SECOND-ORDER DIFFERENTIAL SUBORDINATIONS
IN THE HALF-PLANE

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Received: December 10, 1994
AMS subject classification: 30C80

REZUMAT. - Subordonări diferenţiale de ordinul al doilea în semiplan. În lucrare, folosind subordonările diferenţiale, se obţin proprietăţi ale funcţiilor holomorfe în semiplanul complex care satisfac condiţia de normalizare $f(z) - z \to 0$ pentru $z \to \infty$.

Let $\Delta$ denote the upper half-plane

$$\Delta = \{z \in \mathbb{C} / \text{Im } z > 0\}$$

and let $A(\Delta)$ denote the class of functions $f$ which are holomorphic in $\Delta$ and have the normalization

$$\lim_{\Delta \ni z \to \infty} [f(z) - z] = 0$$

In this paper, using differential subordinations in the half-plane [2], we obtain some properties concerning functions of the class $A(\Delta)$.

DEFINITION 1 [2]. Let $f, g : \Delta \to \mathbb{C}$ be holomorphic functions in $\Delta$. The function $f$ is subordinate to the function $g$ in $\Delta (f \prec g)$ if there is an holomorphic function $\varphi : \Delta \to \Delta$ such that

$$\lim_{\Delta \ni z \to \infty} [\varphi(z) - z] = 0$$

and

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f(z) = g(φ(z)), for all $z \in \Delta$.

**THEOREM 1.** Let $f \in A(\Delta)$, $g \in A(\Delta)$ and $g$ is univalent in $\Delta$. Then the function $f$ is subordinate to the function $g$ in $\Delta$ if and only if $f(\Delta) \subset g(\Delta)$.

*Proof.* If $f < g$ then using Definition 1 and Schwarz's Lemma for the upper half-plane [3], [4], it results $f(\Delta) \subset g(\Delta)$.

If $f(\Delta) \subset g(\Delta)$ then, using the univalence of the function $g$, we obtain that $g^{-1} : g(\Delta) \to \Delta$ is an holomorphic function in $\Delta$ and we can define the function $\varphi : \Delta \to \Delta$, $\varphi(z) = g^{-1}(f(z))$, $z \in \Delta$. We have

$$|\varphi(z) - z| = |g^{-1}(f(z)) - z| \leq |g^{-1}(f(z)) - f(z)| + |f(z) - z|, z \in \Delta$$

and since

$$\lim_{\Delta \ni z \to \infty} [f(z) - z] = \lim_{\Delta \ni z \to \infty} [g(z) - z] = 0$$

it follows that

$$\lim_{\Delta \ni z \to \infty} [\varphi(z) - z] = 0.$$  

**DEFINITION 2** [2]. We denote by $Q(\Delta)$ the set of functions $q \in A(\Delta)$ which are holomorphic and injective on $\overline{\Delta} - E(q)$, where $E(q) = \{\zeta \in \partial \Delta / \lim_{z \to \zeta} q(z) = \infty\}$, and also $q'(\zeta) \neq 0$ for $\zeta \in \partial \Delta \setminus E(q)$.

**DEFINITION 3** [2]. Let $\Omega$ be a set in $\mathbb{C}$ and let $q \in Q(\Delta)$. We define the class of *admissible* functions $\psi_{\Delta}[\Omega, q]$ to be those functions $\psi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\left\{ \begin{array}{ll} \psi(r, s, t; z) \notin \Omega, & \text{when } r = q(\zeta), s = m \cdot q'(\zeta) \\ \Im \frac{t}{s} \geq m \cdot \Im \frac{q''(\zeta)}{q'(\zeta)} & \text{and } z \in \Delta \text{ for } \zeta \in \partial \Delta \setminus E(q), m \in \mathbb{R} \end{array} \right. \quad (1)$$
We shall need the following theorem to prove our results:

**THEOREM 2[2].** Let $\psi \in \psi_\Delta[\Omega, q]$ and $p: \Delta \rightarrow \mathbb{C}$ be an holomorphic function in $\Delta$ such that there exists $a \geq 0$ with $p(\Delta_a) \subset q(\Delta)$, where $\Delta_a = \{z \in \mathbb{C} | \text{Im} z > a\}$. If

$$\psi(p(z), p'(z), p''(z); z) \in \Omega, \text{ for all } z \in \Delta$$

(2) then $p < q$.

**Remark 1.** If $\lim_{\Delta z \rightarrow \infty} [p(z) - z] = \lim_{\Delta z \rightarrow \infty} [q(z) - z] = 0$ then we obtain that there exists $a \geq 0$ such that $p(\Delta_a) \subset q(\Delta)$. Thus, the condition "$p: \Delta \rightarrow \mathbb{C}$ be an holomorphic function in $\Delta$ such that there exists $a \geq 0$ with $p(\Delta_a) \subset q(\Delta)$" from Theorem 2 can be replaced by $p \in A(\Delta)$.

Let $\Omega$ be a set in $\Delta$ and let $q(z) = z$, $z \in \Delta$. We will obtain some applications of the Theorem 2 corresponding to this particular $\Omega$ and $q$.

**THEOREM 3.** Let $p \in A(\Delta)$ and let $\gamma \in \mathbb{R}$, $\gamma \leq 0$. If

$$\text{Im} \left[ p(z) + \gamma \cdot \frac{p''(z)}{p'(z)} \right] > 0, \text{ z } \in \Delta$$

(3) then $\text{Im} p(z) > 0$.

**Proof.** If we let $\psi(r, s, t, z) = r + \gamma \cdot t/s$ then the conclusion will follow from Theorem 2 we show that $\psi \in \psi_\Delta[\Omega, q]$, where $\Omega = \Delta$ and $q(z) = z$. This
follows from Definition 3 since
\[
\text{Im } \psi(r, s, t; z) = \text{Im } (\zeta + \gamma \cdot t/s) = \text{Im } \zeta + \gamma \cdot \text{Im } t/s \leq 0 \text{ for } r = \zeta \in \partial \Delta,
\]
\[
\text{Im } t/s \geq 0 \text{ and } \gamma \leq 0. \text{ Hence } \psi \in \psi_\Delta[\Omega, q], \ p < q \text{ and } \text{Im } p(z) > 0.
\]

**THEOREM 4.** Let \( p \in A(\Delta) \) and let \( \alpha, \beta \in \mathbb{R} \). If
\[
\text{Im } \left[ \alpha p(z) + \beta \frac{p'(z)}{p(z)} \right] > 0, \ z \in \Delta
\]
then \( \text{Im } p(z) > 0 \).

*Proof.* If we let \( \psi(r, s, t; z) = \alpha r + \beta s/r \) then we have \( \text{Im } \psi(r, s, t; z) = \alpha \text{Im } \zeta + \beta \cdot m \text{Im } 1/\zeta = 0 \) for \( r = \zeta \in \partial \Delta, \ s = m \in \mathbb{R} \) and \( \alpha, \beta \in \mathbb{R} \). Hence \( \psi \in \psi_\Delta[\Omega, q], \ p < q \) and \( \text{Im } p(z) > 0 \).

**COROLLARY.** Let \( f: \Delta \to \mathbb{C} \) be an holomorphic function in \( \Delta \) such that \(- \frac{f'}{f}\) satisfies the conditions of Theorem 4 and \( \alpha \in \mathbb{R} \). If
\[
\text{Im } \left[ (1 - \alpha) \frac{f'(z)}{f(z)} + \alpha \frac{f''(z)}{f'(z)} \right] > 0, \ z \in \Delta
\]
then \( \text{Im } \frac{f'(z)}{f(z)} < 0, \ z \in \Delta \).

*Remark 2.* A function \( f \in A(\Delta), \ f(z) \neq 0, \ z \in \Delta \) is starlike in the half-plane \( \Delta \) if and only if
\[
\text{Im } \frac{f'(z)}{f(z)} < 0, \ z \in \Delta.
\]

Using the Corollary, we obtain that a function which satisfies the condition 5 is
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a starlike function in $\Delta$.

THEOREM 5. Let $p \in A(\Delta)$ and $\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \leq 0$. If

$$\text{Im} \left[ \alpha p(z) + \beta \frac{p'(z)}{p(z)} + \gamma \frac{p''(z)}{p'(z)} \right] > 0, \ z \in \Delta$$

then $\text{Im} \ p(z) > 0$.

Proof. If we let $\psi(r, s, t; z) = \alpha r + \beta s/r + \gamma t/s$ then we have

$$\text{Im} \ \psi(r, s, t; z) = \alpha \text{Im} \ \zeta + \beta \cdot m \text{Im} \ \frac{1}{\zeta} + \gamma \cdot \text{Im} \ t/s \leq 0 \ \text{for} \ r = \zeta \in \partial \Delta, s = m \in \mathbb{R}, \ \text{Im} \ t/s \geq 0, \ \alpha, \beta, \gamma \in \mathbb{R}, \ \gamma \leq 0. \ \text{Hence} \ \psi \in \psi_{\Delta}[\Omega, q], \ p < q \ \text{and} \ \text{Im} \ p(z) > 0.$$

Remark 3.

i) If $\gamma = 0$ then Theorem 5 reduces to Theorem 4.

ii) If $\alpha = 1$ and $\beta = 0$ then Theorem 5 reduces to Theorem 3.

REFERENCES


