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Some Bessel type additive inequalities in inner product spaces

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Abstract. In this paper we obtain some additive inequalities related to the celebrated Bessel's inequality in inner product spaces. They complement the results obtained by Boas-Bellman, Bombieri, Selberg and Heilbronn, which have been applied for almost orthogonal series and in Number Theory.

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1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\leq i\leq n}$ are orthonormal vectors in the inner product space H, i.e., $\langle e_i, e_j\rangle = \delta_{ij}$ for all $i, j \in \{1,\dots,n\}$ where δ_{ij} is the Kronecker delta, then the following inequality is well known in the literature as *Bessel's inequality*:

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2 \text{ for any } x \in H. \tag{1.1}$$

For other results related to Bessel's inequality, see [8] – [11] and Chapter XV in the book [14].

In 1941, R. P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalization of Bessel's inequality (see also [14, p. 392]):

Theorem 1.1. *If $x, y_1, \dots, y_n$ are elements of an inner product space $(H;\langle\cdot,\cdot\rangle)$, then the following inequality holds*

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \leq \|x\|^2 \left[\max_{1\leq i\leq n} \|y_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle y_i, y_j\rangle|^2\right)^{\frac{1}{2}}\right]. \tag{1.2}$$

It is obvious that (1.2) will give for orthonormal families the well known Bessel inequality.

In [7] we pointed out the following Boas-Bellman type inequalities:

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \le \|x\| \max_{1\le i\le n} |\langle x, y_i \rangle| \left\{ \sum_{i=1}^{n} \|y_i\|^2 + \sum_{1\le i\ne j\le n} |\langle y_i, y_j \rangle| \right\}^{\frac{1}{2}}, \tag{1.3}$$

for any x, $y_1, \ldots, y_n$ vectors in the inner product space $(H; \langle\cdot,\cdot\rangle)$.

We also have, see [7]

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \le \|x\| \left(\sum_{i=1}^{n} |\langle x, y_i \rangle|^{2p} \right)^{\frac{1}{2p}} \tag{1.4}$$

$$\times \left\{ \left(\sum_{i=1}^{n} \|y_i\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1\le i\ne j\le n} |\langle y_i, y_j \rangle|^{q} \right)^{\frac{1}{q}} \right\}^{\frac{1}{2}},$$

for any x, $y_1, \ldots, y_n \in H$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Further, we recall [7] that

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \le \|x\|^2 \left\{ \max_{1\le i\le n} \|y_i\|^2 + (n-1) \max_{1\le i\ne j\le n} |\langle y_i, y_j \rangle| \right\}, \tag{1.5}$$

for any x, $y_1, \ldots, y_n \in H$. It is obvious that (1.5) will give for orthonormal families the well known Bessel inequality.

In 1971, E. Bombieri [3] gave the following generalization of Bessel's inequality.

Theorem 1.2. *If x, $y_1, \ldots, y_n$ are vectors in the inner product space $(H; (\cdot,\cdot))$, then the following inequality holds:*

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \le \|x\|^2 \max_{1\le i\le n} \left\{ \sum_{j=1}^{n} |\langle y_i, y_j \rangle| \right\}. \tag{1.6}$$

It is obvious that if $(y_i)_{1\le i\le n}$ are orthonormal, then from (1.6) one can deduce Bessel's inequality.

It is not widely known, but it appears in a number of places that, the importance of extensions of the Bombieri and Bessel inequality were first shown by J. Sándor (at a Symposium on Mathematical Inequalities, Sibiu, December, 1984), who proved some generalizations of these inequalities, and who was deeply interested in applications in Number Theory. Also, Bessel's inequality and Gram's inequality have been studied by the author and J. Sándor in [12] as well.

Another generalization of Bessel's inequality was obtained by A. Selberg (see for example [14, p. 394]):

Theorem 1.3. *Let x, $y_1, \dots, y_n$ be vectors in H with $y_i \neq 0 \quad (i = 1, \dots, n)$. Then one has the inequality:*

$$\sum_{i=1}^{n} \frac{|\langle x, y_i \rangle|^2}{\sum_{j=1}^{n} |\langle y_i, y_j \rangle|} \leq \|x\|^2 . \tag{1.7}$$

Another type of inequality related to Bessel's result, was discovered in 1958 by H. Heilbronn [13] (see also [14, p. 395]).

Theorem 1.4. *With the assumptions in Theorem* 1.2*, one has*

$$\sum_{i=1}^{n} |\langle x, y_i \rangle| \leq \|x\| \left(\sum_{i,j=1}^{n} |\langle y_i, y_j \rangle| \right)^{\frac{1}{2}} . \tag{1.8}$$

In [8] we obtained the following Bombieri type inequalities

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\| \max_{1 \leq i \leq n} |\langle x, y_i \rangle| \left(\sum_{i,j=1}^{n} |\langle y_i, y_j \rangle| \right)^{\frac{1}{2}} , \tag{1.9}$$

$$\begin{aligned} &\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \\ &\leq \|x\| \max_{1 \leq i \leq n} |\langle x, y_i \rangle|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\langle x, y_i \rangle|^r \right)^{\frac{1}{2r}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\langle y_i, y_j \rangle| \right)^s \right]^{\frac{1}{2s}} , \end{aligned} \tag{1.10}$$

where $\frac{1}{r} + \frac{1}{s} = 1$, $s > 1$,

$$\begin{aligned} &\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \\ &\leq \|x\| \max_{1 \leq i \leq n} |\langle x, y_i \rangle|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\langle x, y_i \rangle| \right)^{\frac{1}{2}} \left[\max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |\langle y_i, y_j \rangle| \right) \right] , \end{aligned} \tag{1.11}$$

$$\begin{aligned} &\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \\ &\leq \|x\| \max_{1 \leq i \leq n} |\langle x, y_i \rangle|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\langle x, y_i \rangle|^p \right)^{\frac{1}{2p}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\langle y_i, y_j \rangle|^q \right)^{\frac{1}{q}} \right]^{\frac{1}{2}} , \end{aligned} \tag{1.12}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\|^2 \left\{ \sum_{i,j=1}^{n} |\langle y_i, y_j \rangle|^2 \right\}^{\frac{1}{2}} \tag{1.13}$$

for any $x \in H$.

It has been shown that for different selection of vectors the upper bound provided by the inequality (1.13) is some time better other times worse than the one obtained by Bombieri above in (1.6).

In this paper we obtain some inequalities related to the celebrated Bessel's inequality in inner product spaces. They complement the results obtained by Boas-Bellman, Bombieri, Selberg and Heilbronn above, which have been applied for almost orthogonal series and in Number Theory.

2. Some results via CBS inequality

We have:

Theorem 2.1. *Let x, $y_1, ..., y_n \in H$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$. Then*

$$\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) \le \frac{1}{2}\left[\|x\|^2 + \sum_{k=1}^{n} |\alpha_k|^2 \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right]. \tag{2.1}$$

Proof. We have for any x, $y_1, ..., y_n \in H$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$ that

$$\begin{aligned} 0 &\le \left\|\sum_{j=1}^{n} \alpha_j y_j - x\right\|^2 = \left\|\sum_{j=1}^{n} \alpha_j y_j\right\|^2 - 2\operatorname{Re}\left\langle \sum_{j=1}^{n} \alpha_j y_j, x\right\rangle + \|x\|^2 \\ &= \left\langle \sum_{j=1}^{n} \alpha_j y_j, \sum_{k=1}^{n} \alpha_k y_k\right\rangle - 2\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) + \|x\|^2 \\ &= \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k\rangle - 2\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) + \|x\|^2, \end{aligned}$$

which implies the inequality

$$\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) \le \frac{1}{2}\left[\|x\|^2 + \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k\rangle\right] \tag{2.2}$$

for which the term $\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k\rangle$ is obviously nonnegative for any $y_1, ..., y_n \in H$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$.

By using the Cauchy-Buniakowski-Schwarz's inequality for double sums,

$$\sum_{j,k=1}^{n} |a_{jk} b_{jk}| \le \left(\sum_{j,k=1}^{n} |a_{jk}|^2\right)^{1/2} \left(\sum_{j,k=1}^{n} |b_{jk}|^2\right)^{1/2}$$

for complex numbers $a_{jk},\ b_{jk}$ where $j,\ k \in \{1,...,n\}$, then we have

$$\begin{aligned}\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle &= \left| \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle \right| \leq \sum_{j,k=1}^{n} |\alpha_j \overline{\alpha_k}| \, |\langle y_j, y_k \rangle| \\ &\leq \left(\sum_{j,k=1}^{n} |\alpha_j \overline{\alpha_k}|^2 \right)^{1/2} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \\ &= \left(\sum_{j,k=1}^{n} |\alpha_j|^2 |\overline{\alpha_k}|^2 \right)^{1/2} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \\ &= \left(\sum_{j=1}^{n} |\alpha_j|^2 \sum_{k=1}^{n} |\alpha_k|^2 \right)^{1/2} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \\ &= \sum_{k=1}^{n} |\alpha_k|^2 \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2}\end{aligned} \tag{2.3}$$

for any $y_1, ..., y_n \in H$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$.
By making use of (2.2) and (2.3) we get the desired result (2.1). □

Corollary 2.2. *With the assumptions of Theorem* 2.1 *and for* $p \geq 1$ *we have*

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^p \leq \frac{1}{2} \left[\|x\|^2 + \sum_{k=1}^{n} |\langle x, y_k \rangle|^{2(p-1)} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \right]. \tag{2.4}$$

Proof. If we take in (2.1) $\alpha_j = \langle x, y_j \rangle |\langle x, y_j \rangle|^{p-2}$ then we get

$$\begin{aligned}&\operatorname{Re} \left(\sum_{j=1}^{n} \langle x, y_j \rangle |\langle x, y_j \rangle|^{p-2} \langle y_j, x \rangle \right) \\ &\leq \frac{1}{2} \left[\|x\|^2 + \sum_{k=1}^{n} \left| \langle x, y_j \rangle |\langle x, y_j \rangle|^{p-2} \right|^2 \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \right],\end{aligned}$$

which is equivalent to (2.4). □

Remark 2.3. If we take in (2.4) $p = 1$, then we get the following Heilbronn type inequality

$$\sum_{j=1}^{n} |\langle x, y_j \rangle| \leq \frac{1}{2} \left[\|x\|^2 + n \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \right] \tag{2.5}$$

for any $x,\ y_1, ..., y_n \in H$.

If we take in (2.4) $p = 2$, then we get

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \frac{1}{2}\left[\|x\|^2 + \sum_{k=1}^{n} |\langle x, y_k\rangle|^2 \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right], \tag{2.6}$$

that is equivalent to (see also [10])

$$\left[2 - \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right] \sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \|x\|^2 \tag{2.7}$$

for any x, $y_1, ..., y_n \in H$.
The inequality (2.7) is meaningful if

$$2 \ge \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}.$$

Also if

$$1 \ge \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2},$$

then

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \left[2 - \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right] \sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \|x\|^2, \tag{2.8}$$

for any $x \in H$, which improves Bessel's inequality.
We observe that if the family of vectors $\{y_1, ..., y_n\}$ is orthogonal, then

$$\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2 = \sum_{k=1}^{n} \|y\|_k^4,$$

so, if we assume that

$$\sum_{k=1}^{n} \|y\|_k^4 \le 1$$

then by (2.8) we get the refinement of Bessel's inequality

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \left[2 - \left(\sum_{k=1}^{n} \|y\|_k^4\right)^{1/2}\right] \sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \|x\|^2. \tag{2.9}$$

Corollary 2.4. *With the assumptions of Theorem* 2.1 *we have*

$$\sum_{j=1}^{n} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{n} |\langle y_k, y_j\rangle|} \tag{2.10}$$
$$\leq \frac{1}{2}\left[\|x\|^2 + \sum_{j=1}^{n} \frac{|\langle x, y_j\rangle|^2}{\left(\sum_{k=1}^{n} |\langle y_k, y_j\rangle|\right)^2}\left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right],$$

for any $x \in H$.

Proof. We take in (2.1)

$$\alpha_j = \frac{\langle x, y_j\rangle}{\sum_{k=1}^{n} |\langle y_k, y_j\rangle|}, \ j = 1, ..., n$$

to get (2.10). □

Using the Schwarz's inequality we get from (2.4) that

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^p \tag{2.11}$$
$$\leq \frac{1}{2}\|x\|^2\left[1 + \|x\|^{2(p-2)}\sum_{k=1}^{n}\|y_k\|^{2(p-1)}\left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right],$$

for any x, $y_1, ..., y_n \in H$ and $p \geq 1$.
For $p = 2$ we get

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \leq \frac{1}{2}\|x\|^2\left[1 + \sum_{k=1}^{n}\|y_k\|^2\left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right], \tag{2.12}$$

for any x, $y_1, ..., y_n \in H$.
From (2.10) we also get Selberg's type inequality

$$\sum_{j=1}^{n} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{n} |\langle y_k, y_j\rangle|} \tag{2.13}$$
$$\leq \frac{1}{2}\|x\|^2\left[1 + \sum_{j=1}^{n} \frac{\|y_j\|^2}{\left(\sum_{k=1}^{n} |\langle y_k, y_j\rangle|\right)^2}\left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right],$$

for any x, $y_1, ..., y_n \in H$.

Theorem 2.5. *Let* x, $y_1, ..., y_n \in H$ *and* $\alpha_1, ..., \alpha_n \in \mathbb{C}$. *Then*

$$\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) \leq \frac{1}{2}\left[\|x\|^2 + \max_{k\in\{1,...,n\}}\left\{\sum_{j=1}^{n} |\langle y_j, y_k\rangle|\right\}\sum_{k=1}^{n} |a_k|^2\right]. \tag{2.14}$$

Proof. By using the Cauchy-Buniakowski-Schwarz's weighted inequality for double sums,

$$\sum_{j,k=1}^{n} m_{jk} \left|a_{jk} b_{jk}\right| \leq \left(\sum_{j,k=1}^{n} m_{jk} \left|a_{jk}\right|^{2}\right)^{1/2} \left(\sum_{j,k=1}^{n} m_{jk} \left|b_{jk}\right|^{2}\right)^{1/2}$$

for complex numbers a_{jk}, b_{jk} and nonnegative numbers m_{jk} where j, $k \in \{1, ..., n\}$, then we have

$$\begin{aligned} &\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle \qquad (2.15)\\ &= \left|\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle\right| \leq \sum_{j,k=1}^{n} \left|\alpha_j \overline{\alpha_k}\right| \left|\langle y_j, y_k \rangle\right| = \sum_{j,k=1}^{n} \left|\alpha_j\right| \left|a_k\right| \left|\langle y_j, y_k \rangle\right| \\ &\leq \left(\sum_{j,k=1}^{n} \left|\langle y_j, y_k \rangle\right| \left|a_j\right|^{2}\right)^{1/2} \left(\sum_{j,k=1}^{n} \left|\langle y_j, y_k \rangle\right| \left|a_k\right|^{2}\right)^{1/2} \\ &= \sum_{j,k=1}^{n} \left|a_k\right|^{2} \left|\langle y_j, y_k \rangle\right|. \end{aligned}$$

Now, observe that

$$\begin{aligned} \sum_{j,k=1}^{n} \left|a_k\right|^{2} \left|\langle y_j, y_k \rangle\right| &= \sum_{k=1}^{n} \left|a_k\right|^{2} \left(\sum_{j=1}^{n} \left|\langle y_j, y_k \rangle\right|\right) \\ &\leq \max_{k \in \{1,\ldots,n\}} \left\{\sum_{j=1}^{n} \left|\langle y_j, y_k \rangle\right|\right\} \sum_{k=1}^{n} \left|a_k\right|^{2}, \end{aligned}$$

which proves the desired inequality (2.14). □

Corollary 2.6. *With the assumptions of Theorem* 2.5 *and for* $p \geq 1$ *we have*

$$\sum_{j=1}^{n} \left|\langle x, y_j \rangle\right|^{p} \leq \frac{1}{2} \left[\|x\|^{2} + \max_{k \in \{1,\ldots,n\}} \left\{\sum_{j=1}^{n} \left|\langle y_j, y_k \rangle\right|\right\} \sum_{k=1}^{n} \left|\langle x, y_k \rangle\right|^{2(p-1)}\right]. \qquad (2.16)$$

Proof. If we take in (2.14) $\alpha_j = \langle x, y_j \rangle \left|\langle x, y_j \rangle\right|^{p-2}$ then we get

$$\begin{aligned} &\operatorname{Re}\left(\sum_{j=1}^{n} \langle x, y_j \rangle \left|\langle x, y_j \rangle\right|^{p-2} \langle y_j, x \rangle\right) \\ &\leq \frac{1}{2} \left[\|x\|^{2} + \max_{k \in \{1,\ldots,n\}} \left\{\sum_{j=1}^{n} \left|\langle y_j, y_k \rangle\right|\right\} \sum_{k=1}^{n} \left|\langle x, y_j \rangle \left|\langle x, y_j \rangle\right|^{p-2}\right|^{2}\right], \end{aligned}$$

which is equivalent to (2.16). □

Remark 2.7. If we take in (2.16) $p = 1$, then we get the following Heilbronn type inequality

$$\sum_{j=1}^{n} |\langle x, y_j \rangle| \le \frac{1}{2} \left[\|x\|^2 + \max_{k \in \{1,\dots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} \right]. \tag{2.17}$$

for any $x,\ y_1, ..., y_n \in H$.
If we take in (2.16) $p = 2$, then we get

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \le \frac{1}{2} \left[\|x\|^2 + \max_{k \in \{1,\dots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} \sum_{k=1}^{n} |\langle x, y_j \rangle|^2 \right], \tag{2.18}$$

which is equivalent to

$$\left(2 - \max_{k \in \{1,\dots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} \right) \sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \le \|x\|^2 \tag{2.19}$$

for any $x,\ y_1, ..., y_n \in H$.
The inequality (2.19) is meaningful if

$$2 \ge \max_{k \in \{1,\dots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\}.$$

Also if

$$1 \ge \max_{k \in \{1,\dots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\},$$

then

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \le \left[2 - \max_{k \in \{1,\dots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} \right] \sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \le \|x\|^2, \tag{2.20}$$

for any $x \in H$, which improves Bessel's inequality.
We observe that if the family of vectors $\{y_1, ..., y_n\}$ is orthogonal, then

$$\max_{k \in \{1,\dots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} = \max_{k \in \{1,\dots,n\}} \|y\|_k^2,$$

so, if we assume that $\max_{k \in \{1,\dots,n\}} \|y\|_k^2 \le 1$ then by (2.20) we get

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \le \left[2 - \max_{k \in \{1,\dots,n\}} \|y\|_k^2 \right] \sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \le \|x\|^2, \tag{2.21}$$

for any $x \in H$.

Corollary 2.8. *With the assumptions of Theorem* 2.5 *we have*

$$\sum_{j=1}^{n}\frac{\left|\langle x,y_j\rangle\right|^2}{\sum_{k=1}^{n}\left|\langle y_k,y_j\rangle\right|} \tag{2.22}$$
$$\leq \frac{1}{2}\left[\left\|x\right\|^2+\max_{j\in\{1,\dots,n\}}\left\{\sum_{k=1}^{n}\left|\langle y_j,y_k\rangle\right|\right\}\sum_{k=1}^{n}\frac{\left|\langle x,y_k\rangle\right|^2}{\left(\sum_{j=1}^{n}\left|\langle y_k,y_j\rangle\right|\right)^2}\right],$$

for any $x\in H$.

Proof. We take in (2.1)

$$\alpha_k=\frac{\langle x,y_k\rangle}{\sum_{j=1}^{n}\left|\langle y_k,y_j\rangle\right|},\ \ k=1,\dots,n$$

to get (2.10). □

Using the Schwarz's inequality we get from (2.16) that

$$\sum_{j=1}^{n}\left|\langle x,y_j\rangle\right|^p \tag{2.23}$$
$$\leq \frac{1}{2}\left\|x\right\|^2\left[1+\left\|x\right\|^{2(p-2)}\max_{k\in\{1,\dots,n\}}\left\{\sum_{j=1}^{n}\left|\langle y_j,y_k\rangle\right|\right\}\sum_{k=1}^{n}\left\|y_k\right\|^{2(p-1)}\right],$$

for any x, $y_1,\dots,y_n\in H$.
If in this inequality we take $p=2$, then we get

$$\sum_{j=1}^{n}\left|\langle x,y_j\rangle\right|^2\leq\frac{1}{2}\left\|x\right\|^2\left[1+\max_{k\in\{1,\dots,n\}}\left\{\sum_{j=1}^{n}\left|\langle y_j,y_k\rangle\right|\right\}\sum_{k=1}^{n}\left\|y_k\right\|^2\right], \tag{2.24}$$

for any x, $y_1,\dots,y_n\in H$.
From (2.22) we also get the Selberg type inequality

$$\sum_{j=1}^{n}\frac{\left|\langle x,y_j\rangle\right|^2}{\sum_{k=1}^{n}\left|\langle y_k,y_j\rangle\right|} \tag{2.25}$$
$$\leq \frac{1}{2}\left\|x\right\|^2\left[1+\max_{j\in\{1,\dots,n\}}\left\{\sum_{k=1}^{n}\left|\langle y_j,y_k\rangle\right|\right\}\sum_{k=1}^{n}\frac{\left\|y_k\right\|^2}{\left(\sum_{j=1}^{n}\left|\langle y_k,y_j\rangle\right|\right)^2}\right],$$

for any x, $y_1,\dots,y_n\in H$.

3. Related inequalities

We have:

Theorem 3.1. *Let $x,\ y_1,...,y_n \in H$ and $\alpha_1,...,\alpha_n \in \mathbb{C}$. Then*

$$\operatorname{Re}\left(\sum_{j=1}^{n}\alpha_j\langle y_j,x\rangle\right) \le \frac{1}{2}\left[\|x\|^2 + \max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\left(\sum_{j=1}^{n}|\alpha_j|\right)^2\right] \tag{3.1}$$

and

$$\operatorname{Re}\left(\sum_{j=1}^{n}\alpha_j\langle y_j,x\rangle\right) \le \frac{1}{2}\left[\|x\|^2 + \max_{k\in\{1,\dots,n\}}\left\{|\alpha_k|^2\right\}\sum_{j,k=1}^{n}|\langle y_j,y_k\rangle|\right]. \tag{3.2}$$

Proof. From (2.3) we have

$$\begin{aligned}\sum_{j,k=1}^{n}\alpha_j\overline{\alpha_k}\langle y_j,y_k\rangle &= \left|\sum_{j,k=1}^{n}\alpha_j\overline{\alpha_k}\langle y_j,y_k\rangle\right| \le \sum_{j,k=1}^{n}|\alpha_j\overline{\alpha_k}|\,|\langle y_j,y_k\rangle| \\ &\le \max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\sum_{j,k=1}^{n}|\alpha_j\overline{\alpha_k}| \\ &= \max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\sum_{j,k=1}^{n}|\alpha_j|\,|\overline{\alpha_k}| \\ &= \max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\left(\sum_{j=1}^{n}|\alpha_j|\right)^2,\end{aligned}$$

for any $x,\ y_1,...,y_n \in H$ and $\alpha_1,...,\alpha_n \in \mathbb{C}$, which proves (3.1).
Similarly, we have

$$\begin{aligned}\sum_{j,k=1}^{n}\alpha_j\overline{\alpha_k}\langle y_j,y_k\rangle &= \left|\sum_{j,k=1}^{n}\alpha_j\overline{\alpha_k}\langle y_j,y_k\rangle\right| \le \sum_{j,k=1}^{n}|\alpha_j\overline{\alpha_k}|\,|\langle y_j,y_k\rangle| \\ &\le \max_{j,k\in\{1,\dots,n\}}\{|\alpha_j\overline{\alpha_k}|\}\sum_{j,k=1}^{n}|\langle y_j,y_k\rangle| \\ &= \max_{k\in\{1,\dots,n\}}\left\{|\alpha_k|^2\right\}\sum_{j,k=1}^{n}|\langle y_j,y_k\rangle|\end{aligned}$$

for any $x,\ y_1,...,y_n \in H$ and $\alpha_1,...,\alpha_n \in \mathbb{C}$, which proves (3.2). □

Corollary 3.2. *With the assumptions of Theorem* 3.1 *and for $p \ge 1$ we have*

$$\sum_{j=1}^{n}|\langle x,y_j\rangle|^p \le \frac{1}{2}\left[\|x\|^2 + \max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\left(\sum_{j=1}^{n}|\langle x,y_j\rangle|^{p-1}\right)^2\right] \tag{3.3}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^p \leq \frac{1}{2}\left[\|x\|^2 + \max_{k\in\{1,\ldots,n\}}\left\{|\langle x, y_j\rangle|^{2(p-1)}\right\}\sum_{j,k=1}^{n}|\langle y_j, y_k\rangle|\right] \tag{3.4}$$

for any $x,\ y_1, \ldots, y_n \in H$.

Proof. If we take in (3.1) and (3.2) $\alpha_j = \langle x, y_j\rangle\, |\langle x, y_j\rangle|^{p-2}$ then we get (3.3) and (3.4). □

Remark 3.3. If we take in (3.3) and (3.4) $p = 1$, then we get

$$\sum_{j=1}^{n} |\langle x, y_j\rangle| \leq \frac{1}{2}\left[\|x\|^2 + n^2 \max_{j,k\in\{1,\ldots,n\}}\{|\langle y_j, y_k\rangle|\}\right] \tag{3.5}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j\rangle| \leq \frac{1}{2}\left[\|x\|^2 + \sum_{j,k=1}^{n}|\langle y_j, y_k\rangle|\right] \tag{3.6}$$

for any $x,\ y_1, \ldots, y_n \in H$.
If we take in (3.3) and (3.4) $p = 2$, then we get

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \leq \frac{1}{2}\left[\|x\|^2 + \max_{j,k\in\{1,\ldots,n\}}\{|\langle y_j, y_k\rangle|\}\left(\sum_{j=1}^{n}|\langle x, y_j\rangle|\right)^2\right] \tag{3.7}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \leq \frac{1}{2}\left[\|x\|^2 + \max_{k\in\{1,\ldots,n\}}\left\{|\langle x, y_k\rangle|^2\right\}\sum_{j,k=1}^{n}|\langle y_j, y_k\rangle|\right] \tag{3.8}$$

for any $x,\ y_1, \ldots, y_n \in H$.

Using Schwarz's inequality we have from (3.3) and (3.4) that

$$\begin{aligned}&\sum_{j=1}^{n} |\langle x, y_j\rangle|^p \\ &\leq \frac{1}{2}\|x\|^2\left[1 + \|x\|^{2(p-2)}\max_{j,k\in\{1,\ldots,n\}}\{|\langle y_j, y_k\rangle|\}\left(\sum_{j=1}^{n}\|y_j\|^{p-1}\right)^2\right]\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\sum_{j=1}^{n} |\langle x, y_j\rangle|^p \\ &\leq \frac{1}{2}\|x\|^2\left[1 + \|x\|^{2(p-2)}\max_{k\in\{1,\ldots,n\}}\left\{\|y_k\|^{2(p-1)}\right\}\sum_{j,k=1}^{n}|\langle y_j, y_k\rangle|\right]\end{aligned} \tag{3.10}$$

for any $x,\ y_1, \ldots, y_n \in H$.

For $p = 2$ we get

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \frac{1}{2} \|x\|^2 \left[1 + \max_{j,k\in\{1,\dots,n\}} \{|\langle y_j, y_k\rangle|\} \left(\sum_{j=1}^{n} \|y_j\|\right)^2\right] \tag{3.11}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \frac{1}{2} \|x\|^2 \left[1 + \max_{k\in\{1,\dots,n\}} \left\{\|y_k\|^2\right\} \sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|\right] \tag{3.12}$$

for any x, $y_1, \dots, y_n \in H$.
We observe that if $y_1, \dots, y_n \in H$ are such that

$$\max_{j,k\in\{1,\dots,n\}} \{|\langle y_j, y_k\rangle|\} \left(\sum_{j=1}^{n} \|y_j\|\right)^2 \le 1,$$

then (3.1) provides a refinement of Bessel's inequality. Also, if

$$\max_{k\in\{1,\dots,n\}} \left\{\|y_k\|^2\right\} \sum_{j,k=1}^{n} |\langle y_j, y_k\rangle| \le 1,$$

then (3.12) also provides a refinement of Bessel's inequality.
By using Hölder's inequality we can provide other inequalities as follows:

Theorem 3.4. *Let x, $y_1, \dots, y_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Then for $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$*

$$\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) \le \frac{1}{2}\left[\|x\|^2 + \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^r\right)^{1/r} \left(\sum_{j=1}^{n} |\alpha_j|^q\right)^{2/q}\right] \tag{3.13}$$

Proof. From (2.3) and Hölder's inequality we have

$$\begin{aligned}
\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k\rangle &= \left|\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k\rangle\right| \le \sum_{j,k=1}^{n} |\alpha_j \overline{\alpha_k}| \, |\langle y_j, y_k\rangle| \\
&\le \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^r\right)^{1/r} \left(\sum_{j,k=1}^{n} |\alpha_j \overline{\alpha_k}|^q\right)^{1/q} \\
&= \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^r\right)^{1/r} \left(\sum_{j,k=1}^{n} |\alpha_j|^q \, |\alpha_k|^q\right)^{1/q} \\
&= \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^r\right)^{1/r} \left(\sum_{j=1}^{n} |\alpha_j|^q\right)^{2/q},
\end{aligned}$$

for any x, $y_1, \dots, y_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, which proves (3.13). □

Corollary 3.5. *With the assumptions of Theorem* 3.4 *and for* $p \geq 1$ *we have*

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^p \leq \frac{1}{2} \left[\|x\|^2 + \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} |\langle x, y_j \rangle|^{q(p-1)} \right)^{2/q} \right] \tag{3.14}$$

for any $x, y_1, ..., y_n \in H$. *In particular, we have*

$$\sum_{j=1}^{n} |\langle x, y_j \rangle| \leq \frac{1}{2} \left[\|x\|^2 + n^{2/q} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \right] \tag{3.15}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \frac{1}{2} \left[\|x\|^2 + \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} |\langle x, y_j \rangle|^{2q} \right)^{2/q} \right]. \tag{3.16}$$

We observe that, by Schwarz's inequality we get for $p \geq 1$

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^p \tag{3.17}$$
$$\leq \frac{1}{2} \|x\|^2 \left[1 + \|x\|^{2(p-2)} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} \|y_j\|^{q(p-1)} \right)^{2/q} \right],$$

for any $x, y_1, ..., y_n \in H$, where $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$.
For $p = 2$, we get

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \frac{1}{2} \|x\|^2 \left[1 + \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} \|y_j\|^q \right)^{2/q} \right], \tag{3.18}$$

for any $x, y_1, ..., y_n \in H$, where $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$.
We observe that if $y_1, ..., y_n \in H$ are such that

$$\left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} \|y_j\|^q \right)^{2/q} \leq 1,$$

where $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$, then (3.18) provides a refinement of Bessel's inequality.
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