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Local existence and blow up of solutions to a logarithmic nonlinear wave equation with time-varying delay

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Abstract. In this work, we are concerned with a problem of a logarithmic nonlinear wave equation with time-varying delay term. We established the local existence result and we proved a blow up result for the solution with negative initial energy under suitable conditions. This improves earlier results in the literature [11] for time-varying delay.

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1. Introduction

In this paper, we are concerned with the following problem

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t - \tau(t)) = u|u|^{p-2} ln|u|^k \\ u\left(x,t\right) = 0, x \in \partial\Omega, \\ u_t(x, t-\tau(0)) = f_0(x, t-\tau(0)), (x,t) \in \Omega \times (0, \tau(0)) \\ u\left(x,0\right) = u_0\left(x\right), u_t\left(x,0\right) = u_1\left(x\right), x \in \Omega, \end{cases} \tag{1.1}$$

where

$$(x,t) \in \Omega \times (0, +\infty),$$

and $\tau(t) > 0$ represents the time varying delay and $p \geq 2, k, \mu_1$ are positive constants, μ_2 is a real number.

This type of problems is encountered in many branches of physics such as Nuclear Physics, Optics and Geophysics. It is well known, from the Quantum Field Theory, that such kind of nonlinearity appears naturally in inflation cosmology and in super

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symmetric field theories (see [1], [2], [7], [8], [14]).
In [10], the authors considered the following problem

$$\begin{cases} u_{tt}-\Delta u+u-u\log|u|^2+u_t+u|u|^2=0, x\in\Omega, t\in[0,T] \\ u(x,t)=0, x\in\partial\Omega, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega. \end{cases} \tag{1.2}$$

The authors studied the global existence of weak solution. Another related mathematical work involving the logarithmic terms by Cazenave and Haraux [6], where they established the existence and uniqueness of a solution for the following problem in the $(\mathbb{R}^3)$

$$\begin{cases} u_{tt}-\Delta u+u_t-u\log|u|^2=0, \\ u(x,t)=0, x\in\partial\Omega, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \end{cases} \tag{1.3}$$

We can also mention some other works on the logarithmic Schrodinger equation as in [5], [4], [9].
In the case of constant delay, that is for $\tau(t)=\tau$, the system (1.1) has been studied by Kafini and Messaoudi [11], they considered with the following delay wave equation with logarithmic nonlinear source term

$$\begin{cases} u_{tt}-\Delta u+\mu_1 u_t+\mu_2 u_t(x,t-\tau)=u|u|^{p-2}\ln|u|^k \quad , \quad x\in\Omega, \quad t>0 \\ u(x,t)=0, \quad x\in\partial\Omega \\ u_t(x,t-\tau)=f_0(x,t-\tau), \quad t\in(0,\tau) \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \end{cases} \tag{1.4}$$

under the assumption $|\mu_2|\le\mu_1$, they established the local existence by the semigroup theory and proved a finite time blow up result.
The case of time-varying delay in the wave equation has been studied recently by Nicaice et al [13], they proved the exponential stability under the condition

$$\mu_2<\sqrt{1-d}\mu_1$$

where d is a constant satisfies

$$\tau'(t)\le d<1, \forall t>0 \tag{1.5}$$

For the wave equation ant with a time-varying delay, in [13] the authors which considers the system

$$\begin{cases} u_{tt}-\Delta u=0 \\ u(x,t)=0 \\ \dfrac{du}{dv}(x,t)=\mu_1 u_t(x,t)+\mu_2 u_t(x,t-\tau(t)), \end{cases}$$

where the time-varying delay $\tau(t)>0$ satisfies

$$0\le\tau(t)\le\overline{\tau}, \forall t>0 \tag{1.6}$$

$$\tau'(t)\le 1, \forall t>0 \tag{1.7}$$

and

$$\tau(t) \in W^{2,\infty}([0,T]), \forall T > 0 \tag{1.8}$$

They proved the exponential stability, under suitable conditions.
This paper is organized as follows: in the section 2, under the assumption

$$|\mu_2| \leq \sqrt{1-d}\mu_1, \tag{1.9}$$

we establish a local existence and in section 3, we prove a blow-up result under assumption on the delay by the energy method and Lyapunov function.

2. Local existence

In order to prove the existence of a unique solution of problem (1.1)-(2.6), we introduce the new variable

$$z(x,\rho,t) = u_t(x, t - \tau(t)\rho), \tag{2.1}$$

then we obtain

$$\begin{cases} \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0 \\ z(x,0,t) = u_t(x,t) \end{cases} \tag{2.2}$$

consequently, the problem is equivalent to

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = u|u|^{p-2} ln|u|^k. \\ \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0 \end{cases} \tag{2.3}$$

where

$$(x,\rho,t) \in \Omega \times (0,1) \times (0,\infty).$$

with the initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \quad in \ \ \partial\Omega \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) \\ z(x,\rho,0) = f_0(x, -\rho\tau(0)), \end{cases} \tag{2.4}$$

for all $(x,\rho,t) \in \Omega \times (0,1) \times (0,\infty)$, where the function $\tau(t)$ satisfies (1.5), (1.8) and the condition

$$0 < \tau_0 < \tau(t) < \overline{\tau}, \forall t > 0. \tag{2.5}$$

Let $v = u_t$ and denote by

$$U = (u,v,z)^T, \quad and \quad J(U) = (0, u|u|^{p-2}\ln|u|^k, 0)^T$$

Therefore, (1.1) can be rewritten as

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = J(U(t)), \quad t > 0 \\ U(0) = U_0 \end{cases} \tag{2.6}$$

where $U_0 = (u_0, u_1, f_0(., -\rho\tau(0))^T$ and the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} -v \\ -\Delta u + \mu_1 v + \mu_2 z(x,1,t) \\ \frac{(1-\tau'(t))}{\tau(t)} z_\rho. \end{pmatrix} \tag{2.7}$$

We define the energy space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega, (0, 1))$$

$\mathcal{H}$ is a Hilbert space with respect to the inner product

$$< U, \overline{U} >_{\mathcal{H}} \quad = \quad \int_\Omega \nabla u \nabla \overline{u} dx + \int_\Omega v \overline{v} dx + \int_\Omega \int_0^1 z \overline{z} d\rho dx \tag{2.8}$$

for all $U = (u, v, z)^T, \overline{U}(\overline{u}, \overline{v}, \overline{z})^T$.
The domain of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = \begin{pmatrix} (u, v, z)^T \in \mathcal{H} \quad / \quad u \in H^2(\Omega), v \in H_0^1(\Omega), z(x, 1, t) \in L^2(\Omega) \\ z, z_\rho \in L^2(\Omega, (0, 1))), z(x, 0, t) = v. \end{pmatrix} \tag{2.9}$$

Before establishing the local existence result, we need the following lemma

Lemma 2.1. *For any $\varepsilon > 0$, there exist $A > 0$, such that the real function*

$$j(s) = |s|^{p-2} \ln |s|, \qquad p > 2$$

satisfies

$$|j(s)| \leq A + |s|^{p-2+\varepsilon}$$

Proof. Since $\lim\limits_{|s| \to +\infty} \left(\frac{\ln |s|}{|s|^\varepsilon} \right) = 0$, then there exists $B > 0$, such that

$$\frac{\ln |s|}{|s|^\varepsilon} < 1, \qquad \forall |s| > B$$

So

$$|j(s)| \leq |s|^{p-2+\varepsilon}$$

since $p > 2$, then $|j(s)| \leq A$, for some $A > 0$ and for all $|\varepsilon| < B$
thus

$$|j(s)| \leq A + |s|^{p-2+\varepsilon}$$

then, we have following local existence result. □

Theorem 2.2. *Assume that (1.5)-(1.9) and*

$$\begin{cases} 2 < p < \dfrac{2(n-1)}{n-2}, & if \quad n \geq 3 \\ p > 2, & if \quad n = 1, 2 \end{cases} \tag{2.10}$$

then for all $U_0 \in \mathcal{H}$, problem (2.6) has a unique weak solution $U \in C([0, T], \mathcal{H})$.

Proof. We will show that $\mathcal{A}$ is a monotone maximal operator on $\mathcal{H}$ and J is a locally Lipschitz function on $\mathcal{H}$.
First, for all $U \in \mathcal{D}(\mathcal{A})$, we define the time-dependent inner-product on $\mathcal{H}$, (which is equivalent to the classical inner product).

$$\begin{aligned} < U, \overline{U} >_t \quad &= \quad \int_\Omega \nabla u \nabla \overline{u} dx + \int_\Omega v \overline{v} dx \\ &\quad + \xi \tau(t) \int_\Omega \int_0^1 z(x, \rho) \overline{z}(x, \rho) d\rho dx, \end{aligned} \tag{2.11}$$

where ξ satisfies

$$\frac{|\mu_2|}{\sqrt{1-d}} \leq \xi \leq \left(2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}}\right). \tag{2.12}$$

Thanks to hypothesis (1.9).
Let us set

$$\kappa(t) = \frac{(\tau'(t)^2+1)^{\frac{1}{2}}}{2\tau(t)}.$$

In this step, we prove the monotony of the operator $\overline{\mathcal{A}}(t) = \mathcal{A}(t) + \tau(t)I$.
For a fixed t and $U = (u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$, we have

$$\begin{aligned} < \mathcal{A}(t)U, U >_t &= \mu_1 \int_\Omega v^2 dx + \mu_2 \int_\Omega vz(x,1)dx \\ &+\xi \int_\Omega \int_0^1 (1-\tau'(t)\rho) z(x,\rho) z_\rho(x,\rho) d\rho dx. \end{aligned} \tag{2.13}$$

Observe that

$$\begin{aligned} \int_0^1 \int_0^1 (1-\tau'(t)\rho) z(x,\rho) z_\rho(x,\rho) d\rho dx &= \frac{1}{2}\int_0^1 \int_0^1 (1-\tau'(t)\rho) \frac{d}{d\rho} z^2 d\rho dx \\ &= \frac{\tau'(t)}{2} \int_0^1 \int_0^1 z^2(x,\rho) d\rho dx \\ &+\frac{1}{2}\int_0^1 z^2(x,1)(1-\tau'(t))dx \\ &-\frac{1}{2}\int_0^1 z^2(x,0)dx, \end{aligned} \tag{2.14}$$

whereupon

$$\begin{aligned} < \mathcal{A}(t)U, U >_t &= \mu_1 \int_\Omega v^2 dx + \mu_2 \int_\Omega vz(x,1)dx \\ &+\frac{\xi\tau'(t)}{2} \int_0^1 \int_0^1 z^2(x,\rho) d\rho dx \\ &+\frac{\xi}{2}\int_0^1 z^2(x,1)(1-\tau'(t))dx - \frac{\xi}{2}\int_0^1 v^2 dx. \end{aligned} \tag{2.15}$$

By using Cauchy-Schwartz inequality and (1.5), we get

$$\begin{aligned} < \mathcal{A}(t)U, U >_t &= \left(\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\xi}{2}\right) \int_0^1 v^2 dx \\ &+ \left(\xi\frac{(1-d)}{2} - \frac{|\mu_2|\sqrt{1-d}}{2}\right) \int_0^1 z^2(x,1)dx \\ &-\kappa(t) < U, U >_t . \end{aligned}$$

Condition (2.12) allows to write

$$\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\xi}{2} \geq 0 \quad , \quad \xi\frac{(1-d)}{2} - \frac{|\mu_2|\sqrt{1-d}}{2} \geq 0. \tag{2.16}$$

Consequently, the operator $\overline{\mathcal{A}}(t)$ is monotone. To show that $\mathcal{A}$ is maximal, we prove that each

$$F = (f_1, f_2, f_3)^T \in \mathcal{H}$$

there exists $U(u, v, z)^T \in \mathcal{D}(\mathcal{A})$, such that $(I + \mathcal{A})U = F$

$$\begin{cases} u - v = f_1 \\ v - \Delta u + \mu_1 v + \mu_2 z(x, 1, t) = f_2 \\ z + \frac{(1-\tau'(t))}{\tau(t)} z_\rho = f_3. \end{cases} \tag{2.17}$$

Noting that $v = u - f_1$, we have deduce from $(2.17)_3$

$$z(x, 0) = v(x), x \in \Omega. \tag{2.18}$$

Following the same approach as in [11], we obtain

$$\begin{cases} z(x,\rho) = v(x)e^{-\rho\tau(t)} + \tau(t)e^{-\rho\tau(t)} \displaystyle\int_0^\rho f_3(x,y)e^{y\tau(t)}dy, \quad if \quad \tau'(t) = 0 \\ z(x,\rho) = v(x)e^{\eta_\rho(t)} + e^{\eta_\rho(t)} \displaystyle\int_0^\rho \frac{\tau(t)}{1-\tau'(t)y} f_3(x,y)e^{-\eta_y(t)}dy, \quad if \quad \tau'(t) \neq 0, \end{cases}$$

where $\eta_\rho(t) = \frac{\tau(t)}{\tau'(t)} ln(1 - \tau'(t)\rho)$. Whereupon, from $(2.17)_1$, we obtain

$$\begin{cases} z(x,\rho) = u(x)e^{-\rho\tau(t)} - f_1 e^{-\rho\tau(t)} + \tau(t)e^{-\rho\tau(t)} \displaystyle\int_0^\rho f_3(x,y)e^{y\tau(t)}dy, \\ z(x,\rho) = u(x)e^{\eta_\rho(t)} - f_1 e^{\eta_\rho(t)} + e^{\eta_\rho(t)} \displaystyle\int_0^\rho \frac{\tau(t)}{1-\tau'(t)y} f_3(x,y)e^{-\eta_y(t)}dy, \end{cases} \tag{2.19}$$

and in particular

$$\begin{cases} z(x,1) = u(x)e^{-\tau(t)} + z_0(x), \quad if \quad \tau'(t) = 0 \\ z(x,1) = u(x)e^{\eta_1(t)} + z_0(x), \quad if \quad \tau'(t) \neq 0, \end{cases} \tag{2.20}$$

where

$$\begin{cases} z_0(x) = -f_1 e^{-\tau(t)} + \tau(t)e^{-\tau(t)} \displaystyle\int_0^1 f_3(x,y)e^{y\tau(t)}dy, \quad if \quad \tau'(t) = 0 \\ z_0(x) = -f_1 e^{\eta_1(t)} + e^{\eta_1(t)} \displaystyle\int_0^1 \frac{\tau(t)}{1-\tau'(t)y} f_3(x,y)e^{-\eta_y(t)}dy, \quad if \quad \tau'(t) \neq 0, \end{cases}$$

with

$$z_0 \in L^2(\Omega).$$

Substituting (2.20) in $(2.17)_2$, we get

$$\Gamma u - \Delta u = G,$$

where

$$\begin{cases} \Gamma = 1 + \mu_1 + \mu_2 e^{-\tau(t)}, \quad if \quad \tau'(t) = 0 \\ G = f_2 + (1 + \mu_1)f_1 - \mu_2 z_0 \in L^2(\Omega), \end{cases} \tag{2.21}$$

and

$$\begin{cases} \Gamma = 1 + \mu_1 + \mu_2 e^{\eta_1(t)}, \quad if \quad \tau'(t) \neq 0 \\ G = f_2 + (1+\mu_1) f_1 - \mu_2 z_0 \in L^2(\Omega). \end{cases} \tag{2.22}$$

Now, we define, over $H_0^1(\Omega)$, the bilinear and linear forms

$$B(u,\phi) = \Gamma \int_\Omega u\phi + \int_\Omega \nabla u.\nabla\phi, \qquad L(\phi) = G\phi$$

It is easy to verify that B is continuous and coercive and L is continuous on $H_0^1(\Omega)$. Then, Lax-Milgram theorem implies that the equation

$$B(u,\phi) = L(\phi), \quad \forall \phi \in H_0^1(\Omega), \tag{2.23}$$

has a unique solution $u \in H_0^1(\Omega)$. Hence, $v = u - f_1 \in H_0^1(\Omega)$.
Consequently, from (2.19), we have $z, z_\rho \in L^2(\Omega \times (0,1))$. Thus, $U \in \mathcal{H}$.
Using (2.23), we get

$$\Gamma \int_\Omega u\phi + \int_\Omega \nabla u.\nabla\phi = \int_\Omega G\phi, \quad \forall \phi \in H_0^1(\Omega).$$

The elliptic regularity theory implies that $u \in H_0^1(\Omega)$ and, in addition, Green's formula and $(2.17)_2$ give

$$\int_\Omega [(1+\mu_1)v - \Delta u + \mu_2 z(x,1,t) - f_2]\phi = 0, \quad \forall \phi \in H_0^1(\Omega).$$

Hence

$$(1+\mu_1)v - \Delta u + \mu_2 z(x,1,t) = f_2 \in L^2(\Omega).$$

Therefore,

$$U = (u,v,z)^T \in \mathcal{D}(\mathcal{A}).$$

Therefore, the operator $I + \mathcal{A}$ is surjective for any fixed $t > 0$. Since $\tau(t) > 0$ and

$$I + \overline{\mathcal{A}}(t) = (1+\kappa(t))I + \mathcal{A}(t),$$

we deduce that the operator $I + \overline{\mathcal{A}}(t)$ is also surjective for any $t > 0$ and then $\overline{\mathcal{A}}(t)$ is maximal.
Consequently, from the above analysis, we deduce that the problem

$$\begin{cases} \overline{U}_t + \overline{\mathcal{A}}(t)\overline{U} = 0 \\ \overline{U}(0) = U_0, \end{cases} \tag{2.24}$$

has a unique solution $\overline{U} \in C([0,\infty), \mathcal{H})$.
Now, let

$$U(t) = e^{\beta(t)}\overline{U}(t),$$

with $\beta(t) = \int_0^t \tau(s)ds$, then we have using (2.24)

$$\begin{aligned} U_t(t) &= \tau(t)e^{\beta(t)}\overline{U}(t) + e^{\beta(t)}\overline{U}_t(t) \\ &= \tau(t)e^{\beta(t)}\overline{U}(t) - e^{\beta(t)}\overline{\mathcal{A}}(t)\overline{U} \\ &= e^{\beta(t)}(\tau(t)\overline{U}(t) - \overline{\mathcal{A}}(t)\overline{U}) \\ &= e^{\beta(t)}\mathcal{A}(t)\overline{U} \\ &= \mathcal{A}(t)e^{\beta(t)}\overline{U} \\ &= \mathcal{A}(t)U(t). \end{aligned}$$

Consequently, $U(t)$ is the unique solution of problem.
Finally, we show that $J : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz. So, if we set

$$F(s) = |s|^{p-2}s \ln |s|^k,$$

then

$$F'(s) = k[1 + (p-1)\ln |s|]|s|^{p-2}.$$

Hence

$$\begin{aligned} \|J(U) - J(\overline{U})\|^2_{\mathcal{H}} &= \|(0, u|u|^{p-2}\ln |u|^k - \overline{u}|\overline{u}|^{p-2}\ln |\overline{u}|^k, 0, 0)\|^2_{\mathcal{H}} \\ &= \|u|u|^{p-2}\ln |u|^k - \overline{u}|\overline{u}|^{p-2}\ln |\overline{u}|^k\|^2_L \\ &= \|F(U) - F(\overline{U})\|^2_L. \end{aligned} \tag{2.25}$$

As a consequence of the mean value theorem, we have, for $0 \le \theta \le 1$,

$$\begin{aligned} |F(U) - F(\overline{U})| &= |F'(\theta u + (1-\theta)\overline{u})(u - \overline{u})| \\ &\le k[1 + (p-1)\ln|\theta u + (1-\theta)\overline{u}|]|\theta u + (1-\theta)\overline{u}|^{p-2}|u - \overline{u}| \\ &\le k|\theta u + (1-\theta)\overline{u}|^{p-2}|u - \overline{u}| \\ &\quad + k(p-1)|j(\theta u + (1-\theta)\overline{u})||u - \overline{u}|. \end{aligned} \tag{2.26}$$

By recalling Lemma 2.1, we arrive at

$$\begin{aligned} |F(U) - F(\overline{U})| &= k|\theta u + (1-\theta)\overline{u}|^{p-2}|u - \overline{u}| + k(p-1)A|u - \overline{u}| \\ &\quad + k(p-1)|\theta u + (1-\theta)\overline{u}|^{p-2+\varepsilon}|u - \overline{u}| \\ &\le k(|u| + |\overline{u}|)^{p-2}|u - \overline{u}| + k(p-1)A|u - \overline{u}| \\ &\quad + k(p-1)(|u| + |\overline{u}|)^{p-2+\varepsilon}|u - \overline{u}|. \end{aligned} \tag{2.27}$$

As $u, \overline{u} \in H^1_0(\Omega)$, we then use Holder's inequality and the Sobolev embedding

$$H^1_0(\Omega) \hookrightarrow L^r(\Omega), \quad \forall 1 \le r \le \frac{2n}{n-2},$$

to get

$$\begin{aligned}
\int_\Omega [(|u|+|\overline{u}|)^{p-2}|u-\overline{u}|]^2 dx &= \int_\Omega [(|u|+|\overline{u}|)^{2(p-2)}|u-\overline{u}|^2] dx \\
&\leq C\left(\int_\Omega (|u|+|\overline{u}|)^{2(p-2)} dx\right)^{\frac{(p-2)}{(p-1)}} \\
&\quad \times \left(\int_\Omega (|u-\overline{u}|)^{2(p-2)} dx\right)^{1/(p-1)} \\
&\leq C[\|u\|^{2(p-1)}_{L^{2(p-1)}(\Omega)} + \|\overline{u}\|^{2(p-1)}_{L^{2(p-1)}(\Omega)}]^{\frac{(p-2)}{(p-1)}} \\
&\quad \times \|u-\overline{u}\|^2_{L^{2(p-1)}(\Omega)} \\
&\leq C[\|u\|^{2(p-1)}_{H^1_0(\Omega)} + \|\overline{u}\|^{2(p-1)}_{H^1_0(\Omega)}]^{\frac{(p-2)}{(p-1)}} \\
&\quad \times \|u-\overline{u}\|^2_{H^1_0(\Omega)}.
\end{aligned} \tag{2.28}$$

Similarly, we estimate

$$\begin{aligned}
\int_\Omega [(|u|+|\overline{u}|)^{p-2+\varepsilon}|u-\overline{u}|]^2 dx &= \int_\Omega [(|u|+|\overline{u}|)^{2(p-2+\varepsilon)}|u-\overline{u}|^2] dx \\
&\leq C\left(\int_\Omega (|u|+|\overline{u}|)^{\frac{2(p-2+\varepsilon)(p-1)}{(p-2)}} dx\right)^{\frac{(p-2)}{(p-1)}} \\
&\quad \times \left(\int_\Omega (|u-\overline{u}|)^{2(p-2)} dx\right)^{1/(p-1)} \\
&\leq C\left(\int_\Omega (|u|+|\overline{u}|)^{2(p-1)+\frac{2\varepsilon(p-1)}{(p-2)}} dx\right)^{\frac{(p-2)}{(p-1)}} \\
&\quad \times \|u-\overline{u}\|^2_{L^{2(p-1)}(\Omega)}.
\end{aligned} \tag{2.29}$$

Since, $p < (n-1)/(n-2)$, we can choose $\varepsilon > 0$ so small that

$$p^* = 2(p-2) + \frac{2\varepsilon(p-1)}{(p-2)} \leq \frac{2n}{n-2}.$$

Hence, we have

$$\begin{aligned}
\int_\Omega [(|u|+|\overline{u}|)^{p-2+\varepsilon}|u-\overline{u}|]^2 dx &= C[\|u\|^{p^*}_{L^{p^*}(\Omega)} + \|\overline{u}\|^{p^*}_{L^{p^*}(\Omega)}]^{\frac{(p-2)}{(p-1)}} \\
&\qquad \|u-\overline{u}\|^2_{L^{2(p-1)}(\Omega)} \\
&\leq C[\|u\|^{p^*}_{H^1_0(\Omega)} + \|\overline{u}\|^{p^*}_{H^1_0(\Omega)}]^{\frac{(p-2)}{(p-1)}} \\
&\qquad \|u-\overline{u}\|^2_{H^1_0(\Omega)}.
\end{aligned} \tag{2.30}$$

Therefore, by combining (2.25)-(2.30), we obtain

$$\begin{aligned}\|J(U)-J(\overline{U})\|_{\mathcal{H}}^2 &= [k^2(p-1)^2A^2]\|u-\overline{u}\|_{H_0^1(\Omega)}^2\\ &\quad +C[(\|u\|_{H_0^1(\Omega)}^{2(p-1)}+\|\overline{u}\|_{H_0^1(\Omega)}^{2(p-1)})^{(p-2)/(p-1)}\\ &\quad +(\|u\|_{H_0^1(\Omega)}^{p^*}+\|\overline{u}\|_{H_0^1(\Omega)}^{p^*})^{(p-2)/(p-1)}]\|u-\overline{u}\|_{H_0^1(\Omega)}^2\\ &\leq C(\|u\|_{H_0^1(\Omega)},\|\overline{u}\|_{H_0^1(\Omega)})\|u-\overline{u}\|_{H_0^1(\Omega)}^2. \qquad (2.31)\end{aligned}$$

Therefore, J is locally Lipschitz. Thanks to ([12], [15]), the proof is completed. □

3. Blow up

We introduce the energy functional

Lemma 3.1. *Assume that (1.9)holds and the hypotheses (1.5), (1.8) and (2.2) are satisfied, let $u(t)$ be a solution of (1.1), then $E(t)$ is non-increasing, that is*

$$\begin{aligned}E(t) &= \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u_t\|_2^2+\frac{k}{p^2}\|u\|_p^p\\ &\quad +\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\\ &\quad -\frac{1}{p}\int_\Omega |u|^p\ln|u|^k dx. \qquad (3.1)\end{aligned}$$

satisfies

$$E(t)\leq -c_1(\|u_t\|_2^2+\int_\Omega z^2(x,1,t)dx)\leq 0. \qquad (3.2)$$

Proof. By multiplying the equation $(2.3)_1$ by u_t and integrating over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|_2^2+\mu_1\|u_t\|_2^2+\mu_2\int_\Omega u_t z(x,1,t)dx\\ &=\int_\Omega u_t u|u|^{p-2}\ln|u|^k dx. \qquad (3.3)\end{aligned}$$

Now, we multiply $(2.3)_2$ by ξz and integrate the resulting equation over $\Omega\times(0,1)$ with respect to ρ and x, respectively, to obtain

$$\begin{aligned}\frac{\xi}{2}\frac{d}{dt}\int_\Omega\int_0^1\tau(t)z^2(x,\rho,t)d\rho dx &= -\xi\int_\Omega\int_0^1(1-\tau'(t)\rho)zz_\rho d\rho dx\\ &\quad +\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\\ &= -\frac{\xi}{2}\int_\Omega\int_0^1\frac{d}{d\rho}(1-\tau'(t)\rho)z^2(x,\rho,t)d\rho dx\\ &= \frac{\xi}{2}\int_\Omega[z^2(x,0,t)-z^2(x,1,t)]dx\\ &\quad +\frac{\xi\tau'(t)}{2}\int_\Omega z^2(x,1,t)dx. \qquad (3.4)\end{aligned}$$

By (3.3) and (3.4), we get (3.1) and

$$\begin{aligned} \frac{d}{dt}E(t) &= -\left(\mu_1 - \frac{\xi}{2}\right)\|u_t\|_2^2 - \left(\frac{\xi\tau'(t)}{2} - \frac{\xi}{2}\right)\int_\Omega z(x,1,t)dx \\ &\quad -\mu_2\int_\Omega u_t z(x,1,t)dx. \end{aligned} \tag{3.5}$$

Thanks to Young's inequality, the last term in (3.5) can be estimated as follows

$$\mu_2\int_\Omega u_t z(x,1,t)dx \le \frac{|\mu_2|}{2\sqrt{1-d}}\int_\Omega u_t^2 dx + \frac{|\mu_2|\sqrt{1-d}}{2}\int_\Omega z^2(x,1,t)dx,$$

inserting (3.6) into (3.5), we obtain

$$\begin{aligned} \frac{d}{dt}E(t) &\le -\left(\mu_1 - \frac{\xi}{2} - \frac{|\mu_2|}{2\sqrt{1-d}}\right)\int_\Omega u_t^2 dx \\ &\quad -\left(\frac{\xi}{2}(\tau'(t)-1) - \frac{|\mu_2|\sqrt{1-d}}{2}\right)\int_\Omega z(x,1,t)dx. \end{aligned} \tag{3.6}$$

Then, by using (2.16) and (1.5) our conclusion holds. □

Lemma 3.2. *There exists a positive constant $c > 0$, depending on Ω only such that*

$$\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{s/p} \le c\left(\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right), \tag{3.7}$$

for any $u \in L^{p+1}(\Omega)$ and $2 \le s \le p$, provided that

$$\int_\Omega |u|^p \ln|u|^k dx \ge 0.$$

Proof. If $\displaystyle\int_\Omega |u|^p \ln|u|^k dx > 1$, then

$$\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{s/p} \le c[\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2]. \tag{3.8}$$

If $\int_\Omega |u|^p \ln|u|^k dx \le 1$, then we set

$$\Omega_1 = \{x \in \Omega, |u| > 1\}$$

and, for any $\beta \le 2$, we have

$$\begin{aligned} \left(\int_\Omega |u|^p \ln|u|^k dx\right)^{s/p} &\le \left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\beta/p} \le \left(\int_{\Omega_1} |u|^p \ln|u|^k dx\right)^{\beta/p} \\ &\le \left(\int_\Omega |u|^{p+1} dx\right)^{\beta/p} \le \left(\int_{\Omega_1} |u|^{p+1} dx\right)^{\beta/p} = \|u\|_{p+1}^{\beta(p+1)/p}.. \end{aligned}$$

We choose $\beta = 2p/(p+1) < 2$ to get

$$\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{s/p} \le \|u\|_{p+1}^2 \le c\|\nabla u\|_2^2. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain (3.6). □

Lemma 3.3. *There exists a positive constant $C>0$ depending on Ω only such that*

$$\|u\|_p^p \le c\left(\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right), \tag{3.10}$$

for any $u \in L^p(\Omega)$, provided that $\int_\Omega |u|^p \ln|u|^k dx \ge 0$.

Proof. We set

$$\begin{aligned} \Omega_+ &= \{x\in\Omega, |u|>e\} \\ \Omega_- &= \{x\in\Omega, |u|\le e\}, \end{aligned}$$

thus

$$\begin{aligned} \|u\|_p^p &= \int_{\Omega_+} |u|^p dx + \int_{\Omega_-} |u|^p dx \\ &\le \int_{\Omega_+} |u|^p \ln|u|^k dx + \int_{\Omega_-} e^p \left|\frac{u}{e}\right|^p dx \\ &\le \int_{\Omega_+} |u|^p \ln|u|^k dx + e^p \int_{\Omega_-} \left|\frac{u}{e}\right|^p dx \\ &\le \int_{\Omega} |u|^p \ln|u|^k dx + e^p \int_{\Omega} \left|\frac{u}{e}\right|^p dx \\ &\le c\left(\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right). \end{aligned}$$

Using the fact that $\|u\|_2^2 \le c\|u\|_p^2 \le c(\|u\|_p^p)^{2/p}$, we have ☐

Corollary 3.4. *There exists a positive constant $C>0$ depending on Ω only such that*

$$\|u\|_2^2 \le c\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{2/p} + \|\nabla u\|_2^{4/p}, \tag{3.11}$$

provided that $\int_\Omega |u|^p \ln|u|^k dx \ge 0$.

Lemma 3.5. *There exists a positive constant $C>0$ depending on Ω only such that*

$$\|u\|_p^s \le c[\|u\|_p^p + \|\nabla u\|_2^2, \tag{3.12}$$

for any $u\in L^{p+1}(\Omega)$ and $2\le s\le p$.

Proof. If $\|u\|_p \ge 1$ then

$$\|u\|_p^s \le \|u\|_p^p$$

If $\|u\|_p \le 1$ then, $\|u\|_p^s \le \|u\|_p^2$. Using Sobolev embedding theorems, we have

$$\|u\|_p^s \le \|u\|_p^2 \le c\|\nabla u\|_2^2.$$ ☐

Now we are ready to state and prove our main result. For this purpose, we define

$$\begin{aligned} H(t) = -E(t) &= \frac{1}{p}\int_\Omega |u|^p \ln|u|^k dx - \frac{1}{2}\|u_t\|_2^2 - \frac{k}{p^2}\|u\|_p^p \\ &\quad -\frac{1}{2}\|\nabla u_t\|_2^2 - \frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \end{aligned} \tag{3.13}$$

Theorem 3.6. *Assume* (1.5)*-*(1.9) *and* (2.10) *hold. Assume further that* $E(0) < 0$*, then the solution of problem* (1.1) *blow up in finite time.*

Proof. From (3.1), we have

$$E(t) \leq E(0) \leq 0. \tag{3.14}$$

Hence

$$\begin{aligned} H'(t) = -E'(t) &\geq c_1 \left(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)dx \right) \\ &\geq c_1 \int_\Omega z^2(x,1,t)dx \geq 0. \end{aligned} \tag{3.15}$$

and

$$0 \leq H(0) \leq H(t) \leq \frac{1}{p} \int_\Omega |u|^p \ln |u|^k dx. \tag{3.16}$$

We set

$$\mathcal{K}(t) = H^{1-\alpha} + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon \mu_1}{2} \int_\Omega u^2 dx. \tag{3.17}$$

where $\varepsilon > 0$ to be specified later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{3.18}$$

By multiplying $(1.1)_1$ by u and taking a derivative of (3.17), we obtain

$$\begin{aligned} \mathcal{K}'(t) &= (1-\alpha)H^{-\alpha}H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 \\ &+\varepsilon \int_\Omega |u|^p \ln |u|^k dx - \varepsilon\mu_2 \int_\Omega u z(x,1,t)dx. \end{aligned} \tag{3.19}$$

Using

$$\varepsilon\mu_2 \int_\Omega u z(x,1,t)dx \leq \varepsilon|\mu_2|\{\delta_1\|u\|_2^2 + \frac{1}{4\delta_1}\int_\Omega z^2(x,1,t)dx\}. \tag{3.20}$$

we obtain, from (3.19),

$$\begin{aligned} \mathcal{K}'(t) &\geq (1-\alpha)H^{-\alpha}H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 + \varepsilon \int_\Omega |u|^p \ln |u|^k dx \\ &-\varepsilon|\mu_2|\{\delta_1\|u\|_2^2 + \frac{1}{4\delta_1}\int_\Omega z^2(x,1,t)dx\}. \end{aligned} \tag{3.21}$$

Therefore, using (3.15) and by setting δ_1 so that, $\dfrac{|\mu_2|}{4\delta_1 c_1} = \kappa H^{-\alpha}(t)$, substituting in (3.21), we get

$$\begin{aligned} \mathcal{K}'(t) &\geq [(1-\alpha) - \varepsilon\kappa]H^{-\alpha}H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 \\ &+\varepsilon \int_\Omega |u|^p \ln |u|^k dx - \varepsilon\frac{H^\alpha(t)}{4c_1\kappa}|\mu_2|^2\|u\|_2^2. \end{aligned} \tag{3.22}$$

For $0 < a < 1$, from (3.13)

$$\begin{aligned}
\varepsilon\int_\Omega |u|^p \ln|u|^k dx &= \varepsilon p(1-a)H(t) + \frac{\varepsilon p(1-a)}{2}\|u_t\|_2^2 + \frac{\varepsilon(1-a)k}{p}\|u\|_p^p \\
&+\frac{\varepsilon p(1-a)}{2}\|\nabla u_t\|_2^2 + \varepsilon a\int_\Omega |u|^p\ln|u|^k dx \\
&+\frac{\varepsilon p(1-a)}{2}\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \qquad (3.23)
\end{aligned}$$

substituting in (3.22), we get

$$\begin{aligned}
\mathcal{K}'(t) &\geq [(1-\alpha)-\varepsilon\kappa]H^{-\alpha}H'(t) + \varepsilon[\frac{p(1-a)}{2}+1]\|u_t\|_2^2 \\
&+\varepsilon\left[\left(\frac{p(1-a)}{2}-1\right)\right]\|\nabla u\|_2^2 \\
&+a\varepsilon\int_\Omega |u|^p\ln|u|^k dx - \varepsilon\frac{H^\alpha(t)}{4c_1\kappa}|\mu_2|^2\|u\|_2^2 \\
&+\varepsilon p(1-a)H(t) + \frac{\varepsilon(1-a)k}{p}\|u\|_p^p \\
&+\frac{\varepsilon p(1-a)}{2}\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \qquad (3.24)
\end{aligned}$$

Using (3.11), (3.16) and Young's inequality, we find

$$\begin{aligned}
H^\alpha(t)\|u\|_2^2 &\leq \int_\Omega |u|^p\ln|u|^k dx)^\alpha\|u\|_2^2 \\
&\leq c\Big\{\int_\Omega |u|^p\ln|u|^k dx)^{\alpha+2/p} + \left(\int_\Omega |u|^p\ln|u|^k dx\right)^\alpha \|\nabla u\|_{4/p}^2\Big\} \\
&\leq c\Big\{\int_\Omega |u|^p\ln|u|^k dx)^{\frac{(p\alpha+2)}{p}} + \|\nabla u\|_2^2 \\
&\quad + \left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\frac{p\alpha}{(p-2)})}\Big\}
\end{aligned} \qquad (3.25)$$

Exploiting (3.18), we have

$$2 < p\alpha + 2 \leq p, \quad and \quad 2 < \frac{\alpha p^2}{p-2} \leq p.$$

Thus, lemma 3.2 yields

$$H^\alpha(t)\|u\|_2^2 \leq c\left(\int_\Omega |u|^p\ln|u|^k dx + \|\nabla u\|_2^2\right). \qquad (3.26)$$

Combining (3.24) and (3.26), we obtain

$$\begin{aligned}\mathcal{K}'(t) \quad &\geq \quad [(1-\alpha)-\varepsilon\kappa]H^{-\alpha}H'(t)+\varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|_2^2\\ &\quad +\varepsilon\left\{\left(\frac{p(1-a)}{2}-1\right)-\frac{c|\mu_2|^2}{4c_1\kappa}\right\}\|\nabla u\|_2^2\\ &\quad +\varepsilon\left[a-\frac{c|\mu_2|^2}{4c_1\kappa}\right]\int_\Omega |u|^p\ln|u|^k dx\\ &\quad +\varepsilon p(1-a)H(t)+\frac{\varepsilon(1-a)k}{p}\|u\|_p^p\\ &\quad +\frac{\varepsilon p(1-a)}{2}\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \qquad (3.27)\end{aligned}$$

At this point, we choose $a>0$ so small that

$$\alpha_1=\frac{p(1-a)}{2}-1>0$$

then we choose κ so large that

$$\alpha_2=\left(\frac{p(1-a)}{2}-1\right)-\frac{c|\mu_2|^2}{4c_1\kappa}>0$$

and

$$\alpha_3=a-\frac{c|\mu_2|^2}{4c_1\kappa}>0$$

Once κ and a are fixed, we pick ε so small so that

$$\alpha_4=(1-\alpha)-\varepsilon\kappa>0$$

Thus, for some $\beta>0$, estimate (3.27) becomes

$$\begin{aligned}\mathcal{K}'(t) \quad &\geq \quad \beta\{H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\int_\Omega |u|^p\ln|u|^k dx+\|u\|_p^p\\ &\quad +\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\}. \qquad (3.28)\end{aligned}$$

and

$$\mathcal{K}(t)\geq\mathcal{K}(0)>0,\quad t>0. \qquad (3.29)$$

Next, using Holder's and Young's inequalities, we have

$$\|u\|_2=\left(\int_\Omega u^2dx\right)^{\frac{1}{2}}\leq\left[\left(\int_\Omega(|u|^2)^{p/2}dx\right)^{\frac{2}{p}}\left(\int_\Omega 1dx\right)^{1-\frac{2}{p}}\right]^{\frac{1}{2}}\leq c\|u\|_p. \qquad (3.30)$$

and

$$\left|\int_\Omega uu_tdx\right|\leq\|u\|_2.\|u_t\|_2\leq c\|u\|_p.\|u_t\|_2$$

which implies

$$\begin{aligned}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} &\geq c\|u\|_p^{\frac{1}{1-\alpha}}.\|u_t\|_2^{\frac{1}{1-\alpha}}\\ &\leq c\left[\|u\|_p^{\frac{\mu}{1-\alpha}}+\|u_t\|_2^{\frac{\theta}{1-\alpha}}\right]. \end{aligned}\tag{3.31}$$

where $\dfrac{1}{\mu}+\dfrac{1}{\theta}=1.$
we take $\theta=2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha}=\frac{2}{1-2\alpha}\leq p$$

Therefore, for $s=2/(1-2\alpha)$, we obtain

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}}\leq c[\|u\|_p^s+\|u_t\|_2^2].$$

hence, lemma 3.3 gives

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} \leq c\left[\|\nabla u\|_2^2+\|u_t\|_2^2+\|u\|_p^p\right]. \tag{3.32}$$

Therefore,

$$\begin{aligned}\mathcal{K}^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}+\varepsilon\int_\Omega uu_t dx+\frac{\varepsilon\mu_1}{2}\int_\Omega u^2 dx\right)^{\frac{1}{1-\alpha}}\\ &\leq c\left[H(t)+|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}}+\|u\|_2^{\frac{2}{1-\alpha}}\right]\\ &\quad c[H(t)+\|\nabla u\|_2^2+\|u_t\|_2^2+\|u\|_p^p]. \end{aligned}\tag{3.33}$$

According to (3.28) and (3.33), we get

$$\mathcal{K}'(t)\geq\lambda\mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{3.34}$$

where $\lambda>0$, depending only on β and c.
A simple integration of (3.34), we obtain

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t)\geq\frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0)-\lambda\frac{\alpha}{(1-\alpha)}t}$$

Therefore, $\mathcal{K}(t)$ blows up in time

$$T\leq T^*=\frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}$$

This completes the proof. □

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