Bounds for blow-up time in a semilinear parabolic problem with variable exponents

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Abstract. This report deals with a blow-up of the solutions to a class of semilinear parabolic equations with variable exponents nonlinearities. Under some appropriate assumptions on the given data, a more general lower bound for a blow-up time is obtained if the solutions blow up. This result extends the recent results given by Baghaei Khadijeh et al. [8], which ensures the lower bounds for the blow-up time of solutions with initial data $\varphi (0) = \int_{\Omega} u_{0}^{k} dx$, $k = \text{constant}$.

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1. Introduction

In this paper, we are concerned with the following semilinear parabolic equation

$$\begin{cases}
    u_t - \Delta u = u^{p(x)}, & x \in \Omega, \ t > 0 \\
    u = 0 \text{ on } \Gamma, & t \geq 0 \\
    u(x, 0) = u_0(x) \geq 0, & x \in \Omega,
\end{cases}$$

(1.1)

where $\Omega$ be a bounded domain in $\mathbb{R}^n$, with a smooth boundary $\Gamma = \partial \Omega$, $T \in (0, +\infty]$, and the initial value $u_0 \in H^1_0(\Omega)$, the exponent $p(\cdot)$ is given measurable function on $\Omega$ such that:

$$1 < p_1 = \text{ess inf}_{x \in \Omega} p(x) \leq p(x) \leq p_2 = \text{ess sup}_{x \in \Omega} p(x) < \infty,$$

(1.2)

and satisfy the following Zhikov-Fan uniform local continuity condition:

$$|p(x) - p(y)| \leq \frac{M}{|\log|x - y||}, \ \text{for all } x, y \in \Omega \text{ with } |x - y| < \frac{1}{2}, M > 0.$$

(1.3)

The problem (1.1) arises from many important mathematical models in engineering and physical sciences. For example, nuclear science, chemical reactions, heat transfer,
population dynamics, biological sciences, etc., and have interested a great deal of attention in the research, see [4, 7, 9] and the references therein. For problem (1.1), Hua Wang et al. [10] established a blow-up result with positive initial energy under some suitable assumptions on the parameters $p(\cdot)$ and $u_0$. In [9], the authors proved that there are non-negative solutions with a blow-up in finite time if and only if $p_2 > 1$. The authors in [11] obtained the solution of problem (1.1) blows up in finite time when the initial energy is positive. The following problem was considered by R. Abita in [3]

$$u_t - \Delta u_t - \Delta u = u^{p(x)}, \quad x \in \Omega, \ t > 0.$$  

The author proved that the nonnegative classical solutions blow-up in finite time with arbitrary positive initial energy and suitable large initial values. Also, he employed a differential inequality technique to obtain an upper bound for blow-up time if $p(\cdot)$ and the initial value satisfies some conditions. In [8], the authors based exactly on the idea on the one in [6], derived the lower bounds for the time of blow-up, if the solutions blow-up. In order to declare the main results of this paper, we need to add the following energy functional corresponding to the problem (1.1) (see [2])

$$E(t) = \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 \, dx - \int_{\Omega} \frac{1}{p(x) + 1} u(x,t)^{p(x)+1} \, dx.$$  

(1.4)

2. Lower bounds of the blow-up time

In this section, we investigate the lower bound for the blow-up time $T$ in some suitable measure. The idea of the proof of the following theorem is inspired by on the one in [6]. For this goal, we start by the following lemma concerning the energy of the solution.

**Lemma 2.1.** Let $u(x,t)$ be a weak solution of (1.1), then $E(t)$ is a nonincreasing function on $[0, T]$, that is

$$\frac{dE(t)}{dt} = - \int_{\Omega} u^2 (x,t) \, dx \leq 0$$  

(2.1)

and the inequality $E(t) \leq E(0)$ is satisfied.

We consider the following partition of $\Omega$,

$$\Omega^- = \{ x \in \Omega \mid 1 > (k(x) - 1) \ln |u| \}, \quad \Omega^+ = \{ x \in \Omega \mid 1 \leq (k(x) - 1) \ln |u| \},$$

for each $t > 0$ where each $\Omega^\pm$ depends on $t$, and setting

$$\tilde{E}(0) = \frac{1}{2} \int_{\Omega^-} |\nabla u_0|^2 \, dx - \int_{\Omega^-} \frac{1}{p(x) + 1} u_0^{p(x)+1} \, dx.$$  

Now, we are in a position to affirm our principal theorem results.

**Theorem 2.2.** Assume $u_0 \in L^{k(\cdot)}(\Omega)$, and the nonnegative weak solution $u(x,t)$ of problem (1.1) blows up in finite time $T$, then $T$ has a lower bound by:

$$\int_{\varphi(0)}^{+\infty} \frac{d\gamma}{C_1 + C_2 \gamma^{\frac{3n-6}{n-8}}},$$  

(2.2)
where

$$\varphi(0) = \int_\Omega \frac{1}{k(x)(k(x) - 1)} u_0^{k(x)} dx,$$

(2.3)

where \(k(.)\) is a measurable function on \(\Omega\) such that

$$\max(1, 2(n - 2)(p_2 - 1)) < k_1 = \text{ess inf}_{x \in \Omega} k(x) \leq k(x) \leq k_2$$

$$= \text{ess sup}_{x \in \Omega} k(x) < \infty,$$

(2.4)

and

$$\sqrt{C_k} = \sup_{x \in \Omega} |\nabla k(x)| \in L^2(\Omega), \; C_k > 0$$

(2.5)

and \(C_i (i = 1, 2)\) are positive constants will be described later.

**Notation 2.3.** We note that the presence of the variable-exponent nonlinearities in (2.6) below, makes analysis in the paper somewhat harder than that in the related ones, we will establish and give a precise estimate for the lifespan \(T\) of the solution in this case. The method used here is the differential inequality technique. However, our argument is considerably different and it is more abbreviated.

**Proof of Theorem (2.2).** Set

$$\varphi(t) = \int_\Omega \frac{1}{k(x)(k(x) - 1)} u(x,t)^{k(x)} dx.$$  

(2.6)

Multiplying the equation Eq. (1.1) by \(u\) and integrating by parts, we see

$$\varphi'(t) = \int_\Omega \frac{1}{k(x) - 1} u^{k(x) - 1} u_t dx = \int_\Omega \frac{1}{k(x) - 1} u^{k(x) - 1} \left( \Delta u + u^{p(x)} \right) dx$$

$$= \int_\Omega \frac{1}{k(x) - 1} u^{k(x) - 1} \Delta u dx + \int_\Omega \frac{1}{k(x) - 1} u^{k(x) + p(x) - 1} dx$$

$$= - \int_\Omega \nabla \left( \frac{1}{k(x) - 1} u^{k(x) - 1} \right) \nabla u dx + \int_\Omega \frac{1}{k(x) - 1} |u|^{k(x) + p(x) - 1} dx$$

where we have used the divergence theorem, the boundary condition on \(u\).

It is straightforward to check that

$$\nabla \left( \frac{1}{k(x) - 1} u^{k(x) - 1} \right) = u^{k(x)} |u|^{-2} \nabla u + \frac{\nabla k(x)}{k(x) - 1} u^{k(x) - 1} \left( \ln |u| - \frac{1}{k(x) - 1} \right)$$

then, we get

$$\varphi'(t) = - \int_\Omega u^{k(x)} |u|^{-2} |\nabla u|^2 dx + \int_\Omega \frac{1}{k(x) - 1} u^{k(x) + p(x) - 1} dx + Q$$

(2.7)

where

$$Q = \int_\Omega u^{k(x) - 1} \left( \frac{1}{(k(x) - 1)^2} - \frac{1}{(k(x) - 1)^2} \ln |u| \right) \nabla k(x) \cdot \nabla u dx$$
Considering the following properties of the function $G$,

$$G(\lambda) = \frac{\lambda^\gamma}{\gamma^2} (1 - \gamma \ln \lambda), \quad 0 \leq \lambda \leq e^{\frac{1}{\gamma}};$$

$$G(0) = G\left(e^{\frac{1}{\gamma}}\right) = 0, \quad G'(\lambda) = -\lambda^{\gamma - 1} \ln \lambda, \quad \max_{0 \leq \lambda \leq e^{\frac{1}{\gamma}}} G(\lambda) = G(1) = \frac{1}{\gamma^2},$$

and using the fact that

$$\int_{\Omega^-} |\nabla u|^2 \, dx \leq 2E(0) + 2 \int_{\Omega^-} \frac{1}{p(x) + 1} u(x,t)^{p(x)+1} \, dx, \quad \text{by (1.4) and (2.1)}$$

applying the H"older, Young inequalities and (2.5), $Q$ is estimated as follows:

$$Q = \int_{\Omega} u^{k(x)-1} \left( \frac{1}{(k(x) - 1)^2} - \frac{1}{k(x) - 1} \ln |u| \right) \nabla k(x) \cdot \nabla u \, dx$$

$$= \int_{\Omega \cap (1>(k(x)-1) \ln|u(x,t)|)} u^{k(x)-1} \left( \frac{1}{(k(x) - 1)^2} - \frac{1}{k(x) - 1} \ln |u| \right) \nabla k(x) \cdot \nabla u \, dx$$

$$\leq \int_{\Omega} \frac{1}{(k(x) - 1)^2} |u|^{k(x)-1} (1 - (k(x) - 1) \ln |u|) |\nabla u| |\nabla k(x)| \, dx$$

$$\leq \int_{\Omega} \frac{1}{(k(x) - 1)^2} |\nabla k(x)| |\nabla u| \, dx \leq \frac{1}{2(k_1 - 1)^2} \left( C_k + \int_{\Omega} |\nabla u|^2 \, dx \right)$$

$$\leq \frac{1}{2(k_1 - 1)^2} \left( C_k + 2E(0) + 2 \int_{\Omega} \frac{1}{p(x) + 1} u(x,t)^{p(x)+1} \, dx \right)$$

$$\leq \frac{1}{2(k_1 - 1)^2} \left( \frac{1}{2} C_k + E(0) + \frac{1}{p_1 + 1} \max \left( \int_{\Omega} |u|^{p_2+1} \, dx, \int_{\Omega} |u|^{p_1+1} \, dx \right) \right) \leq \frac{1}{(k_1 - 1)^2} \left( \frac{1}{2} C_k + E(0) + \frac{1}{p_1 + 1} e^{\frac{1}{\gamma_1}} |\Omega| \right). \quad (2.8)$$

Because in $\Omega^+$, we have

$$\int_{\Omega^+} |u|^{k(x)-1} \left( \frac{1}{(k(x) - 1)^2} - \frac{1}{k(x) - 1} \ln |u| \right) |\nabla k(x)| \, dx \leq 0,$$

while that of the first term in the right-hand side of (2.7) was estimated as follows

$$-\int_{\Omega} |u|^{k(x)-2} |\nabla u|^2 \, dx \leq - \min \left( \int_{\Omega} |u|^{k_2-2} |\nabla u|^2 \, dx, \int_{\Omega} |u|^{k_1-2} |\nabla u|^2 \, dx \right).$$

Using the fact

$$|\nabla u^\gamma| = \gamma u^{\gamma-1} |\nabla u|$$
to get
\[- \int_{\Omega} |u|^{k(x)-2} |\nabla u|^2 \, dx \leq - \min \left( \frac{4}{(k_2)^2} \int_{\Omega} |\nabla u|^{k_2} \, dx, \frac{4}{(k_1)^2} \int_{\Omega} |\nabla u|^{k_1} \, dx \right) \tag{2.9}\]

Plugging this estimate (2.8) and (2.9) into (2.7), we obtain
\[
\varphi'(t) \leq \min \left( \frac{-4}{(k_2)^2} \int_{\Omega} |\nabla u|^{k_2} \, dx, \frac{-4}{(k_1)^2} \int_{\Omega} |\nabla u|^{k_1} \, dx \right) \\
+ \frac{1}{k_1 - 1} \int_{\Omega} u^{k(x)+p_2-1} \, dx + \frac{1}{k_1 - 1} \int_{\Omega} u^{k(x)+p_1-1} \, dx \\
+ \frac{1}{(k_1 - 1)^2} \left( \frac{1}{2} C_k + E(0) + \frac{1}{p_1 + 1} e^{\frac{p_2+1}{k_1-1} |\Omega|} \right) \tag{2.10}\]

By using (2.4), we can apply the Hölder and Young inequalities to get
\[
\int_{\Omega} u^{k(x)+p_2-1} \, dx \leq \int_{\Omega} 1.\alpha_1 \, dx + \int_{\Omega} \alpha_2 \cdot u^{\frac{k(x)(2n-3)}{2(n-2)}} \, dx \tag{2.11}\]
\[
\leq (\sup_{\Omega} \alpha_1) |\Omega| + (\sup_{\Omega} \alpha_2) \left( \int_{\Omega} u^{\frac{k(x)(2n-3)}{2(n-2)}} \, dx \right),
\]

and
\[
\int_{\Omega} u^{k(x)+p_1-1} \, dx \leq \int_{\Omega} 1.\alpha_3 \, dx + \int_{\Omega} \alpha_4 \cdot u^{\frac{k(x)(2n-3)}{2(n-2)}} \, dx \tag{2.12}\]
\[
\leq (\sup_{\Omega} \alpha_3) |\Omega| + (\sup_{\Omega} \alpha_4) \left( \int_{\Omega} u^{\frac{k(x)(2n-3)}{2(n-2)}} \, dx \right),
\]

where
\[
\alpha_1 = 1 - \frac{2(n-2)(k(x)+p_2-1)}{(2n-3)k(x)}, \quad \alpha_2 = \frac{2(n-2)(k(x)+p_2-1)}{(2n-3)k(x)};
\]
\[
\alpha_3 = 1 - \frac{2(n-2)(k(x)+p_1-1)}{(2n-3)k(x)}, \quad \alpha_4 = \frac{2(n-2)(k(x)+p_1-1)}{(2n-3)k(x)};
\]

observe that \(\alpha_2 \geq \alpha_4\) and \(\alpha_1 \leq \alpha_3\).

Combining (2.11) and (2.12) with (2.10) give
\[
\varphi'(t) \leq -\frac{1}{2} \frac{4}{(k_2)^2} \left( \int_{\Omega} |\nabla u|^{k_2} \, dx + \int_{\Omega} |\nabla u|^{k_1} \, dx \right) \\
+ \frac{2}{k_1 - 1} \left( \sup_{\Omega} \alpha_2 \right) \int_{\Omega} u^{\frac{k(x)(2n-3)}{2(n-2)}} \, dx \\
+ \frac{1}{(k_1 - 1)^2} \left( \frac{1}{2} C_k + E(0) + \frac{1}{p_1 + 1} e^{\frac{p_2+1}{k_1-1} |\Omega|} \right) + \frac{|\Omega|}{k_1 - 1} \sup_{\Omega} (\alpha_3 + \alpha_1) \tag{2.13}\]
We now make use of Schwarz’s inequality to the second term on the right-hand side of (2.13) as follows

\[
\int_{\Omega} u^{\frac{k(x)(n-3)}{2(n-2)}} dx \leq \left( \int_{\Omega} u^{k(x)} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} u^{\frac{k(x)(n-1)}{n-2}} dx \right)^{\frac{1}{2}} \tag{2.14}
\]

Next, by using the Sobolev inequality (see [5]), for \( n \geq 3 \), we get

\[
\left\| u^{\frac{k(x)}{2}} \right\|_{2^{\frac{n}{n-2}}} \leq B^{\frac{n}{2(n-2)}} \max \left( \left\| \nabla u^{\frac{k_2}{2}} \right\|_2, \left\| \nabla u^{\frac{k_1}{2}} \right\|_2 \right) \tag{2.15}
\]

where \( B \) is the best constant in the Sobolev inequality.

By inserting the last inequality in (2.14) and (2.15), we have

\[
\int_{\Omega} u^{\frac{k(x)(n-3)}{2(n-2)}} dx \leq B^{\frac{n}{2(n-2)}} \left( \int_{\Omega} u^{k(x)} dx \right)^{\frac{3}{4}} \left( \left( \int_{\Omega} \left| \nabla u^{\frac{k_2}{2}} \right|^\frac{4}{n-2} dx \right)^{\frac{n}{4(n-2)}} + \left( \int_{\Omega} \left| \nabla u^{\frac{k_1}{2}} \right|^\frac{4}{n-2} dx \right)^{\frac{n}{4(n-2)}} \right).
\]

Now, we can use the Young inequality to get

\[
\int_{\Omega} u^{\frac{k(x)(n-3)}{2(n-2)}} dx \leq 2B^{\frac{2n}{3n-8}} \frac{3n-8}{4(n-2)} \varepsilon^{\frac{n}{3n-8}} \left( \int_{\Omega} u^{k(x)} dx \right)^{\frac{3(n-2)}{4n-8}} \tag{2.16}
\]

\[
+ \frac{\varepsilon n}{4(n-2)} \left( \int_{\Omega} \left| \nabla u^{\frac{k_2}{2}} \right|^2 dx + \int_{\Omega} \left| \nabla u^{\frac{k_1}{2}} \right|^2 dx \right)
\]

where \( \varepsilon \) is a positive constant to be determined later. Combining (2.16) with (2.13), we obtain

\[
\varphi’(t) \leq C_1 + C_2 \varphi(t)^{\frac{3(n-2)}{4n-8}} + C_3 \left( \int_{\Omega} \left| \nabla u^{\frac{k_2}{2}} \right|^2 dx + \int_{\Omega} \left| \nabla u^{\frac{k_1}{2}} \right|^2 dx \right),
\]

where

\[
C_1 = \frac{1}{(k_1 - 1)^2} \left( \frac{1}{2} C_k + E(0) \frac{1}{p_1 + 1} e^{\frac{p_1 + 1}{k_1} |\Omega|} \right) + \frac{|\Omega|}{k_1 - 1} \sup_{\Omega} (\alpha_3 + \alpha_1)
\]

\[
C_2 = \frac{4}{k_1 - 1} \left( \sup_{\Omega} \alpha_2 \right) B^{\frac{2n}{3n-8}} \frac{3n-8}{4(n-2)} \varepsilon^{\frac{n}{3n-8}},
\]

\[
C_3 = \frac{2}{k_1 - 1} \frac{\varepsilon n}{4(n-2)} \left( \sup_{\Omega} \alpha_2 \right) - \frac{2}{(k_2)^2}.
\]
If we choose $\varepsilon > 0$ such that
\[ 0 < \varepsilon \leq \frac{4(n - 2)(k_1 - 1)}{(\sup_{\Omega} \alpha^2)n(k_2)^2}, \]
then, we obtain the differential inequality
\[ \varphi'(t) \leq C_1 + C_2 \varphi(t)^{\frac{3(n - 2)}{3n - 8}} \tag{2.17} \]
Integration of the differential inequality (2.17) from 0 to $t$ leads to
\[ \int_{\varphi(0)}^{\varphi(t)} \frac{d\gamma}{C_1 + C_2 \gamma^{\frac{3(n - 2)}{3n - 8}}} \leq t. \tag{2.18} \]
In fact, let $t \to T^-$, (2.18) leads to
\[ \int_{\varphi(0)}^{+\infty} \frac{d\gamma}{C_1 + C_2 \gamma^{\frac{3(n - 2)}{3n - 8}}} \leq T. \]
where
\[ \varphi(0) = \int_{\Omega} \frac{1}{k(x)(k(x) - 1)} u_0^k(x) dx. \]
Thus, the proof is achieved. \qed

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