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Sufficient conditions for analytic functions defined by Frasin differential operator

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Abstract. Very recently, Frasin [7] introduced the differential operator $\mathcal{I}_{m,\lambda}^{\zeta} f(z)$ defined as

$$\mathcal{I}_{m,\lambda}^{\zeta} f(z) = z + \sum_{n=2}^{\infty} \left(1 + (n-1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \right)^{\zeta} a_n z^n.$$

The current work contributes to give an application of the differential operator $\mathcal{I}_{m,\lambda}^{\zeta} f(z)$ to the differential inequalities in the complex plane.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of all normalized analytic functions in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ that has a Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For a function f in $\mathcal{A}$, and using the binomial series

$$(1-\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j \qquad (m \in \mathbb{N},\ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \cdots\}),$$

let $\mathcal{I}_{m,\lambda}^{\zeta} f(z)$ be the differential operator defined as follows:

$$\begin{aligned}
\mathcal{I}^0 f(z) &= f(z), \\
\mathcal{I}_{m,\lambda}^{1} f(z) &= (1-\lambda)^m f(z) + (1 - (1-\lambda)^m) z f'(z) = \mathcal{I}_{m,\lambda} f(z),\ \lambda > 0; m \in \mathbb{N}, \\
\mathcal{I}_{m,\lambda}^{\zeta} f(z) &= \mathcal{I}_{m,\lambda}(\mathcal{I}^{\zeta-1} f(z)) \qquad (\zeta \in \mathbb{N}).
\end{aligned} \tag{1.2}$$

For $f \in \mathcal{A}$, we see that

$$\mathcal{I}^{\zeta}_{m,\lambda} f(z) = z + \sum_{n=2}^{\infty} \left(1 + (n-1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \right)^{\zeta} a_n z^n, \ \zeta \in \mathbb{N}_0. \tag{1.3}$$

Using (1.3), it is easily verified that

$$C_j^m(\lambda) z (\mathcal{I}^{\zeta}_{m,\lambda} f(z))' = \mathcal{I}^{\zeta+1}_{m,\lambda} f(z) - (1 - C_j^m(\lambda)) \mathcal{I}^{\zeta}_{m,\lambda} f(z), \quad \zeta \in \mathbb{N}_0, \tag{1.4}$$

where $C_j^m(\lambda) := \sum\limits_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j$.

From the identity (1.4), we readily have

$$C_j^m(\lambda) z (\mathcal{I}^{\zeta-1}_{m,\lambda} f(z))' = \mathcal{I}^{\zeta}_{m,\lambda} f(z) - (1 - C_j^m(\lambda)) \mathcal{I}^{\zeta-1}_{m,\lambda} f(z), \quad \zeta \in \mathbb{N}_0 \tag{1.5}$$

and

$$C_j^m(\lambda) z (\mathcal{I}^{\zeta+1}_{m,\lambda} f(z))' = \mathcal{I}^{\zeta+2}_{m,\lambda} f(z) - (1 - C_j^m(\lambda)) \mathcal{I}^{\zeta+1}_{m,\lambda} f(z), \quad \zeta \in \mathbb{N}_0. \tag{1.6}$$

The above differential operator $\mathcal{I}^{\zeta}_{m,\lambda} f(z)$ was introduced and studied by Frasin [7]. Note that for $m = 1$, we obtain the differential operator $\mathcal{I}^{\zeta}_{1,\lambda}$ defined by Al-Oboudi [1] and for $m = \lambda = 1$, we get Sălăgean differential operator $\mathcal{I}^{\zeta}$ [9] (see also Aouf [2, 3]). Our aim in this work is to provide an application of the differential operator $\mathcal{I}^{\zeta}_{m,\lambda} f(z)$, (see for example, [4, 5, 6, 8, 10]).

For our purpose, using the operator $\mathcal{I}^{\zeta}_{m,\lambda} f(z)$, we define the classes Q and G respectively.

Definition 1.1. Let Q be the set of continuous complex functions $q(r, s, t) : \mathbb{C}^3 \to \mathbb{C}$ in $\mathbb{D} \subset \mathbb{C}^3$ such that $(0, 0, 0) \in \mathbb{D}$, $|q(0, 0, 0)| < 1$ and

$$\begin{aligned} & \big| q(e^{i\theta}, [C_j^m(\lambda)\delta + (1 - C_j^m(\lambda))] e^{i\theta}, \\ & [C_j^m(\lambda)]^2 \beta + [C_j^m(\lambda)(2 - C_j^m(\lambda))\delta + (1 - C_j^m(\lambda))^2] e^{i\theta} \big| \\ \geq \ & 1 \end{aligned}$$

whenever

$$\begin{aligned} & \big(e^{i\theta}, [C_j^m(\lambda)\delta + (1 - C_j^m(\lambda))] e^{i\theta}, \\ & [C_j^m(\lambda)]^2 \beta + [C_j^m(\lambda)(2 - C_j^m(\lambda))\delta + (1 - C_j^m(\lambda))^2] e^{i\theta} \big) \\ \in \ & \mathbb{D} \end{aligned}$$

with $\mathrm{Re}\{\beta e^{-i\theta}\} \geq \delta(\delta - 1)$ for real $\theta, \delta \geq 1$.

Definition 1.2. Let G be the set of continuous complex functions $g(r, s, t) : \mathbb{C}^3 \to \mathbb{C}$ in $\mathbb{D} \subset \mathbb{C}^3$ such that $(1, 1, 1) \in \mathbb{D}$, $|g(1, 1, 1)| < L$ $(L > 1)$ and

$$\left| g \left(Le^{i\theta}, Le^{i\theta} + C_j^m(\lambda)\delta, \frac{[C_j^m(\lambda)]^2(\delta + \mu) + 3LC_j^m(\lambda)\delta e^{i\theta} + L^2 e^{2i\theta}}{C_j^m(\lambda)(Le^{i\theta} + C_j^m(\lambda)\delta)} \right) \right| \geq L$$

whenever

$$\left(Le^{i\theta}, Le^{i\theta} + C_j^m(\lambda)\delta, \frac{[C_j^m(\lambda)]^2(\delta + \mu) + 3LC_j^m(\lambda)\delta e^{i\theta} + L^2 e^{2i\theta}}{C_j^m(\lambda)(Le^{i\theta} + C_j^m(\lambda)\delta)} \right) \in \mathbb{D}$$

with $\mathrm{Re}\{\mu\} \geq \delta(\delta-1)$ for real $\theta, \delta \geq \frac{L-1}{L+1}$.

2. Main results

To prove our theorems in this section, we recall two lemmas for Miller and Mocanu.

Lemma 2.1. [8] *Let a function $w(z) \in \mathcal{A}$ with $w(z) \neq 0$ in $\mathbb{U}$. If $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$. Then*

$$z_0 w'(z_0) = \delta w(z_0) \tag{2.1}$$

and

$$Re\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geq \delta, \quad \delta \geq 1. \tag{2.2}$$

Lemma 2.2. [8] *Let $w(z) = a + w_k z^k + \cdots$ be analytic in $\mathbb{U}$ with $w(z) \neq a$ and $k \geq 1$. If $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$. Then*

$$z_0 w'(z_0) = \delta w(z_0) \tag{2.3}$$

and

$$Re\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geq \delta, \quad (\delta \in \mathbb{R}) \tag{2.4}$$

where

$$\delta \geq k \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq k \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}.$$

Applying Lemma 2.1, we prove Theorem 2.3.

Theorem 2.3. *Let $q(r,s,t) \in Q$ and $f(z) \in \mathcal{A}$ such that*

$$\left(\mathcal{I}^{\zeta}_{m,\lambda} f(z), \mathcal{I}^{\zeta+1}_{m,\lambda} f(z), \mathcal{I}^{\zeta+2}_{m,\lambda} f(z)\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.5}$$

and

$$\left|q\left(\mathcal{I}^{\zeta}_{m,\lambda} f(z), \mathcal{I}^{\zeta+1}_{m,\lambda} f(z), \mathcal{I}^{\zeta+2}_{m,\lambda} f(z)\right)\right| < 1 \tag{2.6}$$

for $\zeta \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\lambda > 0$ and $z \in \mathbb{U}$. Then

$$\left|\mathcal{I}^{\zeta}_{m,\lambda} f(z)\right| < 1 \qquad (z \in \mathbb{U}). \tag{2.7}$$

Proof. Let

$$\mathcal{I}^{\zeta}_{m,\lambda} f(z) = w(z),$$

then $w(z) \in \mathcal{A}$ and $w(z) \neq 0$ $(z \in \mathbb{U})$. Using the identity (1.4), we have

$$\mathcal{I}^{\zeta+1}_{m,\lambda} f(z) = C^m_j(\lambda) z w'(z) + (1 - C^m_j(\lambda)) w(z)$$

and

$$\mathcal{I}^{\zeta+2}_{m,\lambda} f(z) = [C^m_j(\lambda)]^2 (z^2 w''(z)) + C^m_j(\lambda)(2 - C^m_j(\lambda)) z w'(z) + (1 - C^m_j(\lambda))^2 w(z).$$

Letting $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$, $|w(z_0)| = \max_{|z|\le r_0} |w(z)| = 1$, $w(z_0) = e^{i\theta}$ and using (2.1), we have

$$\begin{aligned}\mathcal{I}^{\zeta}_{m,\lambda} f(z_0) &= w(z_0) = e^{i\theta},\\ \mathcal{I}^{\zeta+1}_{m,\lambda} f(z_0) &= C^m_j(\lambda)\delta w(z_0) + (1 - C^m_j(\lambda))w(z_0)\\ &= [C^m_j(\lambda)\delta + (1 - C^m_j(\lambda))]e^{i\theta},\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}^{\zeta+2}_{m,\lambda} f(z_0) &= [C^m_j(\lambda)]^2(z_0^2 w''(z_0)) + C^m_j(\lambda)(2 - C^m_j(\lambda))\delta w(z_0) + (1 - C^m_j(\lambda))^2 w(z_0)\\ &= [C^m_j(\lambda)]^2\beta + [C^m_j(\lambda)(2 - C^m_j(\lambda))\delta + (1 - C^m_j(\lambda))^2]e^{i\theta}.\end{aligned}$$

where

$$\beta = z_0^2 w''(z_0) \quad \text{and} \quad \delta \ge 1\,.$$

Moreover, an application of (2.2) gives

$$\mathrm{Re}\left\{\frac{z_0 w''(z_0)}{w'(z_0)}\right\} = \mathrm{Re}\left\{\frac{z_0^2 w''(z_0)}{\delta e^{i\theta}}\right\} \ge \delta - 1,$$

or

$$\mathrm{Re}\{\beta e^{-i\theta}\} \ge \delta(\delta - 1).$$

Since $q(r,s,t) \in Q$, we have

$$\begin{aligned}&\left|q\left(\mathcal{I}^{\zeta}_{m,\lambda} f(z), \mathcal{I}^{\zeta+1}_{m,\lambda} f(z), \mathcal{I}^{\zeta+2}_{m,\lambda} f(z)\right)\right|\\ =\ &\big|q(e^{i\theta}, [C^m_j(\lambda)\delta + (1 - C^m_j(\lambda))]e^{i\theta},\\ &[C^m_j(\lambda)]^2\beta + [C^m_j(\lambda)(2 - C^m_j(\lambda))\delta + (1 - C^m_j(\lambda))^2]e^{i\theta})\big|\\ >\ &1\end{aligned}$$

which opposes the condition (2.6) of Theorem 2.3. So we have

$$\left|\mathcal{I}^{\zeta}_{m,\lambda} f(z)\right| < 1 \qquad (z \in \mathbb{U}). \qquad \square$$

In Theorem 2.3, if $\zeta = 0$, $\lambda = 1$ and $m = 1$ we get

Corollary 2.4. *Let $q(r,s,t) \in Q$ and $f(z) \in \mathcal{A}$ such that*

$$\big(f(z), zf'(z), z^2 f''(z) + zf'(z)\big) \in \mathbb{D} \subset \mathbb{C}^3$$

and

$$\left|q\left(f(z), zf'(z), z^2 f''(z) + zf'(z)\right)\right| < 1, \qquad z \in \mathbb{U}.$$

Then

$$|f(z)| < 1 \qquad (z \in \mathbb{U}).$$

Now, using Lemma 2.2 we will prove the following theorem.

Theorem 2.5. *Let $g(r,s,t) \in G$ and $f(z) \in \mathcal{A}$ satisfy*

$$\left(\frac{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)}, \frac{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}, \frac{\mathcal{I}^{\zeta+2}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.8}$$

and

$$\left| g\left(\frac{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)}, \frac{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}, \frac{\mathcal{I}^{\zeta+2}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}\right)\right| < L \tag{2.9}$$

for $m \in \mathbb{N}$, $\zeta \geq 1$, $\lambda > 0, L > 1$ and all $z \in \mathbb{U}$. Then

$$\left|\frac{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)}\right| < L \qquad (z \in \mathbb{U}).$$

Proof. Let

$$\frac{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)} = w(z) \qquad (\zeta \geq 1). \tag{2.10}$$

Then $w(z)$ is analytic function in $\mathbb{U}$, $w(0) = 1$ and $w(z) \neq 1$. Differentiating (2.10) logarithmically and multiplying by z, we get

$$\frac{z(\mathcal{I}^{\zeta}_{m,\lambda}f(z))'}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)} - \frac{z(\mathcal{I}^{\zeta-1}_{m,\lambda}f(z))'}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)} = \frac{zw'(z)}{w(z)}.$$

Using the identities (1.4) and (1.5), we have

$$\frac{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)} = w(z) + C^m_j(\lambda)\frac{zw'(z)}{w(z)}. \tag{2.11}$$

Differentiating (2.11) logarithmically and multiply by z, we have

$$\begin{aligned}
&\frac{z(\mathcal{I}^{\zeta+1}_{m,\lambda}f(z))'}{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)} - \frac{z(\mathcal{I}^{\zeta}_{m,\lambda}f(z))'}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)} \\
= \;& \frac{z\left[w(z) + C^m_j(\lambda)\frac{zw'(z)}{w(z)}\right]'}{w(z) + C^m_j(\lambda)\frac{zw'(z)}{w(z)}} \\
= \;& \frac{zw'(z) + C^m_j(\lambda)\left[\frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right]}{w(z) + C^m_j(\lambda)\frac{zw'(z)}{w(z)}}.
\end{aligned} \tag{2.12}$$

Using the identities (1.4) and (1.6), it follows from (2.12) that

$$\begin{aligned}
&\frac{1}{C_j^m(\lambda)}\frac{\mathcal{I}_{m,\lambda}^{\zeta+2}f(z)}{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z)}\\
=\;& \frac{1}{C_j^m(\lambda)}\frac{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z)}{\mathcal{I}_{m,\lambda}^{\zeta}f(z)}+\frac{zw'(z)+C_j^m(\lambda)\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{w(z)+C_j^m(\lambda)\frac{zw'(z)}{w(z)}}\\
=\;& \frac{1}{C_j^m(\lambda)}w(z)+\frac{zw'(z)}{w(z)}+\frac{zw'(z)+C_j^m(\lambda)\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{w(z)+C_j^m(\lambda)\frac{zw'(z)}{w(z)}}
\end{aligned}$$

Letting $z_0=r_0e^{i\theta}$ $(0<r_0<1)$, $\max\limits_{|z|\le r_0}|w(z)|=|w(z_0)|=L$, $w(z_0)=Le^{i\theta}$ and using Lemma 2.2 with $a=1$ and $k=1$, we have

$$\begin{aligned}
\frac{\mathcal{I}_{m,\lambda}^{\zeta}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta-1}f(z_0)} &= Le^{i\theta},\\
\frac{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta}f(z_0)} &= Le^{i\theta}+C_j^m(\lambda)\delta,\\
\frac{\mathcal{I}_{m,\lambda}^{\zeta+2}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z_0)} &= \frac{[C_j^m(\lambda)]^2(\delta+\mu)+3LC_j^m(\lambda)\delta e^{i\theta}+L^2e^{2i\theta}}{C_j^m(\lambda)(Le^{i\theta}+C_j^m(\lambda)\delta)},
\end{aligned}$$

where

$$\mu=\frac{z_0^2w''(z_0)}{w(z_0)}\quad\text{and}\quad\delta\ge\frac{L-1}{L+1}.$$

Moreover, an application of (2.2) gives $\mathrm{Re}\{\mu\}\ge\delta(\delta-1)$.
Since $g(r,s,t)\in G$, we have

$$\begin{aligned}
&\left|g\left(\frac{\mathcal{I}_{m,\lambda}^{\zeta}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta-1}f(z_0)},\frac{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta}f(z_0)},\frac{\mathcal{I}_{m,\lambda}^{\zeta+2}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z_0)}\right)\right|\\
=\;&\left|g\left(Le^{i\theta},Le^{i\theta}+C_j^m(\lambda)\delta,\frac{[C_j^m(\lambda)]^2(\delta+\mu)+3LC_j^m(\lambda)\delta e^{i\theta}+L^2e^{2i\theta}}{C_j^m(\lambda)(Le^{i\theta}+C_j^m(\lambda)\delta)}\right)\right|\\
\ge\;& L
\end{aligned}$$

which contradicts the condition (2.9) of Theorem 2.5. Thus

$$|w(z)|=\left|\frac{\mathcal{I}_{m,\lambda}^{\zeta}f(z)}{\mathcal{I}_{m,\lambda}^{\zeta-1}f(z)}\right|<L.$$

for $m\in\mathbb{N}$, $\zeta\ge1$, $\lambda>0$ and all $z\in\mathbb{U}$. The proof is complete. □

In Theorem 2.5, if $\zeta=1$, $\lambda=1$ and $m=1$ we get

Corollary 2.6. *Let* $g(r,s,t) \in G$ *and* $f(z) \in \mathcal{A}$ *satisfy*

$$\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)+f'(z)}{f'(z)}, \frac{z^2f^{(3)}(z)+3zf''(z)+f'(z)}{zf''(z)+f'(z)}\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.13}$$

and

$$\left|g\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)+f'(z)}{f'(z)}, \frac{z^2f^{(3)}(z)+3zf''(z)+f'(z)}{zf''(z)+f'(z)}\right)\right| < L \tag{2.14}$$

for $L > 1$ *and all* $z \in \mathbb{U}$. *Then*

$$\left|\frac{zf'(z)}{f(z)}\right| < L \qquad (z \in \mathbb{U}).$$

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