Finite time blow-up for quasilinear wave equations with nonlinear dissipation

Mohamed Amine Kerker

Abstract. In this paper we consider a class of quasilinear wave equations
\[ u_{tt} - \Delta_\alpha u - \omega_1 \Delta u_t - \omega_2 \Delta \beta u_t + \mu |u_t|^{m-2}u_t = |u|^{p-2}u, \]
associated with initial and Dirichlet boundary conditions. Under certain conditions on \( \alpha, \beta, m, p \), we show that any solution with positive initial energy, blows up in finite time. Furthermore, a lower bound for the blow-up time will be given.

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1. Introduction

In this paper, we would like to study the blow-up of solutions of the following initial boundary value problem of a quasilinear wave equation

\[
\begin{cases}
  u_{tt} - \Delta_\alpha u - \omega_1 \Delta u_t - \omega_2 \Delta \beta u_t + \mu |u_t|^{m-2}u_t = |u|^{p-2}u, & x \in \Omega, \quad t > 0, \\
  u(x, t) = 0, & x \in \partial \Omega, \quad t > 0, \\
  u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega.
\end{cases}
\]

Here, \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \). Additionally, we assume that

\[ u_0 \in W^{1,\alpha}_0(\Omega), \quad u_1 \in L^2(\Omega), \]

and \( \alpha, \beta, \omega_1, \omega_2, \mu, m, p \) are positive constants, with

\[
\begin{cases}
  2 < p \leq \alpha^*, \quad \alpha^* = \frac{\alpha n}{n-\alpha}, & \text{for } n > \alpha, \\
  2 < p < \infty, & \text{for } n = \alpha.
\end{cases}
\]

The operator \( \Delta_\alpha \) is the classical \( \alpha \)-Lapacian given by:

\[ \Delta_\alpha u = \text{div} \left( |\nabla u|^{\alpha-2} \nabla u \right). \]
Notice that $\Delta_{\beta}u_t$ is a quasilinear strong damping term, and it is degenerate when $\beta > 2$.

Nonlinear hyperbolic equations of the type (1.1) have been investigated in the papers [2, 5, 7, 9, 15], and the references therein. Several examples of this type arise in physics, for example, the problem (1.1) represents a longitudinal motion of a viscoelastic rod obeying the nonlinear Voight model.

Zhijiang [14] proved a blow up result for the problem (1.1) when the initial energy is sufficiently negative. This result was extended by Messaoudi and Houari [8] to a situation when the solution has negative initial energy. Liu and Wang [6] studied a more general model including (1.1), and by improving the arguments in [14] and [8] they established a blow-up result in the subcritical initial energy case, i.e. $E(0) < d$, where $E(0)$ is the initial energy and $d$ is the depth of the potential well.

For $\alpha = \beta = m = 2$, equation in (1.1) reduces to the linearly damped wave equation

\[ u_{tt} - \Delta u + \omega \Delta u_t + \mu u_t = |u|^{p-2}u. \quad (1.4) \]

Gazzola and Squassina [3] studied (1.4) and gave a necessary and sufficient conditions for blow-up if $E(0) < d$. Recently, Yang and Xu [13] gave a sufficient condition for blow-up if $E(0) > d$. Sun et al. [12] obtained, for (1.4), an estimate of the lower bound for the blow-up time when $2 < p \leq \frac{2(n-1)}{n-2}$. This work was extended by Guo and Liu [4] to the case when the exponent $p \in \left( \frac{2(n-1)}{n-2}, \frac{2(n^2-2)}{n-2} \right]$. Later, in the case of $\omega > 0$, Baghaei [1] improved the results in [12] and [4] by enlarging the upper bound for $p$ to $2^*$.

In related work, Song and Xue [11] studied the following nonlinear wave equation with strong damping

\[ u_{tt} - \Delta u + \int_0^t g((t - \tau)\Delta u(\tau)d\tau - \Delta u_t = |u|^{p-2}u. \quad (1.5) \]

They introduced a new technique to obtain a finite time blow-up result with arbitrary high initial energy in the case of linear strong damping. By applying the technique similar to that in [11], Song [10] extended the result in [11] to the case of nonlinear weak damping $\mu |u_t|^{m-2}u_t$ in place of $-\Delta u_t$ in (1.5).

In this paper, by using the technique in [10], we give sufficient conditions for finite time blow-up of solutions of (1.1), in the case $E(0) \geq d$. Furthermore, by using the techniques in [4], we obtain a lower bound for the blow-up time.

2. Preliminaries

We denote by $\| \cdot \|_p$ the $L^p(\Omega)$ norm ($2 \leq p < \infty$), and by $(\cdot, \cdot)$ the $L^2$ inner product. We introduce the following functional space

\[ \mathcal{H} := L^\infty([0, T), W^{1, \alpha}_0(\Omega)) \cap W^{1, \infty}([0, T), L^2(\Omega)) \]

\[ \cap W^{1, \beta}([0, T), W^{1, \beta}(\Omega)) \cap W^{1, m}([0, T), L^m(\Omega)), \]
for $T > 0$, and the energy functional

$$E(t) := \frac{1}{2} \|\nabla u\|_{\alpha}^\alpha + \frac{1}{2} \|u_t\|_{2}^2 - \frac{1}{p} \|u\|_{p}^p.$$ 

We define also the following constant

$$\lambda = B_{\ast}^{-\frac{p}{p-\alpha}},$$

where $B_{\ast}$ is the best constant of the Sobolev embedding $W^{1,\alpha}_0(\Omega) \hookrightarrow L^p(\Omega)$. Finally, we characterize the depth of the potential well $d$ as follows:

$$d = \left( \frac{1}{\alpha} - \frac{1}{p} \right) \lambda^2.$$

**Lemma 2.1.** Let $u$ be a global solution to problem (1.1). Then we have

$$E'(t) = -\omega_1 \|\nabla u_t\|_{2}^2 - \omega_2 \|\nabla u_t\|_{\beta}^\beta - \mu \|u_t\|_{m}^m, \quad \forall t \geq 0.$$  

As a consequence, we have the following inequalities:

$$E(t) \leq E(0), \quad \forall t \geq 0,$$  

(2.1)

and

$$-E'(t) \geq \omega_1 \|\nabla u_t\|_{2}^2, \quad -E'(t) \geq \omega_2 \|\nabla u_t\|_{\beta}^\beta, \quad -E'(t) \geq \mu \|u_t\|_{m}^m.$$  

(2.2)

Subsequently, we state the following theorems (see [6]).

**Theorem 2.2 (Local existence).** Assume that conditions (1.2) and (1.3) hold. Then problem (1.1) has a unique local solution $u \in \mathcal{H}$.

**Theorem 2.3 (Blow-up for $E(0) < d$).** Assume (1.2) and (1.3) hold. Assume further that $\alpha, \beta, m \geq 2$ and $p > \alpha > \max \{m, \beta\}$. Suppose $E(0) < d$ and

$$\|\nabla u_0\|_{\alpha} > \lambda.$$  

(3.1)

Then $u$ blows up in finite time.

**3. Finite time blow-up**

In this section we extend the blow-up result in [8] to the case $E(0) \geq d$. Here is our main result:

**Theorem 3.1 (Blow-up for $E(0) \geq d$).** Assume (1.2), (2.3) and (1.3) hold. Assume further that $\alpha, \beta, m > 2$, $\alpha > \beta$ and $p > \max \{m, \alpha\}$. Suppose $E(0) \geq d$ and

$$(u_t(0), u(0)) > ME(0),$$  

(3.1)

where $M > 0$ is defined in (3.7), then the solution $u \in \mathcal{H}$ of (1.1) blows up in finite time.
Proof. Assume by contradiction that $u(t)$ is a global solution of (1.1). Setting

$$F(t) := \frac{1}{2} \|u(t)\|_2^2,$$

it follows from (1.1) that

$$F''(t) = \|u_t\|_2^2 + \|u\|_p^p - \|\nabla u\|_\alpha^\alpha - \omega_1(\nabla u_t, \nabla u) - \omega_2(\|\nabla u_t\|^{\beta-2}\nabla u_t, u) - \mu(\|u_t\|^{m-2}u_t, u). \tag{3.2}$$

By using Hölder’s inequality and Young’s inequality, we estimate the two last terms in the right-hand side of the previous equation, as follows

$$(\nabla u_t, \nabla u) \leq \eta \|\nabla u\|_2^2 + \frac{1}{2\eta} \|\nabla u_t\|_2^2, \quad \eta > 0,$$

$$(\|\nabla u_t\|^{\beta-2}\nabla u_t, u) \leq \frac{1}{\beta} \sigma^\beta \|\nabla u\|_\beta^\beta + \frac{\beta-1}{\beta} \sigma^{\beta/(1-\beta)} \|\nabla u_t\|_\beta^\beta, \quad \sigma > 0,$$

$$\|u_t\|^{m-2}u_t, u) \leq \frac{1}{m} \delta^m \|u\|_m^m + \frac{m-1}{m} \delta^{m/(1-m)} \|u_t\|_m^m, \quad \delta > 0.$$ 

So, that thanks to the convexity of the function $y^\alpha/x$ for $y \geq 0$ and $x > 0$, we have

$$\frac{\delta^m}{m} \|u\|_m^m \leq \frac{s}{2} \delta^m \|u\|_2^2 + \frac{1-s}{p} \delta^m \|u\|_p^p, \quad s = \frac{p-m}{p-2},$$

$$\frac{1}{\beta} \sigma^\beta \|\nabla u\|_\beta^\beta \leq \frac{\theta}{2} \sigma^\beta \|\nabla u\|_2^2 + \frac{1-\theta}{\alpha} \sigma^\beta \|\nabla u\|_\alpha^\alpha, \quad \theta = \frac{\alpha - \beta}{\alpha - 2}.$$ 

Hence, (3.2) becomes

$$F''(t) \geq \|u_t\|_2^2 - \left[1 + \frac{\omega_2(1-\theta)}{\alpha} \sigma^\beta \right] \|\nabla u\|_\alpha^\alpha - \frac{\mu s}{2} \delta^m \|u\|_2^2$$

$$- \left(\omega_1 \eta + \frac{\omega_2 \theta}{2} \sigma^\beta \right) \|\nabla u\|_2^2 + \left[1 - \frac{\mu(1-s)}{p} \delta^m \right] \|u\|_p^p$$

$$- \frac{\omega_1}{4\eta} \|\nabla u\|_2^2 - \frac{\omega_2}{\beta} \|\nabla u_t\|_\beta^{\beta-1} \|\nabla u_t\|_\beta^\beta - \mu \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \|u_t\|_m^m. \tag{3.3}$$

Next, since $u(t)$ is global and $E(0) \geq d$, then by Theorem 2.3, $E(t) \geq d, \forall t \geq 0$. Thus, using the embedding $L^\alpha(\Omega) \hookrightarrow L^2(\Omega)$ and the inequality

$$z^b \leq (z + a) \left(z + \frac{1}{a}\right), \quad z \geq 0, \quad 0 < b \leq 1, \quad a > 0,$$

we obtain

$$\|\nabla u\|_2^2 \leq c \|\nabla u\|_\alpha^\alpha$$

$$= c \left[\|\nabla u\|_\alpha^\alpha\right]^{2/\alpha}$$

$$\leq c \left(1 + \frac{1}{d}\right) \left[\|\nabla u\|_\alpha^\alpha + d\right]$$

$$\leq C \left[\|\nabla u\|_\alpha^\alpha + E(t)\right], \quad \forall t \geq 0. \tag{3.4}$$
By using Lemma 2.1 and (2.2), we get
\[
\frac{d}{dt} \left\{ F'(t) - \left[ \frac{1}{4\eta} + \frac{\beta - 1}{\beta} \sigma^{\frac{\beta}{\beta-1}} + \frac{m-1}{m} \delta^{\frac{-m}{m-1}} \right] E(t) \right\} 
\geq F''(t) + \frac{\omega_1}{4\eta} \| \nabla u_t \|_2^2 + \omega_2 \frac{\beta - 1}{\beta} \sigma^{\frac{\beta}{\beta-1}} \| \nabla u_t \|_\beta^2 + \mu \frac{m-1}{m} \delta^{\frac{-m}{m-1}} \| u_t \|_m^m.
\]

Adding and subtracting \( p(1 - \varepsilon) E(t) \), for \( \varepsilon \in (0, 1) \), in the right-hand side of the last inequality, and using (3.4) and the Poincaré inequality we obtain
\[
\frac{d}{dt} \left\{ F'(t) - \left[ \frac{1}{4\eta} + \frac{\beta - 1}{\beta} \sigma^{\frac{\beta}{\beta-1}} + \frac{m-1}{m} \delta^{\frac{-m}{m-1}} \right] E(t) \right\} 
\geq \| u_t \|_2^2 - \frac{\mu s}{2} \delta^m \| u \|_2^2 - \left[ 1 + \frac{\omega_2(1 - \theta)}{\alpha} \sigma^\beta \right] \| \nabla u \|^\alpha_\alpha
\]
\[
- \left( \frac{\omega_1}{2} + \frac{\omega_2 \theta}{2} \sigma^\beta \right) \| \nabla u \|_2^2 + \left[ 1 - \frac{\mu(1 - s)}{p} \delta^m \right] \| u \|^p_p
\]
\[
\geq \left[ 1 + \frac{p}{2} (1 - \varepsilon) \right] \| u_t \|_2^2 - \frac{\mu s}{2} \delta^m \| u \|_2^2 + k(\varepsilon) \| \nabla u \|_2^2
\]
\[
- \left( \frac{\omega_1}{2} + \frac{\omega_2 \theta}{2} \sigma^\beta \right) \| \nabla u \|_2^2 + \left[ \varepsilon - \frac{\mu(1 - s)}{p} \delta^m \right] \| u \|^p_p - p(1 - \varepsilon) E(t)
\]
\[
\geq \left[ 1 + \frac{p}{2} (1 - \varepsilon) \right] \| u_t \|_2^2 - \frac{\mu s}{2} \delta^m \| u \|_2^2 + \gamma(\varepsilon) \| \nabla u \|_2^2
\]
\[
+ \left[ \varepsilon - \frac{\mu(1 - s)}{p} \delta^m \right] \| u \|^p_p - [k(\varepsilon) + p(1 - \varepsilon)] E(t)
\]
\[
\geq \left[ 1 + \frac{p}{2} (1 - \varepsilon) \right] \| u_t \|_2^2 + \left\{ \gamma(\varepsilon)B - \frac{\mu s}{2} \delta^m \right\} \| u \|_2^2
\]
\[
+ \left[ \varepsilon - \frac{\mu(1 - s)}{p} \delta^m \right] \| u \|^p_p - [k(\varepsilon) + p(1 - \varepsilon)] E(t), \tag{3.5}
\]

where
\[
k(\varepsilon) = \frac{1}{\alpha} \left[ p(1 - \varepsilon) - \alpha - \omega_2(1 - \theta) \sigma^\beta \right],
\]
\[
\gamma(\varepsilon) = \frac{k(\varepsilon)}{C^\varepsilon} - \omega_1 \eta - \frac{\omega_2 \theta}{2} \sigma^\beta,
\]
and \( B \) is the best constant of Poincaré inequality
\[
\| \nabla u \|_2^2 \geq B \| u \|_2^2.
\]

Therefore, taking \( \eta = \varepsilon, \sigma = \varepsilon, \)
\[
\delta = \left[ \frac{p \varepsilon}{\mu(1 - s)} \right]^{1/m},
\]
setting
\[
\gamma_1(\varepsilon) = \frac{1}{4\varepsilon} + \frac{\beta - 1}{\beta} \varepsilon^{\frac{\beta}{\beta-1}} + \frac{m-1}{m} \left( \frac{1 - s}{p \varepsilon} \right)^{-\frac{1}{m-1}},
\]
and substituting in (3.5), we arrive at
\[
\frac{d}{dt} [F'(t) - \gamma_1(\varepsilon)E(t)] \geq \left[1 + \frac{p}{2}(1 - \varepsilon)\right] \|u_t\|^2 \nonumber
\]
\[+ \left[\gamma(\varepsilon)B - \frac{ps}{2(1-s)}\varepsilon\right] \|u\|^2 - [k(\varepsilon) + p(1 - \varepsilon)] E(t). \nonumber\]

By using the Schwarz inequality, we have
\[
2 \left[1 + \frac{p}{2}(1 - \varepsilon)\right]^{1/2} \left[\gamma(\varepsilon)B - \frac{ps}{2(1-s)}\varepsilon\right]^{1/2} (u_t, u) \nonumber
\]
\[\leq \left[1 + \frac{p}{2}(1 - \varepsilon)\right] \|u_t\|^2 + \left[\gamma(\varepsilon)B - \frac{ps}{2(1-s)}\varepsilon\right] \|u\|^2. \nonumber\]

Consequently, we obtain
\[
\frac{d}{dt} [F'(t) - \gamma_1(\varepsilon)E(t)] \geq a(\varepsilon)(u_t, u) - [k(\varepsilon) + p(1 - \varepsilon)] E(t) \nonumber
\]
\[= a(\varepsilon) [F'(t) - \gamma_2(\varepsilon)E(t)], \quad (3.6) \nonumber\]

where
\[
a(\varepsilon) = 2 \left[1 + \frac{p}{2}(1 - \varepsilon)\right]^{1/2} \left[\gamma(\varepsilon)B - \frac{ps}{2(1-s)}\varepsilon\right]^{1/2}, \nonumber\]
\[
\gamma_2(\varepsilon) = \frac{k(\varepsilon)+p(1-\varepsilon)}{a(\varepsilon)}. \nonumber\]

Since
\[
\gamma(\varepsilon)B - \frac{ps}{2(1-s)}\varepsilon \rightarrow \begin{cases} 
\frac{B(p-\alpha)}{\alpha C} > 0 & \text{as } \varepsilon \to 0^+ \\
-\left[\frac{\alpha+\omega_1(1-\theta)}{\alpha C} + \omega_1 + \frac{\omega_2^2}{2}\right] B - \frac{ps}{2(1-s)} < 0 & \text{as } \varepsilon \to 1^-,
\end{cases} \nonumber\]

then, there exists \(\varepsilon_* \in (0,1)\), such that
\[
\begin{cases} 
\gamma_1(\varepsilon) = 0 & \text{and } a(\varepsilon) > 0, \quad \forall \varepsilon \in (0,\varepsilon_*). \\
\end{cases} \nonumber\]

Hence, we have
\[
\gamma_1(\varepsilon) - \gamma_2(\varepsilon) \rightarrow \begin{cases} 
+\infty & \text{as } \varepsilon \to 0^+ \\
-\infty & \text{as } \varepsilon \to \varepsilon_*^-.
\end{cases} \nonumber\]

Therefore, there exists \(\varepsilon_0 \in (0,\varepsilon_*)\), such that \(\gamma_1(\varepsilon_0) = \gamma_2(\varepsilon_0) > 0\). So, by setting
\[
L(t) = F'(t) - \gamma_1(\varepsilon_0)E(t), \\
M = \gamma_1(\varepsilon_0), \quad (3.7) \nonumber\]

and by using (2.3), we obtain
\[
L(0) = (u_t(0), u(0)) - \gamma_1(\varepsilon_0)E(0) > (u_t(0), u(0)) - ME(0) > 0. \nonumber\]

Moreover, with this choice of \(\varepsilon_0\), (3.6) becomes
\[
\frac{d}{dt} L(t) \geq a(\varepsilon_0)L(t), \nonumber\]
which gives
\[ L(t) \geq L(0)e^{a(\varepsilon_0)t}, \quad \forall t \geq 0, \]
and hence
\[ F'(t) \geq L(0)e^{a(\varepsilon_0)t}, \quad \forall t \geq 0. \]
By integrating this last inequality over \((0, t)\), we get
\[
\|u(t)\|^2_2 = 2F(t) \geq 2F(0) + 2\frac{L(0)}{a(\varepsilon_0)} \left[ e^{a(\varepsilon_0)t} - 1 \right], \quad \forall t \geq 0. \tag{3.8}
\]
On the other hand, by using Hölder’s inequality and (2.2), we have
\[
\|u(t)\|_2 \leq \|u(0)\|_2 + \int_0^t \|u_\tau(\tau)\|_2 d\tau \\
\leq \|u(0)\|_2 + C \int_0^t \|u_\tau(\tau)\|_m d\tau \\
\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \int_0^t \|u_\tau(\tau)\|_m^2 d\tau \\
\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \int_0^t \frac{1}{\mu} dE(\tau) d\tau \\
\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \left[ \frac{E(0) - E(t)}{\mu} \right]^{1/m} \\
\leq \|u(0)\|_2 + C \left[ \frac{E(0)}{\mu} \right]^{1/m} t^{\frac{m-1}{m}},
\]
which clearly contradicts (3.8).

4. Lower bound for the blow-up time

In this section, we give a lower bound for the blow-up time \(T_{\text{max}}\). To this end, we define
\[ G(t) := \frac{1}{p} \|u(t)\|_p^p. \]

**Theorem 4.1.** Let \(u\) be the solution of (1.1), and assume that
\[
\begin{cases} 
2 < p \leq \frac{\alpha(n-2)+2n}{2(n-\alpha)}, & \text{for } n > \alpha, \\
2 < p < \infty, & \text{for } n = \alpha.
\end{cases}
\]
Then
\[ T_{\text{max}} \geq \int_{G(0)}^{+\infty} \left\{ \tau + A_1 \tau^{\frac{2}{p-2}}(p-1) + A_2 \right\}^{-1} d\tau, \]
where \(A_1\) and \(A_2\) are positive constants to be determined later in the proof.
Proof. By using inequality (2.1), we have
\[
\frac{1}{2} \| u_t \|_2^2 + \frac{1}{\alpha} \| \nabla u \|_\alpha^\alpha = E(t) + \frac{1}{p} \| u(t) \|_p^p \leq E(0) + G(t). \tag{4.1}
\]
Next, using the Schwarz inequality, the Sobolev-type inequality
\[
\| u \|_q \leq C q \| \nabla u \|_\alpha, \quad \forall q \in [1, \alpha^*], \quad \forall u \in W^{1,\alpha}_0(\Omega), \tag{4.2}
\]
inequality (4.1) yields
\[
G'(t) = (|u|^{p-2}u, u_t)
\leq \frac{1}{2} \| u_t \|_2^2 + \frac{1}{2} \| u \|_{2(p-1)}^{2(p-1)}
\leq \frac{1}{2} \| u_t \|_2^2 + \frac{C_{2(p-1)}^{2(p-1)}}{2} \| \nabla u \|_\alpha^{2(p-1)}
\leq E(0) + G(t) + \frac{2(2(p-1))}{2} [\alpha E(0) + \alpha G(t)]^{\frac{2}{p}(p-1)}. \tag{4.3}
\]
From (4.3) and Jensen’s inequality, we obtain the differential inequality
\[
G'(t) \leq G(t) + A_1 [G(t)]^{\frac{2}{p}(p-1)} + A_2, \tag{4.4}
\]
with
\[
A_1 = C_{2(p-1)}^{2(p-1)} 2^{\frac{n}{p}(p-1)-2} \alpha^{\frac{2}{p}(p-1)} \quad \text{and} \quad A_2 = E(0) + A_1 [E(0)]^{\frac{2}{p}(p-1)}.
\]
Hence, we get
\[
T_{\text{max}} \geq \int_{\text{max}}^{T_{\text{max}}} \left\{ G(s) + A_1 [G(s)]^{\frac{2}{p}(p-1)} + A_2 \right\}^{-1} G'(s) ds.
\]
Since \( \lim_{t \to T_{\text{max}}} G(t) = +\infty \), so the previous inequality implies
\[
T_{\text{max}} \geq \int_{G(0)}^{+\infty} \left\{ \tau + A_1 \tau^{\frac{2}{p}(p-1)} + A_2 \right\}^{-1} d\tau.
\]
\]
In the next theorem, when \( n > \alpha \), the upper bound for \( p \) is enlarged. We define
\[
H(t) := \frac{1}{\sigma} \| u(t) \|_\sigma^\sigma,
\]
where \( \sigma = \frac{\alpha(n-2)+2n}{2(n-\alpha)} \). Then, we have

**Theorem 4.2.** Let \( u \) be the solution of (1.1), and assume that
\[
\frac{\alpha(n-2)+2n}{2(n-\alpha)} < p \leq \frac{\alpha n(n-\alpha+2)-2\alpha^2}{2n(n-\alpha)}. \tag{4.5}
\]
Then
\[
T_{\text{max}} \geq \int_{H(0)}^{+\infty} \left\{ B_1 \tau^{b_1} + B_2 \tau^{b_2} + B_3 \right\}^{-1} d\tau,
\]
where \( B_1, B_2, B_3, b_1 \) and \( b_2 \) are positive constants to be determined later in the proof.
Proof. By using inequality (2.1), we have
\[
\frac{1}{2} \|u_t\|^2_2 + \frac{1}{\alpha} \|\nabla u\|^2_\alpha = E(t) + \frac{1}{p} \|u(t)\|^p_p \leq E(0) + \frac{1}{p} \|u(t)\|^p_p. \tag{4.6}
\]
Using the Schwarz inequality, the Sobolev-type inequality (4.2), with \(q = \alpha^*\), and inequality (4.6) we get
\[
H'(t) = (|u|^{\sigma-2}u, u_t)
\leq \frac{1}{2} \|u_t\|^2_2 + \frac{1}{2} \|u\|^{2(\sigma-1)}_{2(\sigma-1)}
\leq \frac{1}{2} \|u_t\|^2_2 + \frac{C_{\alpha^*}^*}{2} \|\nabla u\|^\alpha_{\alpha^*}
\leq E(0) + \frac{1}{p} \|u\|^p_p + \frac{C_{\alpha^*}^*}{2} \left[ \alpha E(0) + \frac{\alpha}{p} \|u\|^p_p \right]^{\frac{n}{n-\alpha}}. \tag{4.7}
\]
Next, the interpolation inequality, the Sobolev inequality and Young’s inequality give
\[
\|u\|^p_p \leq \|u\|^\theta_{\alpha^*} \|u\|^{(1-\theta)p}_\alpha, \quad \theta = \frac{\alpha^*(p-\sigma)}{p(\alpha^* - \sigma)},
\leq C_{\alpha^*}^* \|\nabla u\|^\theta_{\alpha^*} \|u\|^{(1-\theta)p}_\alpha,
\leq \frac{1}{\alpha} \|\nabla u\|^\alpha_{\alpha^*} + B \|u\|^r_r,
\tag{4.8}
\]
where
\[
B = C_{\alpha^*}^* \left( 1 - \frac{\theta p}{\alpha} \right) \left( p \theta C_{\alpha^*}^* \right)^{\frac{p}{\alpha - \theta p}} \quad \text{and} \quad r = \frac{\alpha p(1-\theta)}{\alpha - \theta p}.
\]
Note that in virtue of (4.5), we have \(\alpha > \theta p\). Hence, by (2.1) we have
\[
\|u\|^p_p \leq E(0) + \frac{1}{p} \|u\|^p_p + B \|u\|^r_r, \tag{4.9}
\]
which gives
\[
\frac{1}{p} \|u\|^p_p \leq \frac{1}{p - 1} \left( E(0) + B \|u\|^r_r \right).
\]
Inserting this last inequality in (4.7), and using Jensen’s inequality, we obtain
\[
H'(t) \leq \frac{pE(0)}{p - 1} + \frac{B}{p - 1} \|u\|^r_r + \frac{C_{\alpha^*}^*}{2} \left[ \frac{\alpha pE(0)}{p - 1} + \frac{\alpha B}{p - 1} \|u\|^r_r \right]^{\frac{n}{n-\alpha}}
= B_1 (H(t))^{b_1} + B_2 (H(t))^{b_2} + B_3, \tag{4.10}
\]
where
\[
B_1 = \frac{B \sigma^r}{p - 1}, \quad B_2 = \frac{C_{\alpha^*}^*}{2} \left[ \frac{\alpha B \sigma^r}{p - 1} \right]^{\frac{n}{n-\alpha}},
B_3 = \frac{pE(0)}{p - 1} + \frac{C_{\alpha^*}^*}{2} \left[ \frac{\alpha pE(0)}{p - 1} \right]^{\frac{n}{n-\alpha}},
\]
\[
b_1 = \frac{r}{\sigma}, \quad b_2 = \frac{rn}{\sigma(n - \alpha)}.
\]
Finally, integrating inequality (4.10) over \((0, T_{\text{max}})\) we get
\[ T_{\text{max}} \geq \int_0^{T_{\text{max}}} \left\{ B_1 (H(s))^{b_1} + B_2 (H(s))^{b_2} + B_3 \right\}^{-1} H'(s) ds, \]
and so
\[ T_{\text{max}} \geq \int_{H(0)}^{+\infty} \left\{ B_1 \tau^{b_1} + B_2 \tau^{b_2} + B_3 \right\}^{-1} d\tau. \]

\[ \square \]

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**References**


Mohamed Amine Kerker
Laboratory of Applied Mathematics,
Badji Mokhtar-Annaba University,
P.O. Box 12, Annaba, 23000, Algeria
e-mail: mohamed-amine.kerker@univ-annaba.dz