On certain class of meromorphic univalent functions with positive coefficients defined by Dziok-Srivastava operator

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Abstract. In this paper, we introduce a new class of meromorphic univalent functions defined by using Dziok-Srivastava operator and obtain some results including coefficient inequality, growth and distortion theorems and modified Hadamard products.

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1. Introduction

Let $\Sigma_m$ denote the class of functions $f$ of the form:

$$f(z) = \frac{1}{z} + \sum_{k=m}^{\infty} a_k z^k \; (m \in \mathbb{N} = \{1, 2, \ldots\})$$

which are analytic and univalent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$. For $g \in \Sigma_m$, given by

$$g(z) = \frac{1}{z} + \sum_{k=m}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of $f$ and $g$ is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z).$$

A function $f \in \Sigma_m$ is said to be meromorphically starlike of order $\lambda$ if

$$-\text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \lambda \; (z \in U; \; 0 \leq \lambda < 1).$$
Denote by $\Sigma S^*_m (\lambda)$ the class of all meromorphically starlike functions of order $\lambda$. A function $f \in \Sigma_m$ is said to be meromorphically convex of order $\lambda$ if
\[
-\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \lambda \quad (z \in U; \ 0 \leq \lambda < 1).
\]
(1.5)

Denote by $\Sigma K_m (\lambda)$ the class of all meromorphically convex functions of order $\lambda$. We note that $f(z) \in \Sigma K_m (\lambda) \iff -zf'(z) \in \Sigma S^*_m (\lambda)$.

The classes $\Sigma S^*_m (\lambda)$ and $\Sigma K_m (\lambda)$ were introduced by Owa et al. [8]. Various subclasses of the class $\Sigma_m$ when $m = 1$ were considered earlier by Pommerenke [9], Miller [6] and others.

For complex parameters
\[
\alpha_1, \ldots, \alpha_q \text{ and } \beta_1, \ldots, \beta_s \quad (\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\} ; \ j = 1, 2, \ldots, s),
\]
the generalized hypergeometric function $qF_s (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is defined by
\[
qF_s (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} \frac{z^k}{k!}
\]

where $(\theta)_v$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma$, by
\[
(\theta)_v = \frac{\Gamma (\theta + v)}{\Gamma (\theta)} = \begin{cases} 1 & \text{if } (v = 0; \ \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \theta (\theta + 1) (\theta + 2) \ldots (\theta + v - 1) & \text{if } (v \in \mathbb{N}; \ \theta \in \mathbb{C}). \end{cases}
\]

Corresponding to the function $h (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$, defined by
\[
h (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-1} qF_s (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),
\]
we consider the linear operator
\[
H (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \Sigma_m \to \Sigma_m,
\]
which is defined by means of the following Hadamard product (or convolution):
\[
H (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f (z) = h_p (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f (z).
\]
(1.9)

We observe that, for a function $f$ of the form (1.1), we have
\[
H (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f (z) = z^{-1} + \sum_{k=m}^{\infty} \frac{(\alpha_1)_{k+1} \ldots (\alpha_q)_{k+1}}{(\beta_1)_{k+1} \ldots (\beta_s)_{k+1}} \frac{a_k}{(k+1)!} z^k.
\]
(1.10)

For convenience, we write
\[
H_{q,s} (\alpha_1) = H (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s).
\]
(1.11)

The linear operator $H_{q,s} (\alpha_1)$ was investigated recently by Liu and Srivastava [5, with $p = 1$] and Aouf [2, with $p = 1$].
For fixed parameters $A, B, \beta$ and $\lambda$ (0 < $\beta$ ≤ 1, -1 ≤ $A < B$ ≤ 1, 0 ≤ $\lambda$ < 1), we say that a function $f \in \Sigma_m$ is in the class $\Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$ of meromorphically univalent functions in if it satisfies the inequality:

$$\left| \frac{z(H_{q,s}(\alpha_1)f(z))' + 1}{B \frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} + [B + (A - B) (1 - \lambda)]} \right| < \beta \quad (z \in U^*).$$

(1.12)

A function $f$ in $\Sigma_m$ is said to belong to the class $C_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$ if and only if $-zf'(z) \in \Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$ that is

$$f \in C_{q,s}^m(\alpha_1; A, B, \lambda, \beta) \iff -zf'(z) \in \Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta).$$

(1.13)

We note that:

(i) $\Sigma_{2,1}^m(1; -1, 1, \lambda, 1) = \Sigma_{2,1} S_1^*(\lambda)$ and $C_{2,1}^m(1; -1, 1, \lambda, 1) = \Sigma_{2,1} K_m(\lambda)$ (0 ≤ $\lambda$ < 1, m ∈ $\mathbb{N}$).

(ii) $\Sigma_{2,1}^m(1; A, B, \lambda, \beta) = \Sigma_{2,1}^* (A, B, \lambda, \beta)$ was studied by Aouf [1];

(iii) $\Sigma_{2,1}^1(1; -1, 1, \lambda, \beta) = \Sigma_{2,1}^* (\lambda, \beta)$ and $C_{2,1}^1(1; -1, 1, \lambda, \beta) = C(\lambda, \beta)$ (Mogra et al. [7]);

(iv) $\Sigma_{2,1}^m(1; A, B, \lambda, \beta) = \Sigma_{2,1} (A, B, \lambda, \beta)$ (Aouf et al. [6]).

We note also that:

$$\frac{\Sigma_{2,1}^1(\alpha_1; \beta, -\beta, \lambda, 1)}{\Sigma_{2,1}^1(\alpha_1; \lambda, \beta)} = \frac{\Sigma_{2,1}^+ (\alpha_1; \lambda, \beta)}{\Sigma_{2,1}^* (\alpha_1; \lambda, \beta)}$$

$$= \left\{ f(z) \in \Sigma_m : \left| \frac{z(H_{q,s}(\alpha_1)f(z))' + 1}{H_{q,s}(\alpha_1)f(z)} - 1 + 2\lambda \right| < \beta \quad (z \in U, \ 0 < \beta \leq 1, \ 0 \leq \lambda < 1) \right\}. $$

2. Coefficient inequality

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ are positive real numbers, 0 < $\beta$ ≤ 1, -1 ≤ $A < B$ ≤ 1, 0 ≤ $\lambda$ < 1, m ∈ $\mathbb{N}$, $\Gamma_{k+1}(\alpha_1)$ is defined by (2.2) and $z \in U^*$.

In order to prove our results we need the following lemma for the class $\Sigma_{q,s}^m(\alpha_1; A, B, \lambda, 1)$ given by Aouf [3, with $p = 1$].

Lemma 2.1. Let a function $f$ defined by (1.1) be in the class $\Sigma_m$. If

$$\sum_{k=m}^{\infty} \{(k + 1) + \beta [(Bk + A) + (B - A) \lambda]\} \Gamma_{k+1}(\alpha_1) \, |a_k| \leq (B - A) \beta (1 - \lambda) \quad (2.1)$$

then $f \in \Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$, where

$$\Gamma_{k+1}(\alpha_1) = \frac{(\alpha_1)_{k+1}, \ldots, (\alpha_q)_{k+1}}{(\beta_1)_{k+1}, \ldots, (\beta_s)_{k+1}} \frac{1}{(k + 1)!}.$$  \hspace{2cm} (2.2)

From Lemma 2.1 and (1.13), we have the following lemma.

Lemma 2.2. Let a function $f$ defined by (1.1) be in the class $\Sigma_m$. If

$$\sum_{k=m}^{\infty} k \{(k + 1) + \beta [(Bk + A) + (B - A) \lambda]\} \Gamma_{k+1}(\alpha_1) \, |a_k| \leq (B - A) \beta (1 - \lambda) \quad (2.3)$$

...
then \( f \in C_{q,s}^m(\alpha_1; A, B, \lambda, \beta) \).

### 3. Growth and distortion theorems

**Theorem 3.1.** If the function \( f \) defined by (1.1) is in the class \( \Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta) \), then

\[
\frac{1}{|z|} - \frac{(B - A) \beta (1 - \lambda)}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]] \Gamma_{m+1}(\alpha_1)} |z|^m \leq |f(z)|
\]

\[
\leq \frac{1}{|z|} + \frac{(B - A) \beta (1 - \lambda)}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]] \Gamma_{m+1}(\alpha_1)} |z|^m,
\]

(3.1)

and

\[
\frac{1}{|z|^2} - \frac{m (B - A) \beta (1 - \lambda)}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]] \Gamma_{m+1}(\alpha_1)} |z|^{m-1} \leq |f'(z)|
\]

\[
\leq \frac{1}{|z|^2} + \frac{m (B - A) \beta (1 - \lambda)}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]] \Gamma_{m+1}(\alpha_1)} |z|^{m-1}.
\]

(3.2)

The bounds in (3.1) and (3.2) are attained for the function \( f \) given by

\[
f(z) = \frac{1}{z} + \frac{(B - A) \beta (1 - \lambda)}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]] \Gamma_{m+1}(\alpha_1)} z^m.
\]

(3.3)

**Proof.** First of all, for \( \Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta) \), it follows from (2.1) that

\[
\sum_{k=m}^{\infty} |a_k| \leq \frac{(B - A) \beta (1 - \lambda)}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]] \Gamma_{m+1}(\alpha_1)},
\]

(3.4)

which, in view of (1.1), yields

\[
|f(z)| \geq \frac{1}{|z|} - |z|^m \sum_{k=m}^{\infty} |a_k| \geq \frac{1}{|z|} - \frac{(B - A) \beta (1 - \lambda)}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]] \Gamma_{m+1}(\alpha_1)} |z|^m,
\]

(3.5)

\[
|f(z)| \leq \frac{1}{|z|} + |z|^m \sum_{k=m}^{\infty} |a_k| \leq \frac{1}{|z|} + \frac{(B - A) \beta (1 - \lambda)}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]] \Gamma_{m+1}(\alpha_1)} |z|^m.
\]

(3.6)
Next, we see from (2.1) that
\[
\left\{ (m + 1) + \beta [(Bm + A) + (B - A) \lambda] \right\} \frac{\Gamma_{m+1}(\alpha_1)}{m} \sum_{k=m}^{\infty} k |a_k| \]
(3.7)
\[
\leq \sum_{k=m}^{\infty} \left\{ (k + 1) + \beta [(Bk + A) + (B - A) \lambda] \right\} \Gamma_{k+1}(\alpha_1) |a_k| \]
\[
\leq (B - A) \beta (1 - \lambda)
\]
then
\[
\sum_{k=m}^{\infty} k |a_k| \leq \frac{m (B - A) \beta (1 - \lambda)}{\left\{ (m + 1) + \beta [(Bm + A) + (B - A) \lambda] \right\} \Gamma_{m+1}(\alpha_1)}.
\]
which, again in view of (1.1), yields
\[
|f'(z)| \geq \frac{1}{|z|^2} - |z|^{m-1} \sum_{k=m}^{\infty} k |a_k| \]
(3.8)
\[
\geq \frac{1}{|z|^2} - \frac{m (B - A) \beta (1 - \lambda)}{\left\{ (m + 1) + \beta [(Bm + A) + (B - A) \lambda] \right\} \Gamma_{m+1}(\alpha_1)} |z|^{m-1},
\]
and
\[
|f'(z)| \leq \frac{1}{|z|^2} + |z|^{m-1} \sum_{k=m}^{\infty} k |a_k| \]
(3.9)
\[
\leq \frac{1}{|z|^2} + \frac{m (B - A) \beta (1 - \lambda)}{\left\{ (m + 1) + \beta [(Bm + A) + (B - A) \lambda] \right\} \Gamma_{m+1}(\alpha_1)} |z|^{m-1}.
\]
Finally, it is easy to see that the bounds in (3.1) and (3.2) are attained for the function 
\( f \) given by (3.3).

**Corollary 3.1.** If the function \( f \) defined by (1.1) is in the class \( C_{q,s}^{m} (\alpha_1; A, B, \lambda, \beta) \), then
\[
\frac{1}{|z|} - \frac{(B - A) \beta (1 - \lambda)}{m \left\{ (m + 1) + \beta [(Bm + A) + (B - A) \lambda] \right\} \Gamma_{m+1}(\alpha_1)} |z|^m \leq |f(z)|
\]
(3.10)
\[
\leq \frac{1}{|z|} + \frac{(B - A) \beta (1 - \lambda)}{m \left\{ (m + 1) + \beta [(Bm + A) + (B - A) \lambda] \right\} \Gamma_{m+1}(\alpha_1)} |z|^m,
\]
and
\[
\frac{1}{|z|^2} - \frac{(B - A) \beta (1 - \lambda)}{\left\{ (m + 1) + \beta [(Bm + A) + (B - A) \lambda] \right\} \Gamma_{m+1}(\alpha_1)} |z|^{m-1} \leq |f'(z)|
\]
(3.11)
\[
\leq \frac{1}{|z|^2} + \frac{(B - A) \beta (1 - \lambda)}{\left\{ (m + 1) + \beta [(Bm + A) + (B - A) \lambda] \right\} \Gamma_{m+1}(\alpha_1)} |z|^{m-1}.
\]
The bounds in (3.1) and (3.2) are attained for the function \( f \) given by
\[
f(z) = \frac{1}{z} + \frac{(B - A) \beta (1 - \lambda)}{m \left\{ (m + 1) + \beta [(Bm + A) + (B - A) \lambda] \right\} \Gamma_{m+1}(\alpha_1)} z^m.
\]
(3.12)
4. Modified Hadamard product

Let each of the functions $f_1$ and $f_2$ defined by

$$f_j (z) = \frac{1}{z} + \sum_{k=m}^{\infty} a_{k,j} z^k \quad (j = 1, 2)$$

belong to the class $\Sigma_m$. We denote by $(f_1 * f_2)$ the modified Hadamard product (or convolution) of the functions $f_1$ and $f_2$, that is,

$$(f_1 * f_2) (z) = \frac{1}{z} + \sum_{k=m}^{\infty} a_{k,1} a_{k,2} z^k.$$  (4.2)

**Theorem 4.1.** Let the functions $f_j \ (j = 1, 2)$ defined by (4.1) be in the class $\Sigma_{q,s}^m (\alpha_1; A, B, \lambda, \beta)$. Then $(f_1 * f_2) (z) \in \Sigma_{q,s}^m (\alpha_1; A, B, \gamma, \beta)$, where

$$\gamma = 1 - \frac{(B - A) \beta (1 - \lambda)^2 (1 + \beta B) (m + 1)}{((m + 1) + \beta [(Bm + A) + (B - A) \lambda]) \Gamma_{m+1}^2 (1 + \beta B) (m + 1)^2}.$$  (4.3)

The result is sharp for the functions $f_j \ (j = 1, 2)$ given by

$$f_j (z) = \frac{1}{z} + \frac{(B - A) \beta (1 - \lambda)}{((m + 1) + \beta [(Bm + A) + (B - A) \lambda]) \Gamma_{m+1}^2 (1 + \beta B) (m + 1)^2} z^m \quad (j = 1, 2).$$  (4.4)

**Proof.** Employing the technique used earlier by Schild and Silverman [10], we need to find the largest $\gamma$ such that

$$\sum_{k=m}^{\infty} \frac{((k + 1) + \beta [(Bk + A) + (B - A) \lambda]) \Gamma_{k+1} \Gamma_{k+1}^2 (1 + \beta B) (m + 1)^2}{((m + 1) + \beta [(Bm + A) + (B - A) \lambda]) \Gamma_{m+1}^2 (1 + \beta B) (m + 1)^2} |a_{k,1}| |a_{k,2}| \leq 1$$  (4.5)

for $(f_1 * f_2) (z) \in \Sigma_{q,s}^m (\alpha_1; A, B, \gamma, \beta)$. Indeed, since each of the functions $f_j \ (j = 1, 2)$ belongs to the class $\Sigma_{q,s}^m (\alpha_1; A, B, \lambda, \beta)$, then

$$\sum_{k=m}^{\infty} \frac{((k + 1) + \beta [(Bk + A) + (B - A) \lambda]) \Gamma_{k+1} \Gamma_{k+1}^2 (1 + \beta B) (m + 1)^2}{((m + 1) + \beta [(Bm + A) + (B - A) \lambda]) \Gamma_{m+1}^2 (1 + \beta B) (m + 1)^2} |a_{k,j}| \leq 1 \quad (j = 1, 2).$$  (4.6)

Now, by the Cauchy-Schwarz inequality, we find from (4.6) that

$$\sum_{k=m}^{\infty} \frac{((k + 1) + \beta [(Bk + A) + (B - A) \lambda]) \Gamma_{k+1} \Gamma_{k+1}^2 (1 + \beta B) (m + 1)^2}{((m + 1) + \beta [(Bm + A) + (B - A) \lambda]) \Gamma_{m+1}^2 (1 + \beta B) (m + 1)^2} \sqrt{|a_{k,1}| |a_{k,2}|} \leq 1.$$  (4.7)

Equation (4.7) implies that we need only to show that

$$\frac{((k + 1) + \beta [(Bk + A) + (B - A) \lambda])}{(1 - \gamma)} \sqrt{|a_{k,1}| |a_{k,2}|} \leq \frac{((k + 1) + \beta [(Bk + A) + (B - A) \lambda])}{(1 - \gamma)} \sqrt{|a_{k,1}| |a_{k,2}|} \quad (k \geq m),$$  (4.8)

that is, that

$$\sqrt{|a_{k,1}| |a_{k,2}|} \leq \frac{((k + 1) + \beta [(Bk + A) + (B - A) \lambda])}{((k + 1) + \beta [(Bk + A) + (B - A) \lambda])} (1 - \gamma) \quad (k \geq m).$$  (4.9)
The result is sharp for the functions \( f \). Let the functions \( \Phi \) which completes the proof of the main assertion of Theorem 4.1.

It follows from (4.10) that

\[
\gamma \leq 1 - \frac{(B - A)\beta (1 + \beta B) (k + 1) (1 - \lambda)^2}{(k + 1) + \beta [(Bk + A) + (B - A) \lambda]} \Gamma_{k+1}(\alpha_1) (1 - \lambda) (k \geq m). \tag{4.11}
\]

Defining the function \( \Phi (k) \) by

\[
\Phi (k) = 1 - \frac{(B - A)\beta (1 + \beta B) (k + 1) (1 - \lambda)^2}{(k + 1) + \beta [(Bk + A) + (B - A) \lambda]} \Gamma_{k+1}(\alpha_1) (1 - \lambda) (k \geq m), \tag{4.12}
\]

we see that \( \Phi (k) \) is an increasing function of \( k \) (\( k \geq m \)). Therefore, we conclude from (4.11) that

\[
\gamma \leq \Phi (m) = 1 - \frac{(B - A)\beta (1 + \beta B) (m + 1) (1 - \lambda)^2}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]} \Gamma_{m+1}(\alpha_1) (1 - \lambda) (k \geq m). \tag{4.13}
\]

which completes the proof of the main assertion of Theorem 4.1.

**Corollary 4.1.** Let the functions \( f_j \) \((j = 1, 2)\) defined by (4.1) be in the class \( C_{q,s}^m (\alpha_1; A, B, \lambda, \beta) \). Then \((f_1 \ast f_2) (z) \in C_{q,s}^m (\alpha_1; A, B, \mu, \beta)\), where

\[
\mu = 1 - \frac{(B - A)\beta (1 + \beta B) (m + 1) (1 - \lambda)^2}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]} \Gamma_{m+1}(\alpha_1) (1 - \lambda). \tag{4.14}
\]

The result is sharp for the functions \( f_j \) \((j = 1, 2)\) given by

\[
f_j (z) = \frac{1}{z} + \frac{(B - A) \beta (1 - \lambda) z^m}{m \{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]\} \Gamma_{m+1}(\alpha_1)} \tag{4.15}
\]

**Theorem 4.2.** Let the functions \( f_j \) \((j = 1, 2)\) defined by (4.1) be in the class \( \Sigma_{q,s}^m (\alpha_1; A, B, \lambda, \beta) \). Then the function \( h(z) \) defined by

\[
h (z) = \frac{1}{z} + \sum_{k=m}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{4.16}
\]

belongs to the class \( \Sigma_{q,s}^m (\alpha_1; A, B, \xi, \beta) \), where

\[
\xi = 1 - \frac{2 (B - A) \beta (1 - \lambda)^2 (1 + \beta B) (m + 1)}{(m + 1) + \beta [(Bm + A) + (B - A) \lambda]} \Gamma_{m+1}(\alpha_1) + 2 (B - A)^2 \beta^2 (1 - \lambda)^2. \tag{4.17}
\]

The result is sharp for the functions \( f_j \) \((j = 1, 2)\) given by (4.4).

**Proof.** Noting that

\[
\sum_{k=m}^{\infty} \left[ \frac{\{(k + 1) + \beta [(Bk + A) + (B - A) \lambda]\} (\Gamma_{k+1}(\alpha_1))}{(B - A) \beta (1 - \lambda)} \right]^2 |a_{k,j}|^2 \tag{4.18}
\]

\[
\leq \left[ \sum_{k=m}^{\infty} \left\{ \frac{\{(k + 1) + \beta [(Bk + A) + (B - A) \lambda]\} (\Gamma_{k+1}(\alpha_1))}{(B - A) \beta (1 - \lambda)} \right| a_{k,j} \right|^2 \right] \leq 1,
\]
Corollary 4.2. Let the functions $f_j \in \Sigma_{q,s}^m (\alpha_1; A, B, \lambda, \beta)$ \ (j = 1, 2), we have
\[
\sum_{k=m}^{\infty} \frac{\{(k + 1) + \beta \left[(Bk + A) + (B - A) \lambda\right]\}^2 (\Gamma_{k+1}(\alpha_1))^2}{2 (B - A)^2 \beta^2 (1 - \lambda)^2} \left(|a_{k,1}|^2 + |a_{k,2}|^2\right) \leq 1.
\] (4.19)

Thus we need to find the largest $\xi$ such that
\[
\frac{\{(k + 1) + \beta \left[(Bk + A) + (B - A) \lambda\right]\}^2 (\Gamma_{k+1}(\alpha_1))^2}{2 (B - A) \beta (1 - \lambda)^2} (k \geq m),
\] (4.20)

that is, that
\[
\xi \leq 1 - \frac{2(B-A)\beta(1-\lambda)^2(1+\beta B)(k+1)}{\{(k+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2 \Gamma_{k+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2} (k \geq m).
\] (4.21)

Defining the function $\Theta (k)$ by
\[
\Theta (k) = 1 - \frac{2(B-A)\beta(1-\lambda)^2(1+\beta B)(k+1)}{\{(k+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2 \Gamma_{k+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2} (k \geq m),
\] (4.22)

we observe that $\Theta (k)$ is an increasing function of $k$ ($k \geq m$). Therefore, we conclude from (4.21) that
\[
\xi \leq \Theta (m) = 1 - \frac{2(B-A)\beta(1-\lambda)^2(1+\beta B)(m+1)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2 \Gamma_{m+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2},
\] (4.23)

which completes the proof of Theorem 4.2.

Corollary 4.2. Let the functions $f_j$ \ (j = 1, 2) defined by (4.1) be in the class $C_{q,s}^m (\alpha_1; A, B, \lambda, \beta)$. Then the function $h(z)$ defined by (4.18) belongs to the class $C_{q,s}^m (\alpha_1; A, B, \rho, \beta)$, where
\[
\rho = 1 - \frac{2 (B - A) \beta (1 - \lambda)^2 (1 + \beta B) (m + 1)}{m \{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2 \Gamma_{m+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2}.
\] (4.24)

The result is sharp for the functions $f_1$ and $f_2$ given by (4.15).

Remarks. (i) Putting $q = 2$ and $s = \alpha_1 = \alpha_2 = \beta_1 = 1$ in the above results, we get the results obtained by Aouf et al. \ [4, Lemmas 1 and 2 and Corollaries 1, 2, 3, 4, 7 and 8, respectively];

(ii) Putting $q = 2$, $s = \alpha_1 = \alpha_2 = \beta_1 = B = 1$ and $A = -1$, in Theorems 4.1, 4.2 and Corollaries 4.1, 4.2, we get the results obtained by Aouf et al. \ [4, Corollaries 5, 9, 6 and 10, respectively].

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References


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