CONVEXITY AND INTEGRAL OPERATORS

Silvia TOADER

Received: January 20, 1995
AMS subject classification: 26A51

REZUMAT. - Convexitate și operatori integrali. În prima parte a lucrării îmbunătățim un rezultat al lui V. Zanelli și dăm o demonstrație ușoară a sa. Apoi considerăm câțiva operatori integrali și studiem proprietățile lor relative la conservarea convexității de ordin superior. Obținem astfel o generalizare a rezultatului din [3].

1. A result of V. Zanelli. In [3] it is proved the following property:

LEMMA 0. Let $f: [a, \infty) \rightarrow \mathbb{R}$ (with $a > 0$) be a positive, decreasing, convex function and

\[
F(x) = \int_a^x f(t) \, dt.
\]

(1)

For $a \leq y$, $k > 0$, $y + k \leq x$, we have the following inequality:

\[
F(y + k) - F(y) - F(x + k) + F(x) \leq k [f(y) - f(x)].
\]

(2)

The proof is based on a rather complicated geometrical method. We want to eliminate some superfluous hypotheses from the enounce and to give a simple proof of it.

LEMMA 1. Let $f: [a,b] \rightarrow \mathbb{R}$ be a convex function and $F$ be defined by

---

* Technical University, Department of Mathematics, 3400 Cluj-Napoca, Romania
(1). For \( a \leq y < x < x + k \leq b \) we have the inequality (2).

**Proof.** Let us consider the auxiliary function:

\[
g(t) = t[f(y) - f(x)] - F(y + t) + F(y) + F(x + t) - F(x), \; t \in [0, k].
\]

We have

\[
g'(t) = \frac{f(x + t) - f(x) - f(y + t) + f(y)}{t} \geq 0
\]

because, by the convexity of \( f \), the conditions \( x > y \) and \( x + t > y + t \) give:

\[
\frac{f(x + t) - f(x)}{t} \geq \frac{f(x + t) - f(y)}{x + t - y} \geq \frac{f(y + t) - f(y)}{t}.
\]

Obviously \( g(0) = 0 \) so that \( g'(t) \geq 0 \) gives \( g(k) \geq 0 \), that is (2).

It can be remarked that we have renounced at the following hypotheses from Lemma 0: \( a > 0, f \) is positive and decreasing and \( y + k \leq x \).

2. **Convex functions of higher order.** We must remind some definitions.

Let \( f: [a, b] \rightarrow R \) be an arbitrary function. For arbitrary distinct points \( x_1, x_2, \ldots, x_{n+1} \in [a, b] \) the divided differences of the function \( f \) are defined by recurrence:

\[
[x_1; f] = f(x_1), \; [x_1, \ldots, x_{n+1}; f] =
\]

\[
= ( [x_1, \ldots, x_{n-1}, x_{n+1}; f] - [x_1, \ldots, x_n; f])/(x_{n+1} - x_n)
\]  \hspace{1cm} (3)

The function \( f \) is called convex of order \( n \) (or shortly \( n \)-convex) if:
where the points are supposed, as in (3), distinct.

For \( n = 1 \) we get convexity and for \( n = 0 \) increasing monotony. It is known (see [2]) that a \( n \)-convex function, with \( n \geq 1 \), is continuous on \( (a,b) \), so it is integrable on any subinterval from \([a,b]\).

The main result that we will use is the following:

**LEMMA 2.** If the function \( f \) is \( n \)-convex then:

\[
[x_1, \ldots, x_{n+1}; f] \leq [y_1, \ldots, y_{n+1}; f], \text{ if } x_i \leq y_i, \forall i.
\]

*Proof.* From (3) and (4) we deduce that:

\[
[x_1, \ldots, x_{n-1}, x_{n+1}; f] \geq [x_1, \ldots, x_n; f] \text{ if } x_{n+1} > x_n. \text{ This gives (5), step by step, because the divided differences are symmetric with respect to the points.}
\]

3. **Arithmetic integral means.** To generalize the result from [3] we consider, for a fixed \( k > 0 \), some operators.

Let \( C[a,b] \) be the set of continuous functions on \([a,b]\). For \( f \in C[a,b] \) we denote by \( F_k(f) \) the function defined by:

\[
F_k(f)(x) = \int_x^{x+k} f(t) \, dt, \forall x \leq b - k.
\]

Then we define:
A_k(f)(x) = \frac{1}{k} F_k(f)(x)
a sort of arithmetic integral mean and:

E_k(f)(x) = A_k(f)(x) - f(x)
an "excess" function. We get so the operators F_k, A_k and E_k defined on C[a,b]
and with values in C[a, b-k]. To study some of their properties, we give simple
representation formulas for them.

As:

F_k(f)(x) = \int_0^k f(x + t) \, dt
making the substitution t = ks, we have:

A_k(f)(x) = \int_0^1 f(x + ks) \, ds

and so

E_k(f)(x) = \int_0^1 [f(x + ks) - f(x)] \, ds.

Thus E_k(f) \geq 0 if f is increasing and Lemma 1 asserts in fact that E_k(f) is
increasing if f is convex. We generalize this result as follows.

THEOREM 1. If the function f is n-convex, then F_k(f) and A_k(f) are
also n-convex but E_k(f) is (n-1)-convex.

Proof. If x_1, ..., x_{n+2} are distinct points from [a, b-k] we have

[x_1, ..., x_{n+2}; A_k(f)] = \int_0^1 [x_1 + ks, ..., x_{n+2} + ks; f] \, ds \geq 0
and

\[ [x_1, \ldots, x_{n+1}; E_k(f)] = \int_0^1 ([x_1 + ks, \ldots, x_{n+1} + ks; f] - [x_1, \ldots, x_{n+1}; f]) \, ds \geq 0 \]

by Lemma 2. So the affirmation follows for \( A_k(f) \) and \( E_k(f) \). As \( F_k(f) = kA_k(f) \), it is true also for \( F_k(f) \).

We remark that the operator \( E_k \) can be defined similarly by:

\[ E_k(f)(x) = f(x + k) - A_k(f)(x) \]

having the same properties.

Let us define also the operators \( F, A, E : C[a, b] \to C[a, b] \) as follows. For \( f \) in \( C[a, b] \) we put:

\[ F(f)(x) = \int_a^x f(t) \, dt \]

\[ A(f)(x) = F(f)(x)/(x-a) \]

and

\[ E(f)(x) = f(x) - A(f)(x). \]

Using the substitution \( t = a + s(x-a) \), we have:

\[ A(f)(x) = \int_0^1 f(a + s(x-a)) \, ds \]

and

\[ E(f)(x) = \int_0^1 [f(x) - f(sx + (1-s)a)] \, ds. \]

Thus, as above, we can prove
THEOREM 2. \textit{If the function }f\textit{ is }n\text{-convex then so is also }A(f),\textit{ but }l:(f)\textit{ is }\left(n-1\right)\text{-convex.}

The first result is well known (see [1]) as it is also known that under the same hypotheses, }F(t)\textit{ is }\left(n+1\right)\text{-convex.}

\textbf{REFERENCES}