A modified Post Widder operators preserving $e^{Ax}$

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Abstract. In the present paper, we discuss the approximation properties of modified Post-Widder operators, which preserve the test function $e^{Ax}$. We establish weighted approximation and a direct quantitative estimate for the modified operators.

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1. Post-Widder operators

In the recent years some sequences of linear positive operators and the operators of integral type have been studied in [2], [3] and [4] etc. Also the moments of several operators have been provided in [8]. In the present article, we discuss the variant of an integral operators viz. Post-Widder operators. Post-Widder operators are defined for $f \in C[0, \infty)$ as (see [13]):

$$P_n(f, x) := \frac{1}{n!} \left( \frac{n}{x} \right)^{n+1} \int_0^\infty t^n e^{-\frac{nt}{x}} f(t) \, dt.$$ 

Following [7], we have

$$P_n(e^{\theta t}, x) = \left( 1 - \frac{x\theta}{n} \right)^{-(n+1)}.$$  \hspace{1cm} (1.1)


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Let us consider that the Post-Widder operators preserve the test function $e^{Ax}$, then we start with the following form

$$\tilde{P}_n(f, x) := \frac{1}{n!} \left( \frac{n}{a_n(x)} \right)^{n+1} \int_0^\infty t^n e^{-\frac{nt}{a_n(x)}} f(t) \, dt.$$  

Then using (1.1), we have

$$\tilde{P}_n(e^{At}, x) = e^{Ax} = \left( 1 - \frac{a_n(x)A}{n} \right)^{-(n+1)},$$

implying

$$a_n(x) = \frac{n}{A} \left( 1 - e^{-Ax/(n+1)} \right).$$

Thus our modified operators $\tilde{P}_n$ take the following form

$$\tilde{P}_n(f, x) := \frac{1}{n!} \left[ \frac{A}{\left( 1 - e^{-Ax/(n+1)} \right)} \right]^{(n+1)} \int_0^\infty t^n e^{-\frac{t}{(1-e^{-Ax/(n+1)})}} f(t) \, dt,$$

with $x \in (0, \infty)$ and $\tilde{P}_n(f, 0) = f(0)$, which preserve constant and the test function $e^{Ax}$.

2. Lemmas

**Lemma 2.1.** We have for $\theta > 0$ that

$$\tilde{P}_n(e^{\theta t}, x) = \left( 1 - \frac{(1-e^{-Ax/(n+1)})\theta}{A} \right)^{-(n+1)}.$$  

It may be observed that $\tilde{P}_n(e^{\theta t}, x)$ may be treated as m.g.f. of the operators $\tilde{P}_n$, which may be utilized to obtain the moments of (1.2). Let $\mu_r^{\tilde{P}_n}(x) = \tilde{P}_n(e^r, x)$, where $e_r(t) = t^r, r \in N \cup \{0\}$. The moments are given by

$$\mu_r^{\tilde{P}_n}(x) = \left[ \frac{\partial^r}{\partial \theta^r} \tilde{P}_n(e^{\theta t}, x) \right]_{\theta=0}$$

$$= \left[ \frac{\partial^r}{\partial \theta^r} \left( 1 - \frac{(1-e^{-Ax/(n+1)})\theta}{A} \right)^{-(n+1)} \right]_{\theta=0}.$$  

Few moments are given below:

$$\mu_0^{\tilde{P}_n}(x) = 1,$$

$$\mu_1^{\tilde{P}_n}(x) = \frac{(n+1)}{A} \left( 1 - e^{-Ax/(n+1)} \right),$$

$$\mu_2^{\tilde{P}_n}(x) = \frac{(n+1)(n+2)}{A^2} \left( 1 - e^{-Ax/(n+1)} \right)^2.$$
Lemma 2.2. The moments of arbitrary order, satisfy the following
\[ \mu_{kn}^\tilde{P}(x) = \frac{(n + 1)k}{A^k} (1 - e^{-Ax/(n+1)})^k, \] for \( k = 0, 1, \ldots \),
where the Pochhammer symbol is defined by
\[ (c)_0 = 1, \quad (c)_k = c(c + 1) \cdots (c + k - 1). \]
Further by linearity property and using Lemma 2.2, we have the following lemma:

Lemma 2.3. The central moments \( \tilde{P}_n^r((t - x)^r, x) \) are given below:
\[ \tilde{P}_n^r(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} x^{k-j} (1 - e^{-Ax/(n+1)})^j \frac{(n + 1)j}{A^j}, \] \( k = 0, 1, \ldots \).

Also, for each \( n \in \mathbb{N} \), we have
\[ \tilde{P}_1^r(x) = \frac{(n + 1)}{A} (1 - e^{-Ax/(n+1)} - 1) - x, \]
\[ \tilde{P}_2^r(x) = \frac{(n + 1)(n + 2)}{A^2} (1 - e^{-Ax/(n+1)})^2 + x^2 - 2x \frac{(n + 1)}{A} (1 - e^{-Ax/(n+1)}). \]

Lemma 2.4. For the central moments \( \tilde{P}_{2k}^r(x) = \tilde{P}_n((t - x)^{2k}, x) \), we have
\[ \tilde{P}_{2k}^r(x) = O(n^{-k}), n \to \infty, k = 1, 2, 3, \ldots \]

Proof. We observe that
\[ \tilde{P}_n(f, x) = P_n(f, \alpha_n(x)), \]
where
\[ \alpha_n(x) = \frac{n}{A} (1 - e^{-Ax/(n+1)}). \]
It is easy to verify \( y > 1 - e^{-y} > y - \frac{y^2}{2} \) for \( y \in [0, \infty) \). We set \( y = Ax/(n+1) \) and get
\[ x \left( \frac{n}{n+1} \right) > \alpha_n(x) > x \left( \frac{n}{n+1} \right) - \left( \frac{Ax}{n+1} \right)^2 \frac{n}{2A}. \]
Hence
\[ \frac{x}{n+1} < x - \alpha_n(x) < \frac{x}{n+1} + \frac{Ax^2n}{2(n+1)^2} = O(n^{-1}), \]
by fixed \( x \in [0, \infty) \). Therefore
\[ \tilde{P}_n((t - x)^{2k}, x) = P_n((t - x)^{2k}, \alpha_n(x)) \]
\[ = P_n((t - \alpha_n(x) + \alpha_n(x) - x)^{2k}, \alpha_n(x)) \]
\[ \leq C(k) P_n((t - \alpha_n(x))^{2k}, \alpha_n(x)) + P_n((x - \alpha_n(x))^{2k}, \alpha_n(x)) \]
\[ \leq C(k) \frac{1}{n^k} + (x - \alpha_n(x))^{2k} = O(n^{-k}). \]
This completes the proof of Lemma 2.4.
3. Weighted approximation

We also analyse the behaviour of the operators on some weighted spaces. Set \( \phi(x) = 1 + e^{Ax}, \ x \in R^+ \) and consider the following weighted spaces:

\[
B_\phi(R^+) = \{ f : R^+ \to R : |f(x)| \leq C_1(1 + e^{Ax}) \}, \\
C_\phi(R^+) = B_\phi(R^+) \cap C(R^+), \\
C^k_\phi(R^+) = \left\{ f \in C_\phi(R^+) : \lim_{x \to \infty} \frac{f(x)}{1 + e^{Ax}} = C_2 < \infty \right\},
\]

where \( C_1, C_2 \) are constants depending on \( f \). The norm is defined as

\[
||f||_\phi = \sup_{x \in R^+} \frac{|f(x)|}{1 + e^{Ax}}.
\]

**Theorem 3.1.** For each \( f \in C^k_\phi(R^+) \), we have

\[
\lim_{n \to \infty} ||\tilde{P}_n f - f||_\phi = 0.
\]

**Proof.** Following [1, Th. 1] in order to prove the result we have to prove

\[
\lim_{n \to \infty} ||\tilde{P}_n (e^{iAt/2}) - e^{iAx/2}||_\phi = 0, \ i = 0, 1, 2.
\]

The result is true for \( i = 0, i = 2 \). It remains to verify it for \( i = 1 \). By Lemma 2.1 we have

\[
||\tilde{P}_n (e^{A/2}) - e^{A/2}||_\phi = \sup_{x \in R^+} \left| \left(1 - \frac{(1 - e^{-Ax/(n+1)})}{2}\right)^{n+1} - e^{Ax/2} \right|
\]

\[
= \sup_{x \in R^+} \left| \frac{(1 + e^{-Ax/(n+1)})^{n+1}}{2} - e^{Ax/2} \right|
\]

\[
= \sup_{x \in R^+} \left| \frac{e^{Ax}(1 + e^{Ax/(n+1)})^{n+1}}{1 + e^{Ax}} - e^{Ax/2} \right|
\]

\[
= \sup_{x \in R^+} \left[ \frac{e^{Ax}}{1 + e^{Ax}} \right] \cdot \left| \frac{2}{1 + e^{Ax/(n+1)}}^{n+1} - e^{-Ax/2} \right|. \quad (3.1)
\]

Obviously \( \frac{e^{Ax}}{1 + e^{Ax}} \in \left[ \frac{1}{2}, 1 \right], \ A > 0, \ x > 0. \) We set \( t = e^{Ax/2}, \ t \in [1, \infty) \) for \( x \in (0, \infty) \). Then (3.1) implies

\[
\left| \left( \frac{2}{1 + t^{2/(n+1)}} \right)^n - t^{-1} \right| = t^{-1} \left| \left( \frac{2t^{1/(n+1)}}{1 + t^{2/(n+1)}} \right)^n - 1 \right| = g(t). \quad (3.2)
\]
In (3.2), we set \( t^{1/(n+1)} = y \in [1, \infty) \). Hence
\[
g(t) = h(y) = y^{-(n+1)} \left| \left( \frac{2y}{1+y^2} \right)^{n+1} - 1 \right|
\]
\[
= \left| \left( \frac{2}{1+y^2} \right)^{n+1} - y^{-(n+1)} \right|
\]
\[
= y^{-(n+1)} - \left( \frac{2}{1+y^2} \right)^{n+1}.
\]

We have \( h(1) = 0, h(+\infty) = \lim_{y \to \infty} h(y) = 0 \). To find the global maxima of \( h(y) \) we solve the equation \( h'(y) = 0 \). Simple calculations imply that \( h'(y_0) = 0 \) for \( y_0 \) satisfying the equation
\[
\frac{2}{1+y_0^2} = y_0^{-(n+3)/(n+2)}, y_0 \in (1, \infty).
\]

The equations (3.3) and (3.4) imply
\[
h(y) \leq h(y_0) = y_0^{-(n+1)} - y_0^{-(n+3)(n+1)/(n+2)}.
\]

The proof will be completed if we show
\[
h(y_0) < \frac{1}{2(n+3)}, n \to \infty.
\]

We set in (3.5) \( y_0^{n+1} = z_0 \in (1, +\infty) \). Then \( h(y_0) = z_0^{-1} - z_0^{-(n+3)/(n+2)} < \max p(z) \) with \( p(z) = z^{-1} - z^{-(n+3)/(n+2)} \). We compute that \( p'(z_1) \) for \( z_1 = \left( \frac{n+3}{n+2} \right)^{n+2} \).

Therefore
\[
p(z_1) = \left( \frac{n+3}{n+2} \right)^{-(n+2)} - \left( \frac{n+3}{n+2} \right)^{-(n+3)}
\]
\[
= \left( \frac{n+3}{n+2} \right)^{-(n+2)} \left[ 1 - \left( \frac{n+3}{n+2} \right)^{-1} \right]
\]
\[
= \left( 1 + \frac{1}{n+2} \right)^{-(n+2)} \frac{1}{n+3} < \frac{1}{2(n+3)},
\]
due to \( \lim_{n \to \infty} \left( 1 + \frac{1}{n+2} \right)^{-(n+2)} = e^{-1} < 1/2 \).

\[\square\]

4. A direct quantitative estimate

Our goal in this section is to obtain a quantitative form of the statement in Theorem 3.1. For the sake of simplicity we slightly modify the weight function and instead of \( \phi(x) = 1 + e^{Ax}, x \in R^+ \) we consider \( \phi(x) = e^{Ax}, x \in R^+ \), For continuous functions on \([0, \infty)\) with exponential growth i.e.
\[
||f||_A := \sup_{x \in [0, \infty)} |f(x) \cdot e^{-Ax}| < \infty, A > 0,
\]
(4.1)
it is easy to observe that

\[ ||\tilde{P}_n f||_A \leq ||f||_A. \] (4.2)

Consequently if the following function series is uniformly convergent on \([0, \infty)\)

\[ S(x) = \sum_{k=0}^{\infty} u_k(x), x \in [0, \infty), \]

then

\[ \tilde{P}_n(S(t), x) = \sum_{k=0}^{\infty} \tilde{P}_n(u_k(t), x), x \in [0, \infty), \] (4.3)

where the last series is also uniformly convergent. For our goals in this section we
need the first order exponential modulus of continuity, studied by Ditzian in [5] and
declared as

\[ \omega_1(f, \delta, A) := \sup_{h \leq \delta, 0 \leq x < \infty} |f(x) - f(x + h)e^{-Ax}|. \]

We consider the sequence of operators \( \tilde{P}_n : E \to C[0, \infty) \), where the domain of the
operator \( \tilde{P}_n \) contains the space of functions \( f \) with exponential growth, i.e. \( ||f||_A < \infty \).
Our main result states the following:

**Theorem 4.1.** Let \( \tilde{P}_n : E \to C[0, \infty) \) be sequence of linear positive operators of Post-
Widder type defined in (1.2). Then

\[ |\tilde{P}_n(f, x) - f(x)| \leq e^{Ax}[3 + C(n,x)\omega_1(f, \sqrt{U_{2n}P_n}(x), A)], \]

where

\[ C(n, x) = 2 \sum_{k=1}^{\infty} \frac{A^k}{k!} \sqrt{U_{2k}P_n}(x), \ n \to \infty \text{ for fixed } x \in [0, \infty). \]

**Proof.** We observe that

\[ |f(t) - f(x)| \leq \begin{cases} e^{Ax}\omega_1(f, \delta, A), & |t - x| \leq \delta \\ e^{Ax}\omega_1(f, k\delta, A), & \delta \leq |t - x| \leq k\delta, \end{cases} \] (4.4)

where \( k \) is the smallest natural number in the above upper bound. Now [12, Lemma 2.2] (also see [10]) implies

\[ \omega_1(f, k\delta, A) \leq k e^{A(k-1)\delta}\omega_1(f, \delta, A) \]

\[ \leq \omega_1(f, \delta, A) \left[ \frac{|t - x|}{\delta} + 1 \right] e^{A|t-x|}. \] (4.5)

Now (4.4) and (4.5) imply

\[ |f(t) - f(x)| \leq \left[ 1 + \left( \frac{|t - x|}{\delta} + 1 \right) e^{A|t-x|} \right] e^{Ax}\omega_1(f, \delta, A). \] (4.6)
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For fixed $x \in [0, \infty)$ the following series is uniformly convergent for $t \in [0, \infty)$

$$S_1(t, x) = e^{A|t-x|} = \sum_{k=0}^{\infty} \frac{(A|t-x|)^k}{k!}$$

$$\frac{|t-x|}{\delta} S_1(t, x) = \frac{|t-x|}{\delta} + \frac{1}{\delta} \sum_{k=1}^{\infty} \frac{A^k|t-x|^{k+1}}{k!}. \quad (4.7)$$

Obviously for linear positive operators $\tilde{P}_n$ using (4.4), (4.6) and (4.7), we obtain

$$|\tilde{P}_n(f(t) - f(x))| \leq e^{Ax} \left( 1 + \tilde{P}_n(S_1(t, x), x) + \frac{1}{\delta} \tilde{P}_n(|t-x|, x) + \frac{1}{\delta} \sum_{k=1}^{\infty} \frac{A^k \tilde{P}_n(|t-x|^{k+1}, x)}{k!} \right) \omega_1(f, \delta, A). \quad (4.8)$$

From Cauchy Schwarz inequality, we have

$$\tilde{P}_n(|t-x|^{k+1}, x) \leq \sqrt{\tilde{P}_n((t-x)^2, x)} \sqrt{\tilde{P}_n((t-x)^{2k}, x)} = \sqrt{U_{2k}^\tilde{P}_n(x)} \sqrt{U_{2k}^\tilde{P}_n(x)}. \quad (4.9)$$

Further

$$S_1(t, x) = 1 + A|t-x| + \sum_{k=2}^{\infty} \frac{(A|t-x|)^k}{k!}.$$  

Hence

$$\tilde{P}_n(S_1(t, x), x) \leq 1 + A \sqrt{U_{2k}^\tilde{P}_n(x)} + \sum_{k=2}^{\infty} \frac{A^k \sqrt{U_{2k}^\tilde{P}_n(x)}}{k!}. \quad (4.10)$$

From Lemma 2.4, for fixed $x \in [0, \infty)$, we have

$$U_{2k}^\tilde{P}_n(x) = O(n^{-k}), n \to \infty. \quad (4.11)$$

We set in (4.8) that

$$\delta = \sqrt{U_{2k}^\tilde{P}_n(x)} = O(n^{-1/2}), n \to \infty. \quad (4.12)$$

Therefore estimates (4.8)-(4.12) imply

$$|\tilde{P}_n(f, x) - f(x)| \leq e^{Ax} [3 + C(n, x)] \omega_1(f, \sqrt{U_{2k}^\tilde{P}_n(x)}, A),$$

where

$$C(n, x) = A \sqrt{U_{2k}^\tilde{P}_n(x)} + \sum_{k=2}^{\infty} \frac{A^k \sqrt{U_{2k}^\tilde{P}_n(x)}}{k!} + \sum_{k=1}^{\infty} \frac{A^k \sqrt{U_{2k}^\tilde{P}_n(x)}}{k!} = O(n^{-1/2}), n \to \infty,$$

by fixed $x \in [0, \infty)$. This completes the proof of theorem.
References


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