_2^T} K'_\alpha(T, s) \]

\[ = \frac{1}{\xi_\alpha} \max_{s \in \mathbb{N}_2^T} \left[ H_{\alpha-1}(T, \rho(s)) + 2H_{\alpha-1}(T, 0)H_{\alpha-2}(T, \rho(s)) \right. \]

\[ + \left. H_{\alpha-1}(T, \rho(s))H_{\alpha-2}(T, 0) \right] \]

\[ \leq \frac{1}{\xi_\alpha} \left[ \max_{s \in \mathbb{N}_2^T} H_{\alpha-1}(T, \rho(s)) + 2H_{\alpha-1}(T, 0) \max_{s \in \mathbb{N}_2^T} H_{\alpha-2}(T, \rho(s)) \right. \]

\[ + \left. H_{\alpha-2}(T, 0) \max_{s \in \mathbb{N}_2^T} H_{\alpha-1}(T, \rho(s)) \right] \]

\[ = \frac{1}{\xi_\alpha} \left[ H_{\alpha-1}(T, \rho(2)) + 2H_{\alpha-1}(T, 0)H_{\alpha-2}(T, \rho(T)) + H_{\alpha-2}(T, 0)H_{\alpha-1}(T, \rho(2)) \right] \]

\[ = \frac{1}{\xi_\alpha} \left[ H_{\alpha-1}(T, 1) + 2H_{\alpha-1}(T, 0) + H_{\alpha-2}(T, 0)H_{\alpha-1}(T, 1) \right]. \]

The proof is complete.

\[ \square \]

**4. Main results**

Let \( X = \mathbb{R}^{T+1} \) be the Banach space of all real \((T + 1)\)-tuples equipped with the maximum norm

\[ |u| = \max_{t \in \mathbb{N}_0^T} |u(t)|. \]

Obviously, the product space \( X^2 \) is also a Banach space with the vector-norm

\[ \|u\| = \left( \frac{|u_1|}{|u_2|} \right), \]
for \( u = (u_1, u_2) \in X^2 \).

For our convenience, denote by
\[
\Lambda_i = \frac{1}{\xi_{\alpha_i}} \left[ H_{\alpha_i-1}(T, 1) + 2H_{\alpha_i-1}(T, 0) + H_{\alpha_i-2}(T, 0)H_{\alpha_i-1}(T, 1) \right],
\]
(4.1)

\[
a_i = l_i [\Lambda_i(T - 1) + H_{\alpha_i}(T, 1)],
\]
(4.2)

\[
b_i = m_i [\Lambda_i(T - 1) + H_{\alpha_i}(T, 1)],
\]
(4.3)

\[
c_i = n_i [\Lambda_i(T - 1) + H_{\alpha_i}(T, 1)],
\]
(4.4)

for \( i = 1, 2 \).

Define the operator \( T : X^2 \to X^2 \) by
\[
T(u_1, u_2)(t) = \left( \begin{array}{c} T_1(u_1, u_2)(t) \\ T_2(u_1, u_2)(t) \end{array} \right), \quad t \in \mathbb{N}_0^T,
\]
(4.5)

where \( T_1(u_1, u_2)(t) \)
\[
= \sum_{s=2}^{T} G_{\alpha_1}(t, s)f_1(u_1(s), u_2(s))
= \sum_{s=2}^{T} K_{\alpha_1}(t, s)f_1(u_1(s), u_2(s)) - \sum_{s=2}^{T} H_{\alpha_1-1}(t, s)f_1(u_1(s), u_2(s)),
\]
(4.6)

and \( T_2(u_1, u_2)(t) \)
\[
= \sum_{s=2}^{T} G_{\alpha_2}(t, s)f_2(u_1(s), u_2(s))
= \sum_{s=2}^{T} K_{\alpha_2}(t, s)f_2(u_1(s), u_2(s)) - \sum_{s=2}^{T} H_{\alpha_2-1}(t, s)f_2(u_1(s), u_2(s)),
\]
(4.7)

**Theorem 4.1.** A couple \((u_1, u_2) \in X^2\) is a solution of (1.6) if, and only if,
\[
\begin{cases}
T_1(u_1, u_2) = u_1, \\
T_2(u_1, u_2) = u_2.
\end{cases}
\]
(4.8)

In view of Theorem 4.1 it is enough to apply Theorems 1.1 - 1.3 to the system (4.8).

**Theorem 4.2.** Assume that
\[
(1) \text{ for each } i \in \{1, 2\}, \text{ there exist nonnegative numbers } l_i \text{ and } m_i \text{ such that }
\]
\[
|f_i(u_1, u_2) - f_i(v_1, v_2)| \leq l_i |u_1 - v_1| + m_i |u_2 - v_2|,
\]
(4.9)

for all \((u_1, u_2), (v_1, v_2) \in X^2\);

In addition, assume that condition (H2) is satisfied. Then, (4.8) has a unique solution \((u_1, u_2) \in X^2\).
Proof. For each $i \in \{1, 2\}$ and for all $(u_1, u_2), (v_1, v_2) \in X^2$, consider
\[
|T_i(u_1, u_2) - T_i(v_1, v_2)| \leq \sum_{s=2}^{T} |K_{a_i}(t, s)| |f_i(u_1(s), u_2(s)) - f_i(v_1(s), v_2(s))| \\
+ \sum_{s=2}^{t} H_{a_i-1}(t, s) |f_i(u_1(s), u_2(s)) - f_i(v_1(s), v_2(s))| \\
\leq [l_i|u_1 - v_1| + m_i|u_2 - v_2|] \left[ \sum_{s=2}^{T} |K_{a_i}(t, s)| + \sum_{s=2}^{t} H_{a_i-1}(t, s) \right] \\
\leq [l_i|u_1 - v_1| + m_i|u_2 - v_2|] [\Lambda_i(T - 1) + H_{a_i}(T, 1)] \\
\leq [l_i|u_1 - v_1| + m_i|u_2 - v_2|] [\Lambda_i(T - 1) + H_{a_i}(T, 1)] \\
\leq a_i|u_1 - v_1| + b_i|u_2 - v_2|,
\]
implying that (H1) holds. Thus, by Theorem 1.1, the system (4.8) has a unique solution $(u_1, u_2) \in X^2$. \hfill \Box

**Theorem 4.3.** Assume that

(II) for each $i \in \{1, 2\}$, there exist nonnegative numbers $a_i, b_i$ and $c_i$ such that
\[
|f_i(u_1, u_2)| \leq l_i|u_1| + m_i|u_2| + n_i,
\]
for all $(u_1, u_2) \in X^2$.

In addition, assume that condition (H2) is satisfied. Then, (4.8) has at least one solution $(u_1, u_2) \in X^2$ satisfying (1.5).

Proof. Since $T_i, i = 1, 2$, is a summation operator on a discrete finite set, it is trivially completely continuous on $X^2$. For each $i \in \{1, 2\}$ and for all $(u_1, u_2) \in X^2$, consider
\[
|T_i(u_1, u_2)| \leq \sum_{s=2}^{T} |K_{a_i}(t, s)| |f_i(u_1(s), u_2(s))| + \sum_{s=2}^{t} H_{a_i-1}(t, s) |f_i(u_1(s), u_2(s))| \\
\leq [l_i|u_1| + m_i|u_2| + n_i] \left[ \sum_{s=2}^{T} |K_{a_i}(t, s)| + \sum_{s=2}^{t} H_{a_i-1}(t, s) \right] \\
\leq [l_i|u_1| + m_i|u_2| + n_i] [\Lambda_i(T - 1) + H_{a_i}(T, 1)] \\
\leq [l_i|u_1| + m_i|u_2| + n_i] [\Lambda_i(T - 1) + H_{a_i}(T, 1)] \\
\leq a_i|u_1| + b_i|u_2| + c_i,
\]
implying that (H3) holds. Thus, by Theorem 1.2, the system (4.8) has at least one solution $(u_1, u_2) \in X^2$ satisfying (1.5). \hfill \Box

**Definition 4.4.** [25] Let $X$ be a Banach space and $T_1, T_2 : X \times X \to X$ be two operators. Then, the system (4.8) is said to be Ulam–Hyers stable if there exist $C_1,$
The spectral radius of $M$ is 0.158, which is less than one, implying that $M$ converges to zero. Hence, by Theorem 4.2, the system (5.1) has a unique solution $(u_1^*, u_2^*) \in X^2$. Also, by Theorem 4.5, the unique solution of (5.1) is Ulam–Hyers stable.
Example 5.2. Consider the following boundary value problem for a coupled system of fractional difference equations

\[
\begin{cases}
\left(\nabla_0^{0.5}(\nabla u_1)\right)(t) + (0.01) \left[ 1 + \frac{1}{\sqrt{1+u_1^2(t)}} + u_2(t) \right] = 0, & t \in \mathbb{N}_2^4, \\
\left(\nabla_0^{0.5}(\nabla u_2)\right)(t) + (0.02) \left[ 1 + u_1(t) + \frac{1}{\sqrt{1+u_2^2(t)}} \right] = 0, & t \in \mathbb{N}_2^4, \\
u_1(0) + u_1(4) = 0, & (\nabla u_1)(1) + (\nabla u_1)(4) = 0, \\
u_2(0) + u_2(4) = 0, & (\nabla u_2)(1) + (\nabla u_2)(4) = 0.
\end{cases}
\]

Comparing (1.6) and (5.2), we have \(T = 4, \alpha_1 = \alpha_2 = 1.5,\)

\[
f_1(u_1, u_2) = (0.01) \left[ 1 + \frac{1}{\sqrt{1+u_1^2(t)}} + u_2(t) \right],
\]

and

\[
f_2(u_1, u_2) = (0.02) \left[ 1 + u_1(t) + \frac{1}{\sqrt{1+u_2^2(t)}} \right],
\]

for all \((u_1, u_2) \in \mathbb{R}^2.\) Clearly, \(f_1\) and \(f_2\) are continuous on \(\mathbb{R}^2.\) Next, \(f_1\) and \(f_2\) satisfy assumption (II) with \(l_1 = 0.01, m_1 = 0.01, l_2 = 0.02, m_2 = 0.02, n_1 = 0.01\) and \(n_2 = 0.02.\) We have,

\[
\begin{align*}
a_1 &= l_1 [\Lambda_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.1219, \\
a_2 &= l_2 [\Lambda_2(T - 1) + H_{\alpha_2}(T, 1)] = 0.2438, \\
b_1 &= m_1 [\Lambda_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.1219, \\
b_2 &= m_2 [\Lambda_2(T - 1) + H_{\alpha_2}(T, 1)] = 0.2438, \\
c_1 &= n_1 [\Lambda_1(T - 1) + H_{\alpha_1}(T, 1)] = 0.1219, \\
c_2 &= n_2 [\Lambda_2(T - 1) + H_{\alpha_2}(T, 1)] = 0.2438.
\end{align*}
\]

Further,

\[
M = \begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{pmatrix} = \begin{pmatrix}
0.1219 & 0.1219 \\
0.2438 & 0.2438
\end{pmatrix}.
\]

The spectral radius of \(M\) is 0.3657, which is less than one, implying that \(M\) converges to zero. Hence, by Theorem 4.3, the system (5.2) has at least one solution \((u_1, u_2) \in X^2\) satisfying

\[
\left(\begin{array}{c}
u_1 \\
u_2
\end{array}\right) \leq (I - M)^{-1} \left(\begin{array}{c}
c_1 \\
c_2
\end{array}\right) = \begin{pmatrix}
0.1757 \\
0.2658
\end{pmatrix}.
\]

Conclusion

In this article, we obtained sufficient conditions on existence, uniqueness and Ulam–Hyers stability of solutions of the system (1.6) using the approaches of Precup and Urs. We also provided two examples to demonstrate the applicability of established results. Observe that Theorem 4.2 is not applicable to the system (5.1).
A coupled system of fractional difference equations

References


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