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On Fejér type inequalities for products convex and s-convex functions

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Abstract. In this paper, we first obtain some new Fejér type inequalities for products of convex and s-convex mappings. Then, some Fejér type inequalities for products of two s-convex function are established.

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1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [6], [14, p. 137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Over the years, many studies have focused on to establish generalization of the inequality (1.1) and to obtain new bounds for left hand side and right hand side of the inequality (1.1).

The overall structure of the paper takes the form of five sections including introduction. The remainder of this work is organized as follows: we first give some Hermite-Hadamard and Fejér type inequalities.. Moreover, we give some Hermite-Hadamard type inequalities for products two convex functions. In Section 2 and Section 3, we obtain some integral inequalities of Hermite-Hadamard-Fejér type for products convex and s-convex functions and for products two s-convex functions. We give also some special cases of these inequalities. Finally, conclusions and future directions of research are discussed in Section 4.

The weighted version of the inequalities (1.1), so-called Hermite-Hadamard-Fejér inequalities, was given by Fejer in [7] as follow:

Theorem 1.1. *$f:[a,b]\to\mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \le \int_a^b f(x)w(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.2}$$

holds, where $w:[a,b]\to\mathbb{R}$ is non-negative, integrable, and symmetric about $x=\frac{a+b}{2}$ (i.e. $w(x)=w(a+b-x)$).

In [13], Pachpatte established the Hermite-Hadamard type inequalities for products of two convex functions.

Theorem 1.2. *Let f and g be real-valued, non-negative and convex functions on $[a,b]$. Then we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b), \tag{1.3}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b) \tag{1.4}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

In recent years, the generalized versions of inequalities (1.3) and (1.4) for several convexity have been proved. For some of them please refer to ([4]-[5], [8], [16], [17]). Kirmaci et al. gave the proved inequalities (1.3) and (1.4) for products of convex and s-convex functions in [9]. On the other hand, Budak and Bakış [1] proved the weighted versions of the inequalities (1.3) and (1.4) which generalize the several obtained inequalities. Moreover in [10], Latif and Alomari proved some inequalities for product of two co-ordinated convex function. Furthermore in [11] and [12], Ozdemir et al. gave some generalizations of results given by Latif and Alomari using the product of two coordinated s-convex mappings and product of two coordinated h-convex mappings, respectively. In [2], Budak and Sarıkaya proved Hermite-Hadamard type inequalities for products of two co-ordinated convex mappings via fractional integrals.

2. Fejér type inequalities for products convex and s-convex functions

In this section, we present some Fejér type inequalities for products convex and s-convex functions.

Theorem 2.1. *Suppose that $w:I\to\mathbb{R}$ is non-negative, integrable, and symmetric about $x=\frac{a+b}{2}$ (i.e. $w(x)=w(a+b-x)$). If $f:I\to\mathbb{R}$ is a real-valued, non-negative* *and convex functions on I and if $g : I \to \mathbb{R}$ is a s-convex on I for some fixed $s \in (0,1]$, then for any $a, b \in I$, we have*

$$\int_a^b f(x)g(x)w(x)dx \leq \frac{M(a,b)}{(b-a)^{s+1}} \int_a^b (b-x)^{s+1} w(x)dx + \frac{N(a,b)}{(b-a)^{s+1}} \int_a^b (b-x)(x-a)^s w(x)dx \tag{2.1}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b) \text{ and } N(a,b) = f(a)g(b) + f(b)g(a).$$

Proof. Since f is convex and g is s-convex functions on $[a,b]$, then we have

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{2.2}$$

and

$$g(ta + (1-t)b) \leq t^s g(a) + (1-t)^s g(b). \tag{2.3}$$

By adding the inequalities (2.2) and (2.3), we get

$$\begin{aligned} & f(ta + (1-t)b)\, g(ta + (1-t)b) \\ \leq\ & t^{s+1} f(a)g(a) + (1-t)^{s+1} f(b)g(b) \\ +\ & t(1-t)^s f(a)g(b) + t^s(1-t) f(b)g(a). \end{aligned} \tag{2.4}$$

Multiplying both sides of (2.4) by $w(ta + (1-t)b)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$\begin{aligned} & \int_0^1 f(ta + (1-t)b)\, g(ta + (1-t)b)\, w(ta + (1-t)b)\, dt \\ \leq\ & f(a)g(a) \int_0^1 t^{s+1} w(ta + (1-t)b)\, dt \\ & + f(b)g(b) \int_0^1 (1-t)^{s+1} w(ta + (1-t)b)\, dt \\ & + f(a)g(b) \int_0^1 t(1-t)^s w(ta + (1-t)b)\, dt \\ & + f(b)g(a) \int_0^1 t^s (1-t) w(ta + (1-t)b)\, dt. \end{aligned} \tag{2.5}$$

By change of variable $x = ta + (1-t)\,b$ with $dx = -(b-a)dt$, we get

$$\int_0^1 f\left(ta + (1-t)\,b\right) g\left(ta + (1-t)\,b\right) w\left(ta + (1-t)\,b\right) dt \tag{2.6}$$

$$= \frac{1}{b-a} \int_a^b f(x)g(x)w(x)dx.$$

Moreover, it is easily observe that

$$\int_0^1 t^{s+1} w\left(ta + (1-t)\,b\right) dt = \frac{1}{(b-a)^{s+2}} \int_a^b (b-x)^{s+1}\, w(x)dx \tag{2.7}$$

and since w is symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned} \int_0^1 (1-t)^{s+1}\, w\left(ta + (1-t)\,b\right) dt &= \frac{1}{(b-a)^{s+2}} \int_a^b (x-a)^{s+1}\, w(x)dx \\ &= \frac{1}{(b-a)^{s+2}} \int_a^b (b-u)^{s+1}\, w(a+b-u)du. \\ &= \frac{1}{(b-a)^{s+2}} \int_a^b (b-u)^{s+1}\, w(u)du. \end{aligned} \tag{2.8}$$

We also have

$$\int_0^1 t\,(1-t)^s\, w\left(ta + (1-t)\,b\right) dt = \frac{1}{(b-a)^{s+2}} \int_a^b (b-x)\,(x-a)^s\, w(x)dx \tag{2.9}$$

and

$$\int_0^1 t^s\,(1-t)\, w\left(ta + (1-t)\,b\right) dt \tag{2.10}$$

$$\begin{aligned} &= \frac{1}{(b-a)^{s+2}} \int_a^b (b-x)^s\,(x-a)\, w(x)dx \\ &= \frac{1}{(b-a)^{s+2}} \int_a^b (b-u)\,(u-a)^s\, w(a+b-u)du \\ &= \frac{1}{(b-a)^{s+2}} \int_a^b (b-u)\,(u-a)^s\, w(u)du. \end{aligned}$$

By substituting the equalities (2.6)-(2.10) in (2.5), then we have the following inequality

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)w(x)dx \qquad (2.11)\\
&\leq \frac{[f(a)g(a)+f(b)g(b)]}{(b-a)^{s+2}}\int_a^b (b-x)^{s+1}w(x)dx\\
&+\frac{f(a)g(b)+f(b)g(a)}{(b-a)^{s+2}}\int_a^b (b-x)(x-a)^s w(x)dx.
\end{aligned}$$

If we multiply both sides of (2.11) by $(b-a)$, then we obtain the desired result. □

Remark 2.2. If we choose $w(x)=1$ for all $x\in[a,b]$ in Theorem 2.1, then we have the following inequality

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{s+2}M(a,b)+\frac{1}{(s+1)(s+2)}N(a,b)$$

which is proved by Kırmacı et al. in [9].

Remark 2.3. If we choose $s=1$ in Theorem 2.1, then we have the following inequality

$$\begin{aligned}
\int_a^b f(x)g(x)w(x)dx &\leq \frac{M(a,b)}{(b-a)^2}\int_a^b (b-x)^2 w(x)dx\\
&+\frac{N(a,b)}{(b-a)^2}\int_a^b (b-x)(x-a)w(x)dx
\end{aligned}$$

which is proved by Budak and Bakış in [1].

Remark 2.4. If we choose $f(x)=1$ for all $x\in[a,b]$ in Theorem 2.1, then we have the following inequality

$$\int_a^b g(x)w(x)dx \leq \frac{g(a)+g(b)}{2(b-a)^s}\int_a^b \left[(b-x)^s+(x-a)^s\right]w(x)dx$$

which is proved by Sarıkata et al. in [15, for $h(t)=t^s$].

Proof. From the inequality (2.1) for $f(x)=1$ for all $x\in[a,b]$, we have

$$\begin{aligned}
&\int_a^b g(x)w(x)dx \\
\leq\ & \frac{g(a)+g(b)}{(b-a)^{s+1}}\int_a^b (b-x)^{s+1}w(x)dx \qquad (2.12)\\
&+\frac{g(a)+g(b)}{(b-a)^{s+1}}\int_a^b (b-x)(x-a)^{s}w(x)dx \\
=\ & \frac{g(a)+g(b)}{(b-a)^{s+1}}\left[\int_a^b (b-x)^{s+1}w(x)dx+\int_a^b (b-x)(x-a)^{s}w(x)dx\right].
\end{aligned}$$

Since w is symmetric about $\frac{a+b}{2}$, we have

$$\int_a^b (b-x)^{s+1}w(x)dx=\int_a^b (x-a)^{s+1}w(x)dx.$$

Using this equality in (2.12), we get

$$\begin{aligned}
&\int_a^b g(x)w(x)dx \\
\leq\ & \frac{g(a)+g(b)}{(b-a)^{s+1}}\left[\int_a^b (x-a)^{s+1}w(x)dx+\int_a^b (b-x)(x-a)^{s}w(x)dx\right] \\
=\ & \frac{g(a)+g(b)}{(b-a)^{s}}\int_a^b (x-a)^{s}w(x)dx \\
=\ & \frac{g(a)+g(b)}{2(b-a)^{s}}\int_a^b \left[(x-a)^{s}+(b-x)^{s}\right]w(x)dx
\end{aligned}$$

which completes the proof. □

Theorem 2.5. *Suppose that conditions of Theorem* 2.1 *hold, then we have the following inequality*

$$2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \int_a^b w(x)\, dx \tag{2.13}$$

$$\leq \int_a^b f(x)g(x)w(x)dx + \frac{M(a,b)}{(b-a)^{s+1}} \int_a^b (x-a)^s (b-x)\, w(x)dx$$

$$+\frac{N(a,b)}{(b-a)^{s+1}} \int_a^b (b-x)^{s+1} w(x)dx.$$

where $M(a,b)$ *and* $N(a,b)$ *are defined as in Theorem* 2.1.

Proof. For $t \in [0,1]$, we can write

$$\frac{a+b}{2} = \frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}.$$

Using the convexity of f and s-convexity of g, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ = \; & f\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) g\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) \\ \leq \; & \frac{1}{2^{s+1}} \left[f((1-t)a+tb) + f(ta+(1-t)b)\right] \\ & \times \left[g((1-t)a+tb) + g(ta+(1-t)b)\right] \\ = \; & \frac{1}{2^{s+1}} \left[f((1-t)a+tb)g((1-t)a+tb) + f(ta+(1-t)b)g(ta+(1-t)b)\right] \\ & + \frac{1}{2^{s+1}} \left[f((1-t)a+tb)g(ta+(1-t)b) + f(ta+(1-t)b)g((1-t)a+tb)\right]. \end{aligned}$$

By using again the convexity of f and s-convexity of g, we obtain

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \tag{2.14}$$

$$\begin{aligned} \leq \; & \frac{1}{2^{s+1}} \left[f((1-t)a+tb)g((1-t)a+tb) + f(ta+(1-t)b)g(ta+(1-t)b)\right] \\ & + \frac{1}{2^{s+1}} \left[t^s (1-t) + t(1-t)^s\right] \left[f(a)g(a) + f(b)g(b)\right] \\ & + \frac{1}{2^{s+1}} \left[t^{s+1} + (1-t)^{s+1}\right] \left[f(a)g(b) + f(b)g(a)\right]. \end{aligned}$$

Multiplying both sides of (2.14) by $w\left((1-t)\,a+tb\right)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_0^1 w\left((1-t)\,a+tb\right)dt \tag{2.15}$$

$$\leq \frac{1}{2^{s+1}}\int_0^1 [f((1-t)a+tb)g((1-t)a+tb)$$

$$+f(ta+(1-t)b)g(ta+(1-t)b)]\,w\left((1-t)\,a+tb\right)dt$$

$$+\frac{M(a,b)}{2^{s+1}}\int_0^1 \left[t^s\,(1-t)+t(1-t)^s\right]w\left((1-t)\,a+tb\right)dt$$

$$+\frac{N(a,b)}{2^{s+1}}\int_0^1 \left[t^{s+1}+(1-t)^{s+1}\right]w\left((1-t)\,a+tb\right)dt.$$

Using the change of variable, we have

$$\int_0^1 w\left((1-t)\,a+tb\right)dt=\frac{1}{b-a}\int_a^b w\left(x\right)dx, \tag{2.16}$$

$$\int_0^1 f((1-t)a+tb)g((1-t)a+tb)w\left((1-t)\,a+tb\right)dt \tag{2.17}$$

$$+\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)w\left((1-t)\,a+tb\right)dt$$

$$=\frac{1}{b-a}\int_a^b f(x)g(x)w(x)dx+\frac{1}{b-a}\int_a^b f(x)g(x)w(a+b-x)dx$$

$$=\frac{2}{b-a}\int_a^b f(x)g(x)w(x)dx,$$

$$\int_0^1 \left[t^s\,(1-t)+t(1-t)^s\right]w\left((1-t)\,a+tb\right)dt \tag{2.18}$$

$$=\int_0^1 \left[t^s\,(1-t)\,w\left((1-t)\,a+tb\right)+t(1-t)^s w\left((1-t)\,a+tb\right)\right]dt$$

$$= \frac{1}{(b-a)^{s+2}} \int_a^b (x-a)^s (b-x) w(x) dx$$

$$+ \frac{1}{(b-a)^{s+2}} \int_a^b (x-a)^s (b-x) w(a+b-x) dx$$

$$= \frac{2}{(b-a)^{s+2}} \int_a^b (x-a)^s (b-x) w(x) dx$$

and

$$\int_0^1 \left[t^{s+1} + (1-t)^{s+1} \right] w((1-t)a+tb)\, dt \tag{2.19}$$

$$= \int_0^1 \left[t^{s+1} w((1-t)a+tb) + (1-t)^{s+1} w((1-t)a+tb) \right] dt$$

$$= \frac{1}{(b-a)^{s+2}} \int_a^b (b-x)^{s+1} w(a+b-x) dx$$

$$+ \frac{1}{(b-a)^{s+2}} \int_a^b (b-x)^{s+1} w(x) dx$$

$$= \frac{2}{(b-a)^{s+2}} \int_a^b (b-x)^{s+1} w(x) dx.$$

If we substitute the equalities (2.16)-(2.19) in (2.15), then we have the following inequality

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \frac{1}{b-a} \int_a^b w(x)\, dx \tag{2.20}$$

$$\leq \frac{1}{2^s (b-a)} \int_a^b f(x)g(x)w(x) dx + \frac{M(a,b)}{2^s (b-a)^{s+2}} \int_a^b (x-a)^s (b-x) w(x) dx$$

$$+ \frac{N(a,b)}{2^s (b-a)^{s+2}} \int_a^b (b-x)^{s+1} w(x) dx.$$

By multiplying the both sides of (2.20) by $2^s(b-a)$ then we obtain the desired result (2.13). □

Remark 2.6. If we choose $w(x) = 1$ for all $x \in [a,b]$ in Theorem 2.5, then we have the following inequality

$$2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{M(a,b)}{(s+1)(s+2)} + \frac{N(a,b)}{s+2}$$

which is proved by Kırmacı et al. in [9].

Remark 2.7. If we choose $s = 1$ in Theorem 2.1, then we have the following inequality

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx &\leq \int_a^b f(x)g(x)w(x)dx \\ &+ \frac{M(a,b)}{(b-a)^2}\int_a^b (x-a)(b-x)\,w(x)dx \\ &+ \frac{N(a,b)}{(b-a)^2}\int_a^b (b-x)^2\,w(x)dx. \end{aligned}$$

which is proved by Budak and Bakış in [1].

Corollary 2.8. *If we choose $f(x) = 1$ for all $x \in [a,b]$ in Theorem 2.5, then we have the following the following Fejér type inequality*

$$2^s g\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx \leq \int_a^b g(x)w(x)dx + \frac{g(a)+g(b)}{2(b-a)^s}\int_a^b \left[(x-a)^s + (b-x)\right]^s w(x)dx.$$

Proof. From inequality (2.13) for $f(x) = 1$ for all $x \in [a,b]$, we have

$$\begin{aligned} 2g\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx &\leq \int_a^b g(x)w(x)dx + \frac{g(a)+g(b)}{(b-a)^{s+1}}\int_a^b (x-a)^s(b-x)\,w(x)dx \\ &+ \frac{g(a)+g(b)}{(b-a)^{s+1}}\int_a^b (b-x)^{s+1}\,w(x)dx \\ &= \int_a^b g(x)w(x)dx \\ &+ \frac{g(a)+g(b)}{(b-a)^{s+1}}\left[\int_a^b (x-a)^s(b-x)w(x)dx + \int_a^b (b-x)^{s+1}w(x)dx\right] \\ &= \int_a^b g(x)w(x)dx + \frac{g(a)+g(b)}{2(b-a)^s}\int_a^b \left[(x-a)^s + (b-x)^s\right]w(x)dx. \end{aligned}$$

This completes the proof. ∎

3. Fejér type inequalities for products two s-convex functions

In this section, we present some Fejér type inequalities for products two s-convex functions which generalize the results in Section 2.

Theorem 3.1. *Suppose that $w : I \to \mathbb{R}$ is non-negative, integrable, and symmetric about $x = \frac{a+b}{2}$ (i.e. $w(x) = w(a+b-x)$). If $f : I \to \mathbb{R}$ is s_1-convex functions on I and if $g : I \to \mathbb{R}$ is s_2-convex on I for some fixed $s_1, s_2 \in (0,1]$, then for any $a, b \in I$, we have*

$$\int_a^b f(x)g(x)w(x)dx \leq \frac{M(a,b)}{(b-a)^{s_1+s_2}} \int_a^b (b-x)^{s_1+s_2} w(x)dx + \frac{N(a,b)}{(b-a)^{s_1+s_2}} \int_a^b (b-x)^{s_1} (x-a)^{s_2} w(x)dx. \tag{3.1}$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 2.1.

Proof. Since f is s_1-convex and g is s_2-convex functions on $[a,b]$, then we have

$$f(ta + (1-t)b) \leq t^{s_1} f(a) + (1-t)^{s_1} f(b) \tag{3.2}$$

and

$$g(ta + (1-t)b) \leq t^{s_2} g(a) + (1-t)^{s_2} g(b). \tag{3.3}$$

By (3.2) and (3.3), we have

$$\begin{aligned} & f(ta + (1-t)b)\, g(ta + (1-t)b) \\ \leq\ & t^{s_1+s_2} f(a)g(a) + (1-t)^{s_1+s_2} f(b)g(b) \\ & + t^{s_1}(1-t)^{s_2} f(a)g(b) + t^{s_2}(1-t)^{s_1} f(b)g(a). \end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $w(ta+(1-t)b)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$\int_0^1 f(ta+(1-t)b)\, g(ta+(1-t)b)\, w(ta+(1-t)b)\, dt \tag{3.5}$$

$$\begin{aligned} \leq\ & f(a)g(a)\int_0^1 t^{s_1+s_2} w(ta+(1-t)b)\, dt \\ & +f(b)g(b)\int_0^1 (1-t)^{s_1+s_2}\, w(ta+(1-t)b)\, dt \\ & +f(a)g(b)\int_0^1 t^{s_1}(1-t)^{s_2}\, w(ta+(1-t)b)\, dt \\ & +f(b)g(a)\int_0^1 t^{s_2}(1-t)^{s_1}\, w(ta+(1-t)b)\, dt. \end{aligned}$$

By change of variable $x=ta+(1-t)b$, we get

$$\int_0^1 t^{s_1+s_2} w(ta+(1-t)b)\, dt = \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (b-x)^{s_1+s_2}\, w(x)dx \tag{3.6}$$

and since w is symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned} \int_0^1 (1-t)^{s_1+s_2}\, w(ta+(1-t)b)\, dt &= \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (x-a)^{s_1+s_2}\, w(x)dx \\ &= \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (b-u)^{s_1+s_2}\, w(a+b-u)du. \\ &= \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (b-u)^{s_1+s_2}\, w(u)du. \end{aligned}$$

We also have

$$\int_0^1 t^{s_1}(1-t)^{s_2} w(ta+(1-t)b)dt = \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (b-x)^{s_1}(x-a)^{s_2} w(x)dx \tag{3.7}$$

and

$$\begin{aligned}
& \int_0^1 t^{s_2}(1-t)^{s_1} w(ta+(1-t)b)\,dt && (3.8)\\
= & \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_2}(x-a)^{s_1} w(x)dx \\
= & \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-u)^{s_1}(u-a)^{s_2} w(a+b-u)du \\
= & \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-u)^{s_1}(u-a)^{s_2} w(u)du
\end{aligned}$$

By substituting the equalities (3.6)-(3.8) in (3.5), then we have the following inequality

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b f(x)g(x)w(x)dx && (3.9)\\
& \leq \frac{f(a)g(a)+f(b)g(b)}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(x)dx \\
& + \frac{f(a)g(b)+f(b)g(a)}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1}(x-a)^{s_2} w(x)dx.
\end{aligned}$$

If we multiply both sides of (3.9) by $(b-a)$, then we obtain the desired result. □

Remark 3.2. If we choose $w(x) = 1$ for all $x \in [a, b]$ in Theorem 3.1, then we have the following inequality

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{s_1+s_2+1}M(a,b) + B(s_1+1, s_2+1)N(a,b)$$

which is proved by Kırmacı et al. in [9]. Here $B(x, y)$ is the Beta Euler function.

Remark 3.3. If we choose $s_1 = 1$ and $s_2 = s$ in Theorem 3.1, then the inequality (3.1) reduces to the inequality (2.1).

Corollary 3.4. *If we choose $s_1 = s_2 = s$ in Theorem* 3.1*, then we have the following inequality*

$$\int_a^b f(x)g(x)w(x)dx \leq \frac{M(a,b)}{(b-a)^{2s}} \int_a^b (b-x)^{2s} w(x)dx + \frac{N(a,b)}{(b-a)^{2s}} \int_a^b (b-x)^s (x-a)^s w(x)dx.$$

Theorem 3.5. *Suppose that conditions of Theorem* 3.1 *hold, then we have the following inequality*

$$2^{s_1+s_2-1} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \int_a^b w(x)\,dx \tag{3.10}$$
$$\leq \int_a^b f(x)g(x)w(x)dx + \frac{M(a,b)}{(b-a)^{s_1+s_2}} \int_a^b (x-a)^{s_1} (b-x)^{s_2} w(x)dx + \frac{N(a,b)}{(b-a)^{s_1+s_2}} \int_a^b (b-x)^{s_1+s_2} w(x)dx.$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 2.1*.*

Proof. Using the s_1-convexity of f and s_2-convexity of g, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ = & f\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) g\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) \\ \leq & \frac{1}{2^{s_1+s_2}} \left[f((1-t)a+tb) + f(ta+(1-t)b)\right] \\ & \times \left[g((1-t)a+tb) + g(ta+(1-t)b)\right] \\ = & \frac{1}{2^{s_1+s_2}} \left[f((1-t)a+tb)g((1-t)a+tb) + f(ta+(1-t)b)g(ta+(1-t)b)\right] \\ & + \frac{1}{2^{s_1+s_2}} \left[f((1-t)a+tb)g(ta+(1-t)b) + f(ta+(1-t)b)g((1-t)a+tb)\right]. \end{aligned}$$

By using again the s_1-convexity of f and s_2-convexity, we obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \tag{3.11}$$

$$\leq \frac{1}{2^{s_1+s_2}}\left[f((1-t)a+tb)g((1-t)a+tb)+f(ta+(1-t)b)g(ta+(1-t)b)\right]$$

$$+\frac{1}{2^{s_1+s_2}}\left[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\right]\left[f(a)g(a)+f(b)g(b)\right]$$

$$+\frac{1}{2^{s_1+s_2}}\left[t^{s_1+s_2}+(1-t)^{s_1+s_2}\right]\left[f(a)g(b)+f(b)g(a)\right].$$

Multiplying both sides of (3.11) by $w\left((1-t)\,a+tb\right)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_0^1 w\left((1-t)\,a+tb\right)dt \tag{3.12}$$

$$\leq \frac{1}{2^{s_1+s_2}}\int_0^1 [f((1-t)a+tb)g((1-t)a+tb)$$

$$+f(ta+(1-t)b)g(ta+(1-t)b)]\,w\left((1-t)\,a+tb\right)dt$$

$$+\frac{M(a,b)}{2^{s_1+s_2}}\int_0^1\left[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\right]w\left((1-t)\,a+tb\right)dt$$

$$+\frac{N(a,b)}{2^{s_1+s_2}}\int_0^1\left[t^{s_1+s_2}+(1-t)^{s_1+s_2}\right]w\left((1-t)\,a+tb\right)dt.$$

Using the change of variable, we have

$$\int_0^1\left[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\right]w\left((1-t)\,a+tb\right)dt \tag{3.13}$$

$$=\int_0^1\left[t^{s_1}(1-t)^{s_2}\,w\left((1-t)\,a+tb\right)+t^{s_2}(1-t)^{s_1}w\left((1-t)\,a+tb\right)\right]dt$$

$$=\frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (x-a)^{s_1}(b-x)^{s_2}\,w(x)dx$$

$$+\frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (x-a)^{s_1}(b-x)^{s_2}\,w(a+b-x)dx$$

$$= \frac{2}{(b-a)^{s_1+s_2+1}} \int_a^b (x-a)^{s_1} (b-x)^{s_2} w(x)dx$$

and

$$\int_0^1 \left[t^{s_1+s_2} + (1-t)^{s_1+s_2} \right] w\left((1-t)a+tb\right) dt \tag{3.14}$$

$$= \int_0^1 \left[t^{s_1+s_2} w\left((1-t)a+tb\right) + (1-t)^{s_1+s_2} w\left((1-t)a+tb\right) \right] dt$$

$$= \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(a+b-x)dx$$

$$+ \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(x)dx$$

$$= \frac{2}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(x)dx.$$

If we substitute the equalities (2.16), (2.17), (3.13) and (3.14) in (3.12), then we have the following inequality

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \frac{1}{b-a} \int_a^b w(x)\, dx \tag{3.15}$$

$$\leq \frac{1}{2^{s_1+s_2-1}(b-a)} \int_a^b f(x)g(x)w(x)dx$$

$$+ \frac{M(a,b)}{2^{s_1+s_2-1}(b-a)^{s_1+s_2+1}} \int_a^b (x-a)^{s_1} (b-x)^{s_2} w(x)dx$$

$$+ \frac{N(a,b)}{2^{s_1+s_2-1}(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(x)dx.$$

By multiplying the both sides of (3.15) by $2^{s_1+s_2-1}(b-a)$ then we obtain the desired result (3.10). □

Corollary 3.6. *If we choose $w(x) = 1$ for all $x \in [a,b]$ in Theorem 3.5, then we have the following inequality*

$$2^{s_1+s_2-1} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)$$

$$\leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + B(s_1+1, s_2+1)M(a,b) + \frac{1}{s_1+s_2+1}N(a,b).$$

Remark 3.7. If we choose $s_1 = 1$ and $s_2 = s$ in Theorem 3.5, then the inequality (3.10) reduces to the inequality (2.13).

Corollary 3.8. *If we choose $s_1 = s_2 = s$ in Theorem 3.5, then we have the following inequality*

$$2^{2s-1} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \int_a^b w(x)\,dx$$

$$\leq \int_a^b f(x)g(x)w(x)dx$$

$$+\frac{M(a,b)}{(b-a)^{2s}} \int_a^b (x-a)^s (b-x)^s w(x)dx + \frac{N(a,b)}{(b-a)^{2s}} \int_a^b (b-x)^{2s} w(x)dx.$$

4. Concluding remarks

In this paper, we present some Hermite-Hadamard-Fejér type inequalities for products convex and s-convex functions. For further investigations we propose to consider the Fejér type inequalities for products other type convex functions or for fractional integral operators.

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