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Coupled fixed point theorems for rational type contractions

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Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. In this paper, we will consider the coupled fixed problem in b-metric space for single-valued operators satisfying a generalized contraction condition of rational type. First part of the paper concerns with some fixed point theorems, while the second part presents a study of the solution set of the coupled fixed point problem. More precisely, we will present some existence and uniqueness theorems for the coupled fixed point problem, as well as a qualitative study of it (data dependence of the coupled fixed point set, well-posedness, Ulam-Hyers stability and the limit shadowing property of the coupled fixed point problem) under some rational type contraction assumptions on the mapping.

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1. Introduction and preliminaries

The notion of b-metric spaces and discussion on the topological structure of it appeared in several papers, such as L.M. Blumenthal [2], S. Czerwik [6], N. Bourbaki [5], Heinonen [10].

On the other hand, the concept of coupled fixed point problem, was considered, for the first time, by Opoitsev in [14]-[15], but a very fruitful approach in this field was proposed by D. Guo, V. Lashmikantham [9] and T. Gnana Bhaskar and V. Lashmikantham [7]. Later on, many results related to this kind of problem appeared (see, for example [8], [13],...).

Moreover, starting with the paper of Dass and Gupta [9], several extensions of the contraction principle considered the case of self mappings satisfying some rational type contraction assumptions, see, for example, [7].

Our aim is to consider both of the above research directions. More precisely, we will prove, using some adequate fixed point theorems for monotone rational contractions in ordered b-metric spaces, some coupled fixed point theorems for operators $T : X \times X \to X$ satisfying some rational type assumptions on comparable elements.

We shall recall some well known notions and definition of the b-metric spaces.

Definition 1.1. Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if the following axioms are satisfied:

1. if $x, y \in X$, then $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

A pair (X, d) with the above properties is called a b-metric space.

Let $(X, \leq)$ be a partially ordered set and d a metric on X. Notice that we can endow the product space $X \times X$ with the partial order $\leq_p$ given by

$$(x, y) \leq_p (u, v) \Leftrightarrow x \leq u, y \geq v.$$

Definition 1.3. Let $(X, \leq)$ be a partially ordered set and let $T : X \times X \to X$. We say that T has the mixed monotone property if $T(\cdot, y)$ is monotone increasing for any $y \in X$ and $T(x, \cdot)$ is monotone decreasing for any $x \in X$.

Lemma 1.4. Let (X, d) be a b-metric space. Then the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called:

i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case we write $\lim\limits_{n\to\infty} x_n = x$;

ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

If (X, d) is a metric space and $T : X \times X \to X$ is an operator, then by definition, a coupled fixed point for T is a pair $(x^*, y^*) \in X \times X$ satisfying

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*) \end{cases} \qquad (P_1)$$

We will denote by $CFix(T)$ the coupled fixed point set for T.

The aim of this paper is to present, in the framework of complete ordered b-metric spaces, some existence and uniqueness theorems for the coupled fixed point problem, as well as, a qualitative study of this problem (data dependence of the coupled fixed point set, well-posedness, Ulam-Hyers stability and the limit shadowing property of the coupled fixed point problem) under some rational type contraction assumptions on the mapping. Our results extend and complement some theorems given in the recent literature, see e.g. [21], [22].

2. Fixed point theorems

In this part of the paper, we will present a fixed point theorems in ordered b-metric spaces for a single-valued operstor satisfying a rational type contraction condition.

Theorem 2.1. Let $(X, \leq)$ be a partially ordered set and $d : X \times X \to \mathbb{R}_+$ be a complete b-metric with constant $s \geq 1$. Let $f : X \to X$ be an operator which has closed graph with respect to d and it is increasing with respect to "$\leq$". Suppose that there exists $\alpha, \beta \geq 0$ with $\alpha + \beta s < 1$ satisfying

$$d(f(x), f(y)) \leq \frac{\alpha \cdot d(y, f(y))[1 + d(x, f(x))]}{1 + d(x, y)} + \beta \cdot d(x, y), \tag{2.1}$$

for $x, y \in X$ with $x \leq y$.

If there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$, then there exists $x^* \in X$ such that $x^* = f(x^*)$ and $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, as $n \to \infty$.

Proof. We have two cases:

Case 1. If $f(x_0) = x_0$, then $Fix(f) \neq \emptyset$.

Case 2. Suppose that $x_0 < f(x_0)$.

Using that f is an increasing operator and by mathematical induction, we have

$$x_0 < f(x_0) \leq f^2(x_0) \leq \ldots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \ldots$$

By this method we get a sequence $(x_n)_{n\subset\mathbb{N}} \in X$ defined by

$$x_{n+1} = f(x_n) = f(f(x_{n-1})) = f^2(x_{n-1}) = \ldots = f^n(x_1) = f^{n+1}(x_0).$$

If there exists $n \geq 1$ such that $x_{n+1} = x_n$, then $f(x_n) = x_n$. So we get that x_n is a fixed point of f, which implies $Fix(f) \neq \emptyset$.

Suppose that $x_{n+1} \neq x_n$ for $n \geq 0$.

Since $x_n \leq x_{n+1}$ for any $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n))$$
$$\leq \frac{\alpha \cdot d(x_n, f(x_n))[1 + d(x_{n-1}, f(x_{n-1}))]}{1 + d(x_{n-1}, x_n)} + \beta \cdot d(x_{n-1}, x_n)$$
$$= \frac{\alpha \cdot d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta \cdot d(x_{n-1}, x_n)$$
$$= \alpha \cdot d(x_n, x_{n+1}) + \beta \cdot d(x_{n-1}, x_n).$$

So we obtain

$$d(x_n, x_{n+1}) \leq \frac{\beta}{1-\alpha} \cdot d(x_{n-1}, x_n) \text{ for any } n \in \mathbb{N}.$$

Using mathematical induction we get that

$$d(x_n, x_{n+1}) \leq \frac{\beta}{1-\alpha} \cdot d(x_{n-1}, x_n) \leq \ldots \leq \left(\frac{\beta}{1-\alpha}\right)^n \cdot d(x_0, x_1)$$

or

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \left(\frac{\beta}{1-\alpha}\right)^n \cdot d(x_0, f(x_0)) \quad for \quad any \quad n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$. We will prove that $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = f^n(x_0)$ is a Cauchy sequence in X.

$$d(f^n(x_0), f^{n+p}(x_0)) \leq s \cdot d(f^n(x_0), f^{n+1}(x_0)) + s^2 \cdot d(f^{n+1}(x_0), f^{n+2}(x_0)) + \ldots$$
$$+ s^{p-1} \cdot d(f^{n+p-2}(x_0), f^{n+p-1}(x_0)) + s^{p-1} \cdot d(f^{n+p-1}(x_0), f^{n+p}(x_0)).$$

We denote

$$A = \frac{\beta}{1-\alpha}.$$

So we obtain

$$d(f^n(x_0), f^{n+p}(x_0)) \leq s \cdot A^n \cdot d(x_0, f(x_0)) + s^2 \cdot A^{n+1} \cdot d(x_0, f(x_0)) + \dots$$
$$+ s^{p-1} \cdot A^{n+p-2} \cdot d(x_0, f(x_0)) + s^p \cdot A^{n+p-1} \cdot d(x_0, f(x_0))$$
$$= s \cdot A^n [1 + s \cdot A + \dots + (s \cdot A)^{p-1}] \cdot d(x_0, f(x_0)) = s \cdot A^n \cdot \frac{1 - (s \cdot A)^p}{1 - s \cdot A} \cdot d(x_0, f(x_0)).$$

But $A = \frac{\beta}{1-\alpha} < \frac{1}{s}$, then we get that

$$d(f^n(x_0), f^{n+p}(x_0)) \leq s \cdot A^n \cdot \frac{1 - (s \cdot A)^p}{1 - s \cdot A} \cdot d(x_0, f(x_0)) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence on X. We also know that (X, d) is a complete b-metric space. So there exists $x^* \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$. Because f has closed graph, then $x^* \in Fix(f)$, which implies $Fix(f) \neq \emptyset$.

Or f is continuous, we have

$$f(x^*) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*. \qquad \square$$

A uniqueness result concerning the fixed point equation is the following.

Theorem 2.2. Suppose that all the hypotheses of Theorem 2.1. take place. Additionally, suppose that the following condition holds: for all $x, y \in X$ there exists $z \in X$ such that $z \leq x$ and $z \leq y$.

Then $Fix(f) = \{x^*\}$.

Proof. Suppose that $x^*, y^* \in X$ are two fixed points of f. We have two cases:

Case 1. x^* and y^* are comparable. Suppose $x^* \leq y^*$(or $y^* \leq x^*$ is the same)

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \frac{\alpha \cdot d(y^*, f(y^*))[1 + d(x^*, f(x^*))]}{1 + d(x^*, y^*)} + \beta \cdot d(x^*, y^*)$$
$$= \beta \cdot d(x^*, y^*).$$

Since $\beta < 1$, this is only possible when $d(x^*, y^*) = 0$. This implies $x^* = y^*$, so $Fix(f) = \{x^*\}$.

Case 2. x^* and y^* are not comparable.

By our additional assumption, we have that there exists $z \in X$ with $z \leq x^*$ and $z \leq y^*$.

Since $z \leq x^*$, then $f^n(z) \leq f^n(x^*) = x^*$ for any $n \in \mathbb{N}$.

We obtain

$$d(f^n(z), x^*) = d(f^n(z), f^n(x^*)) \leq \frac{\alpha \cdot d(f^{n-1}(x^*), f^n(x^*))[1 + d(f^{n-1}(z), f^n(z)]}{1 + d(f^{n-1}(z), f^{n-1}(x^*))}$$
$$+ \beta \cdot d(f^{n-1}(z), f^{n-1}(x^*)) = \beta \cdot d(f^{n-1}(z), f^{n-1}(x^*)) = \beta \cdot d(f^{n-1}(z), x^*)$$

So we have

$$d(f^n(z), x^*) \leq \beta \cdot d(f^{n-1}(z), x^*) \leq \beta^2 \cdot d(f^{n-2}(z), x^*) \leq \dots \leq \beta^n \cdot d(z, x^*)$$

and since $\beta < 1$, $\beta^n \to 0$ then we get that

$$\lim_{n\to\infty} d(f^n(z), x^*) = 0$$

This implies $\lim_{n\to\infty} f^n(z) = x^*$. Using a similar argument, we get that $\lim_{n\to\infty} f^n(z) = y^*$. Then $x^* = y^*$.

A global version of the previous result is the following:

Theorem 2.3. Let (X, d) be a complete b- metric space with constant $s \geq 1$, $f : X \to X$ be an operator of X with the following condition: there exists $\alpha, \beta \geq 0$ with $\max\{\alpha, \frac{\beta}{1-\alpha}\} < \frac{1}{s}$ such that

$$d(f(x), f(y)) \leq \frac{\alpha \cdot d(y, f(y))[1 + d(x, f(x))]}{1 + d(x, y)} + \beta \cdot d(x, y), \tag{2.2}$$

for $x, y \in X$. Then f has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary chosen. Using the same method as in previous proof, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$, which is a Cauchy sequence.

Since (X, d) is a complete b-metric space, we get that there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Then, we have

$$\begin{aligned} d(x^*, f(x^*)) &\leq s \cdot d(x^*, f(x_n)) + s \cdot d(f(x_n), f(x^*)) \\ &\leq s \cdot d(x^*, f(x_n)) + s \cdot \frac{\alpha \cdot d(x^*, f(x^*))[1 + d(x_n, f(x_n))]}{1 + d(x_n, x^*)} + s \cdot \beta \cdot d(x_n, x^*) \\ &= s \cdot d(x^*, x_{n+1}) + s \cdot \frac{\alpha \cdot d(x^*, f(x^*))[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x^*)} + s \cdot \beta \cdot d(x_n, x^*). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} d(x^*, f(x^*)) &\left[\frac{1 + d(x_n, x^*) - s \cdot \alpha - s \cdot \alpha \cdot d(x_n, x_{n+1})}{1 + d(x_n, x^*)}\right] \\ &\leq s \cdot d(x^*, x_{n+1}) + s \cdot \beta \cdot d(x_n, x^*). \end{aligned}$$

Letting $n \to \infty$ we have $d(x^*, f(x^*))(1 - s \cdot \alpha) \leq 0$. Thus $d(x^*, f(x^*)) = 0$, i.e., $x^* \in Fix(f)$.

We prove that x^* is the unique fixed point of f. Suppose that y^* is a fixed point of f, i.e. $f(y^*) = y^*$. Then

$$d(y^*, x^*) = d(f(y^*), f(x^*)) \leq \frac{\alpha \cdot d(x^*, f(x^*))[1 + d(y^*, f(y^*))]}{1 + d(x^*, y^*)} + \beta \cdot d(y^*, x^*)$$

Hence $d(y^*, x^*) \leq \beta \cdot d(y^*, x^*)$ and thus $y^* = x^*$.

Therefore x^* is the unique fixed point of f.

3. Coupled fixed point theorems

In this section, using the fixed point theorems proved in Section 2, we will obtain some existence and uniqueness theorems for the coupled fixed point problem.

Theorem 3.1. Let $(X, \leq)$ be a partially ordered set and $d : X \times X \to \mathbb{R}_+$ be a complete b-metric on X with constant $s \geq 1$. Let $T : X \times X \to X$ be an operator with closed graph (or in particular, it is continuous) which has the mixed monotone property on $X \times X$. Assume that the following conditions are satisfied:

i) Suppose that there exists $\alpha, \beta \geq 0$ with $\frac{\beta}{1-\alpha} < \frac{1}{s}$ such that

$$d(T(x,y),T(u,v)) + d(T(y,x),T(v,u))$$
$$\leq \frac{\alpha \cdot [d(u,T(u,v)) + d(v,T(v,u))][1 + d(x,T(x,y)) + d(y,T(y,x))]}{1 + d(x,u) + d(y,v)}$$
$$+\beta \cdot [d(x,u) + d(y,v)], \tag{3.1}$$

for all $(x,y),(u,v) \in X \times X$ with $x \leq u, y \geq v$;

ii) there exists $x_0, y_0 \in X$ such that $x_0 \leq T(x_0,y_0), y_0 \geq T(y_0,x_0)$, i.e. $(x_0,y_0) \leq_p (T(x_0,y_0),T(y_0,x_0))$.

Then, the following conclusions hold:

a) there exists $(x^*,y^*) \in X \times X$ a solution of the coupled fixed point problem (P_1), such that the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in X defined by

$$\begin{cases} x_{n+1} = T(x_n,y_n), \\ y_{n+1} = T(y_n,x_n), \quad \textit{for} \quad \textit{all} \quad n \in \mathbb{N}. \end{cases}$$

have the property that $(x_n)_{n\in\mathbb{N}} \to x^*, (y_n)_{n\in\mathbb{N}} \to y^*$ as $n \to \infty$.

b) in particular, if d is a continuous b-metric on X, then

$$d(x_n,x^*) + d(y_n,y^*) \leq \frac{s \cdot A^n}{1 - s \cdot A}[d(x_0,x_1) + d(y_0,y_1)]$$

where $A = \frac{2\beta}{1-2\alpha}$ and $\begin{cases} x_1 = T(x_0,y_0) \\ y_1 = T(y_0,x_0). \end{cases}$

Proof. By ii) we have that $z_0 = (x_0,y_0) \leq_p (T(x_0,y_0),T(y_0,x_0)) = (x_1,y_1) = z_1$. So we have $z_0 \leq_p z_1$.

If we consider $x_2 = T(x_1,y_1)$ and $y_2 = T(y_1,x_1)$, then we get $x_2 = T(x_1,y_1) = T^2(x_0,y_0)$ and $y_2 = T(y_1,x_1) = T^2(y_0,x_0)$. Using the mixed monotone property of T, we get

$$x_2 = T(x_1,y_1) \geq T(x_0,y_0) = x_1 \quad \text{implies} \quad x_1 \leq x_2$$
$$y_2 = T(y_1,x_1) \leq T(y_0,x_0) = y_1 \quad \text{implies} \quad y_1 \geq y_2$$

Hence $z_1 = (x_1,y_1) \leq_p (x_2,y_2) = z_2$.

By this approach we obtain the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in X with

$$\begin{cases} x_{n+1} = T(x_n,y_n) \\ y_{n+1} = T(y_n,x_n) \end{cases}$$

and by mathematical induction we obtain $z_n = (x_n,y_n) \leq_p (x_{n+1},y_{n+1}) = z_{n+1}$, which implies $(z_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence in $(Z,\leq_p)$, where $Z = X \times X$.

Consider the metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, defined by

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v).$$

Then, $\widetilde{d}$ is a b-metric on Z with the same constant $s \geq 1$ and if (X,d) is complete, we have $(Z,\widetilde{d})$ is complete, too.

Let $F : Z \to Z$ be an operator defined by $F(x,y) = (T(x,y),T(y,x)), \ \forall (x,y) \in Z$.

We have $z_{n+1} = F(z_n)$, for $n \geq 0$ where $z_0 = (x_0, y_0)$. Using the mixed monotone property of T, then the operator F is monotone increasing with respect to " $\leq_p$ " i.e. $(x, y), (u, v) \in Z$, with $(x, y) \leq_p (u, v) \Rightarrow F(x, y) \leq_p F(u, v)$.

Because T has a closed graph (or, in particulat it is continuous on $X \times X$), then F has a closed graph (or respectively is continuous on Z).

F is a contraction in $(Z, \widetilde{d})$ on all comparable elements of Z. Let $z = (x, y) \leq_p (u, v) = w \in Z$, so we have

$$\widetilde{d}(F(z), F(w)) = \widetilde{d}((T(x, y), T(y, x)), (T(u, v), T(v, u))$$
$$= d(T(x, y), T(u, v)) + d(T(y, x), T(v, u))$$
$$\leq \frac{\alpha \cdot [d(u, T(u, v)) + d(v, T(v, u))][1 + d(x, T(x, y)) + d(y, T(y, x))]}{1 + d(x, u) + d(y, v)}$$
$$+\beta \cdot [d(x, u) + d(y, v)]$$
$$= \frac{\alpha \cdot \widetilde{d}(w, F(w))[1 + \widetilde{d}(z, F(z))]}{1 + \widetilde{d}(z, w)} + \beta \cdot \widetilde{d}(z, w).$$

The operator $F : Z \to Z$ has the following properties:

1) $F : Z \to Z$ has a closed graph;

2) $F : Z \to Z$ is increasing on Z;

3) there exist $z_0 = (x_0, y_0) \in Z$ such that $z_0 \leq_p F(z_0)$;

4) there exists $\alpha, \beta \geq 0$ with $\frac{\beta}{1-\alpha} < \frac{1}{s}$ such that

$$\widetilde{d}(F(z), F(w)) \leq \frac{\alpha \cdot \widetilde{d}(w, F(w))[1 + \widetilde{d}(z, F(z))]}{1 + \widetilde{d}(z, w)} + \beta \cdot \widetilde{d}(z, w).$$

We can apply the conclusion of the Theorem 2.1. and we get that F has at least one fixed point. Hence, there exists $z^* \in Z$ with $F(z^*) = z^*$. Let $z^* = (x^*, y^*) \in Z$, so we have $F(x^*, y^*) = (x^*, y^*)$.

This implies

$$(T(x^*, y^*), T(y^*, x^*)) = (x^*, y^*) \Rightarrow \begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*) \end{cases}$$

and the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in X defined by

$$\begin{cases} x_{n+1} = T(x_n, y_n) \\ y_{n+1} = T(y_n, x_n) \end{cases} \quad \text{for } n \in \mathbb{N}$$

have the property that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$.

We know that $z_{n+1} = F(z_n) = F(x_n, y_n)$ for $n \geq 0$. This yields to

$$\widetilde{d}(z_n, z_{n+1}) = \widetilde{d}(F(z_{n-1}), F(z_n))$$
$$= \widetilde{d}((T(x_{n-1}, y_{n-1}), T(y_{n-1}, x_{n-1})), (T(x_n, y_n), T(y_n, x_n)))$$
$$= d(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) + d(T(y_{n-1}, x_{n-1}), T(y_n, x_n))$$
$$\leq \frac{\alpha[d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))][1 + d(x_{n-1}, T(x_{n-1}, y_{n-1})) + d(y_{n-1}, T(y_{n-1}, x_{n-1}))]}{1 + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}$$
$$+\beta[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]$$

$$= \frac{\alpha \cdot \widetilde{d}(z_n, F(z_n))[1 + \widetilde{d}(z_{n-1}, F(z_{n-1}))]}{1 + \widetilde{d}(z_{n-1}, z_n)} + \beta \cdot \widetilde{d}(z_{n-1}, z_n)$$

$$= \frac{\alpha \cdot \widetilde{d}(z_n, F(z_n))[1 + \widetilde{d}(z_{n-1}, z_n)]}{1 + \widetilde{d}(z_{n-1}, z_n)} + \beta \cdot \widetilde{d}(z_{n-1}, z_n) = \alpha \cdot \widetilde{d}(z_n, z_{n+1}) + \beta \cdot \widetilde{d}(z_{n-1}, z_n).$$

This yields to

$$\widetilde{d}(z_n, z_{n+1}) \leq \frac{\beta}{1-\alpha} \cdot \widetilde{d}(z_{n-1}, z_n) \leq \left(\frac{\beta}{1-\alpha}\right)^2 \cdot \widetilde{d}(z_{n-2}, z_{n-1})$$
$$\leq \ldots \leq \left(\frac{\beta}{1-\alpha}\right)^n \cdot \widetilde{d}(z_0, z_1)$$

where $\frac{\beta}{1-\alpha} < \frac{1}{s} < 1$.

We denote $A = \frac{\beta}{1-\alpha} < 1$. Moreover, for $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we have

$$\widetilde{d}(z_n, z_{n+p}) \leq s \cdot \widetilde{d}(z_n, z_{n+1}) + s^2 \cdot \widetilde{d}(z_{n+1}, z_{n+2}) + \ldots + +s^{p-1} \cdot \widetilde{d}(z_{n+p-1}, z_{n+p})$$
$$\leq s \cdot A^n \cdot \widetilde{d}(z_0, z_1) + s^2 \cdot A^{n+1} \cdot \widetilde{d}(z_0, z_1) + \ldots + s^{p-1} \cdot A^{n+p-1} \cdot \widetilde{d}(z_0, z_1)$$
$$\leq s \cdot A^n \cdot [1 + s \cdot A + \ldots + (s \cdot A)^{p-1}] \cdot \widetilde{d}(z_0, z_1)$$
$$= s \cdot A^n \cdot \frac{1 - (s \cdot A)^{p-1}}{1 - s \cdot A} \cdot \widetilde{d}(z_0, z_1) \leq s \cdot A^n \cdot \frac{1}{1 - s \cdot A} \cdot \widetilde{d}(z_0, z_1).$$

If the b-metric is continuous, letting $p \to \infty$ we obtain

$$\widetilde{d}(z_n, z^*) \leq \frac{s \cdot A^n}{1 - s \cdot A} \cdot \widetilde{d}(z_0, z_1).$$

But $z_n = (x_n, y_n)$, so we get

$$\widetilde{d}((x_n, y_n), z^*) \leq \frac{s \cdot A^n}{1 - s \cdot A} \cdot \widetilde{d}((x_0, y_0), (x_1, y_1))$$

and, by definition of $\widetilde{d}$, we finally get

$$d(x_n, z^*) + d(y_n, z^*) \leq \frac{s \cdot A^n}{1 - s \cdot A} \cdot [d(x_0, x_1) + d(y_0, y_1)].$$ □

The following theorem gives the uniqueness of the coupled fixed point.

Theorem 3.2. Consider that we have the hypotheses of Theorem 3.1. and the following condition holds:

for all $(x, y), (u, v) \in X \times X$ there exists $(z, w) \in X \times X$ such that

$$(z, w) \leq_p (x, y) \text{ and } (z, w) \leq_p (u, v).$$

Then $CFix(T) = \{(x^*, y^*)\}$.

Proof. The operator T verifies the hypotheses of Theorem 3.1. Hence there exists $(x^*, y^*) \in Z := X \times X$ such that

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*) \end{cases}$$

Let $(\overline{x}, \overline{y}) \in CFix(T)$ and $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, defined by

$$\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v),$$

where $Z = X \times X$.

We have two cases:

Case 1. $(x^*, y^*) \leq_p (\overline{x}, \overline{y})$, which implies

$$\begin{aligned} \widetilde{d}((x^*, y^*), (\overline{x}, \overline{y})) &= \widetilde{d}((T(x^*, y^*), T(y^*, x^*)), (T(\overline{x}, \overline{y}), T(\overline{y}, \overline{x}))) \\ &= d(T(x^*, y^*), T(\overline{x}, \overline{y})) + d(T(y^*, x^*), T(\overline{y}, \overline{x})) \\ \leq \frac{\alpha \cdot [d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x}))][1 + d(x^*, T(x^*, y^*)) + d(y^*, T(y^*, x^*))]}{1 + d(x^*, \overline{x}) + d(y^*, \overline{y})} \\ +\beta \cdot [d(x^*, \overline{x}) + d(y^*, \overline{y})] &= \beta \cdot [d(x^*, \overline{x}) + d(y^*, \overline{y})] = \beta \cdot \widetilde{d}((x^*, y^*), (\overline{x}, \overline{y})) \end{aligned}$$

This yields to

$$\widetilde{d}((x^*, y^*), (\overline{x}, \overline{y})) \leq \beta \cdot \widetilde{d}((x^*, y^*), (\overline{x}, \overline{y}))$$

or

$$(1 - \beta) \cdot \widetilde{d}((x^*, y^*), (\overline{x}, \overline{y})) \leq 0 \quad (\text{but} \quad 1 - \beta > 0)$$

Hence, we have

$$(x^*, y^*) = (\overline{x}, \overline{y}).$$

Case 2. $(x^*, y^*), (\overline{x}, \overline{y})$ are not comparable.

Let $F : Z \to Z$ be defined by $F(x, y) = (T(x, y), T(y, x)) \quad \forall (x, y) \in Z$. There exists $(z, w) \in Z$, such that $(z, w) \leq_p (x^*, y^*)$, implies $F^n(z, w) \leq_p F^n(x^*, y^*)$ because F is an increasing operator and $(z, w) \leq_p (\overline{x}, \overline{y})$, implies $F^n(z, w) \leq_p F(\overline{x}, \overline{y})$, F is an increasing operator.

We have

$$\begin{aligned} \widetilde{d}(F^n(z, w), (x^*, y^*)) &= \widetilde{d}(F^n(z, w), F^n(x^*, y^*)) = \widetilde{d}(F(F^{n-1}(z, w)), F(F^{n-1}(x^*, y^*))) \\ &\leq \frac{\alpha \cdot \widetilde{d}(F^{n-1}(x^*, y^*), F^n(x^*, y^*))[1 + \widetilde{d}(F^{n-1}(z, w), F^n(z, w))]}{1 + \widetilde{d}(F^{n-1}(z, w), F^{n-1}(x^*, y^*))} \\ &\quad +\beta \cdot \widetilde{d}(F^{n-1}(z, w), F^{n-1}(x^*, y^*)) \\ &= \beta \cdot \widetilde{d}(F^{n-1}(z, w), F^{n-1}(x^*, y^*)). \end{aligned}$$

By mathematical induction we get

$$\begin{aligned} \widetilde{d}(F^n(z, w), F^n(x^*, y^*)) &\leq \beta \cdot \widetilde{d}(F^{n-1}(z, w), F^{n-1}(x^*, y^*)) \\ &\leq \ldots \leq \beta^n \cdot \widetilde{d}((z, w), (x^*, y^*)) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence

$$\lim_{n \to \infty} F^n(z, w) = (x^*, y^*). \tag{3.2}$$

But, we also know,

$$(z, w) \leq_p (\overline{x}, \overline{y})$$

implies

$$F^n(z, w) \leq_p F^n(\overline{x}, \overline{y}) = (\overline{x}, \overline{y}).$$

Similarly, we obtain that

$$\widetilde{d}(F^n(z, w), (\overline{x}, \overline{y})) \leq \beta^n \cdot \widetilde{d}((z, w), (\overline{x}, \overline{y})) \to 0 \quad \text{as} \quad n \to \infty$$

Hence,

$$\lim_{n \to \infty} F^n(z, w) = (\overline{x}, \overline{y}). \tag{3.3}$$

By (3.1)+(3.2) we obtain that

$$(x^*, y^*) = (\overline{x}, \overline{y}). \qquad \square$$

A global version of the previous results is the following.

Theorem 3.3. Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T : X \times X \to X$ be an operator such that there exist $\alpha, \beta \geq 0$ with $\max\{\alpha, \frac{\beta}{1-\alpha}\} < \frac{1}{s}$ such that

$$\begin{aligned} &d(T(x,y), T(u,v)) + d(T(y,x), T(v,u)) \\ \leq\ &\frac{\alpha \cdot [d(u, T(u,v)) + d(v, T(v,u))][1 + d(x, T(x,y)) + d(y, T(y,x))]}{1 + d(x,u) + d(y,v)} \\ &+\beta \cdot [d(x,u) + d(y,v)] \quad \text{for} \quad (x,y), (u,v) \in X \times X. \end{aligned}$$

Then, there exists an unique solution $(x^*, y^*) \in X \times X$ of the coupled fixed point problem (P_1), and for any initial element $(x_0, y_0) \in X \times X$ the sequence $z_{n+1} = (x_{n+1}, y_{n+1}) = (T(x_n, y_n), T(y_n, x_n)) \in X \times X$ converges to (x^*, y^*).

Proof. Let $Z = X \times X$ and the functional $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, such that

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v).$$

We know that $\widetilde{d}$ is a b-metric on Z with the same constant $s \geq 1$. Moreover, if (X, d) is a complete b-metric space, then $(Z, \tilde{d})$ is a complete b-metric space too.

Consider the operator $F : Z \to Z$ defined by $F(x,y) = (T(x,y), T(y,x))$ for $(x,y) \in Z$.

Let $z = (x,y) \in Z$ and $w = (u,v) \in Z$.

We have

$$\begin{aligned} \widetilde{d}(F(z), F(w)) &= \widetilde{d}((T(x,y), T(y,x)), (T(u,v), T(v,u))) \\ &= d(T(x,y), T(u,v)) + d(T(y,x), T(v,u)) \\ &\leq \frac{\alpha \cdot [d(u, T(u,v)) + d(v, T(v,u))][1 + d(x, T(x,y)) + d(y, T(y,x))]}{1 + d(x,u) + d(y,v)} \\ &\quad +\beta \cdot [d(x,u) + d(y,v)] \\ &= \frac{\alpha \cdot \widetilde{d}((u,v),(T(u,v),T(v,u)))[1 + \widetilde{d}((x,y),(T(x,y),T(y,x)))]}{1 + \widetilde{d}((x,y),(u,v))} + \beta \cdot \widetilde{d}((x,y),(u,v)) \\ &= \frac{\alpha \cdot \widetilde{d}(w, F(w))[1 + \widetilde{d}(z, F(z))]}{1 + \widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w). \end{aligned}$$

Therefore

$$\widetilde{d}(F(z), F(w)) \leq \frac{\alpha \cdot \widetilde{d}(w, F(w))[1 + \widetilde{d}(z, F(z))]}{1 + \widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w).$$

From Theorem 2.3. we have that $Fix(F) = \{(x^*, y^*)\}$, so the coupled fixed point problem (P_1) has a unique solution $(x^*, y^*) \in Z$.

An existence and uniqueness result for the fixed point of T is given now.

Theorem 3.4. If we suppose that we have the hypotheses of Theorem 3.2., then for the unique coupled fixed point (x^*, y^*) of T we have that $x^* = y^*$ i.e. $T(x^*, x^*) = x^*$.

Proof. From Theorem 3.2., there exists an unique coupled fixed point of T, $(x^*, y^*) \in X \times X$.

We have two cases:

Case 1. If x^* and y^* are comparable, $x^* \leq y^*$.

Then we have

$$d(T(x,y),T(u,v)) + d(T(y,x),T(v,u))$$
$$\leq \frac{\alpha \cdot [d(u,T(u,v)) + d(v,T(v,u))][1 + d(x,T(x,u)) + d(y,T(y,x))]}{1 + d(x,u) + d(y,v)}$$
$$+\beta \cdot [d(x,u) + d(y,v)].$$

Let

$$x = v = x^* \quad \text{and} \quad y = u = y^*.$$

Thus we obtain

$$2 \cdot d(T(x^*,y^*),T(y^*,x^*))$$
$$\leq \frac{\alpha \cdot [d(y^*,T(y^*,x^*)) + d(x^*,T(x^*,y^*))][1 + d(x^*,T(x^*,y^*)) + d(y^*,T(y^*,x^*))]}{1 + 2d(x^*,y^*)}$$
$$+\beta \cdot 2 \cdot d(x^*,y^*).$$

This yields to

$$d(x^*,y^*) \leq \beta \cdot d(x^*,y^*).$$

So

$$(1-\beta) \cdot d(x^*,y^*) \leq 0,$$

follows that $x^* = y^*$.

Case 2. Suppose that x^* and y^* are not comparable.

Hence, there exists $z \in X$ such that $z \leq x^*$ and $z \leq y^*$. Thus, the following relations are satisfied:

$$(z,y^*) \leq_p (y^*,z), \; (z,y^*) \leq_p (x^*,y^*), \; (y^*,x^*) \leq_p (y^*,z).$$

Let $F : Z \to Z$ be defined by $F(x,y) = (T(x,y),T(y,x)) \quad \forall (x,y) \in Z$. Then,

$$d(x^*,y^*) = \frac{1}{2} \cdot \widetilde{d}((y^*,x^*),(x^*,y^*)) = \frac{1}{2} \cdot \widetilde{d}(F^n(y^*,x^*),F^n(x^*,y^*))$$
$$\leq \frac{s}{2} \cdot \widetilde{d}(F^n(y^*,x^*),F^n(y^*,z)) + \frac{s}{2} \cdot \widetilde{d}(F^n(y^*,z),F^n(x^*,y^*))$$
$$\leq \frac{s}{2} \cdot \widetilde{d}(F^n(y^*,x^*),F^n(y^*,z)) + \frac{s^2}{2} \cdot \widetilde{d}(F^n(y^*,z),F^n(z,y^*)) + \frac{s^2}{2} \cdot \widetilde{d}(F^n(z,y^*),F^n(x^*,y^*)).$$

But we know that

$$\widetilde{d}(F^n(y^*,x^*),F^n(y^*,z)) \leq \beta^n \cdot \widetilde{d}((y^*,x^*),(y^*,z)) = \beta^n \cdot d(x^*,z)$$
$$\widetilde{d}(F^n(y^*,z),F^n(z,y^*)) \leq \beta^n \cdot \widetilde{d}((y^*,z),(z,y^*)) = 2\beta^n \cdot d(y^*,z)$$
$$\widetilde{d}(F^n(z,y^*),F^n(x^*,y^*)) \leq \beta^n \cdot \widetilde{d}((z,y^*),(x^*,y^*)) = \beta^n \cdot d(z,x^*).$$

Using this assumptions, we get that

$$d(x^*,y^*) \leq \frac{s}{2} \cdot \beta^n \cdot d(x^*,z) + \frac{s^2}{2} \cdot \beta^n \cdot 2 \cdot d(y^*,z) + \frac{s^2}{2} \cdot \beta^n \cdot d(z,x^*)$$
$$= \frac{s}{2} \cdot \beta^n \cdot [(1+s)d(x^*,z) + 2 \cdot s \cdot d(y^*,z)] \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, we have that $x^* = T(x^*, x^*)$.

4. Properties of the coupled fixed point problem

This section presents data dependence, well-posedness, Ulam-Hyers stability and limit shadowing property for the coupled fixed point problem.

The following theorem is a data dependence result of a coupled fixed point problem.

Theorem 4.1. Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T_i : X \times X \to X \quad (i \in \{1, 2\})$ be two operators which satisfy the following conditions:

i) there exist $\alpha, \beta \geq 0$ with $\max\{\alpha, \frac{\beta}{1-\alpha}\} < \frac{1}{s}$ such that

$$d(T_1(x,y), T_1(u,v)) + d(T_1(y,x), T_1(v,u))$$
$$\leq \frac{\alpha \cdot [d(u, T_1(u,v)) + d(v, T_1(v,u))][1 + d(x, T_1(x,y)) + d(y, T_1(y,x))]}{1 + d(x,u) + d(y,v)}$$
$$+\beta \cdot [d(x,u) + d(y,v)] \quad \text{all for} \quad (x,y), (u,v) \in X \times X;$$

ii) $CFix(T_2) \neq \emptyset$;

iii) there exists $\eta > 0$ such that $d(T_1(x,y), T_2(x,y)) \leq \eta$ for all $(x,y) \in X \times X$.

In the above conditions, if $(x^*, y^*) \in X \times X$ is the unique coupled fixed point for T_1, then $d(x^*, \overline{x}) + d(y^*, \overline{y}) \leq \frac{2s(1+\alpha)}{1-s\beta} \cdot \eta$, where $(\overline{x}, \overline{y}) \in CFix(T_2)$.

Proof. By Theorem 3.3, there exists $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_1(y^*, x^*). \end{cases}$$

Let $(\overline{x}, \overline{y}) \in CFix(T_2)$, i.e. $\begin{cases} \overline{x} = T_2(\overline{x}, \overline{y}) \\ \overline{y} = T_2(\overline{y}, \overline{x}). \end{cases}$

Consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, defined by

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v)$$

for $(x,y),(u,v) \in Z$, where $Z = X \times X$.

Consider two operators $F_i : Z \to Z$ defined by $F_i(x,y) = (T_i(x,y), T_i(y,x))$, for $(x,y) \in Z$, $i \in \{1,2\}$.

We denote by $z = (x^*, y^*) \in Z$, which means $F_1(z) = z$ and $w = (\overline{x}, \overline{y}) \in Z$, which means $F_2(w) = w$. Then,

$$\widetilde{d}(F_1(z), F_1(w)) = \frac{\alpha \cdot \widetilde{d}(w, F_1(w))[1 + \widetilde{d}(z, F_1(z))]}{1 + \widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w)$$

$$= \frac{\alpha \cdot \widetilde{d}(w, F_1(w))}{1 + \widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w) \leq \alpha \cdot \widetilde{d}(w, F_1(w)) + \beta \cdot \widetilde{d}(z,w) \leq 2\alpha \cdot \eta + \beta \cdot \widetilde{d}(z,w).$$

Since

$$\widetilde{d}(z,w) = \widetilde{d}(F_1(z), F_2(w)) \leq s \cdot [\widetilde{d}(F_1(z), F_1(w)) + \widetilde{d}(F_1(w), F_2(w))]$$
$$\leq s \cdot [2\alpha \cdot \eta + \beta \cdot \widetilde{d}(z,w)] + 2s \cdot \eta,$$

we will obtain that $(1 - s\beta) \cdot \widetilde{d}(z, w) \leq 2s \cdot (1 + \alpha) \cdot \eta$.

Since $\max\{\alpha, \frac{\beta}{1-\alpha}\} < \frac{1}{s}$, we get that $1 - s\beta > 0$. Therefore $\widetilde{d}(z, w) \leq \frac{2s(1+\alpha)}{1-s\beta} \cdot \eta$ and by definition of the metric $\widetilde{d}$, we have

$$d(x^*, \overline{x}) + d(y^*, \overline{y}) \leq \frac{2s(1 + \alpha)}{1 - s\beta} \cdot \eta. \qquad \square$$

Definition 4.2. Let (X, d) be a b-metric space with constant $s \geq 1$ and $T : X \times X \to X$ be an operator. By definition, the coupled fixed point problem (P_1) is said to be well-posed if:

i) $CFix(T) = \{(x^*, y^*)\}$;

ii) for any sequence $(x_n, y_n)_{n\in\mathbb{N}} \in X \times X$ for which $d(x_n, T(x_n, y_n)) \to 0$ and $d(y_n, T(y_n, x_n)) \to 0$ as $n \to \infty$, we have that $(x_n)_{n\in\mathbb{N}} \to x^*$ and $(y_n)_{n\in\mathbb{N}} \to y^*$ as $n \to \infty$.

Theorem 4.3. Assume that all the hypotheses of Theorem 3.3. take place. Then the coupled fixed problem (P_1) is well-possed.

Proof. We denote by $Z = X \times X$. By Theorem 3.3. we have $CFix(T) = \{(x^*, y^*)\}$.

Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence on Z. We know that $d(x_n, T(x_n, y_n)) \to 0$ and $d(y_n, T(y_n, x_n)) \to 0$ as $n \to \infty$.

Consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, such that $\widetilde{d}((x, y),(u, v)) = d(x, u) + d(y, v)$ for all $(x, y),(u, v) \in Z$.

Let $F : Z \to Z$ be an operator defined by $F(x, y) = (T(x, y), T(y, x))$ for all $(x, y) \in Z$. We know that $F(x^*, y^*) = (x^*, y^*)$, so we have

$$\widetilde{d}((x_n, y_n),(x^*, y^*)) = d(x_n, x^*) + d(y_n, y^*)$$

$$\leq s \cdot d(x_n, T(x_n, y_n)) + s \cdot d(T(x_n, y_n), T(x^*, y^*))$$

$$+s \cdot d(y_n, T(y_n, x_n)) + s \cdot d(T(y_n, x_n), T(y^*, x^*))$$

$$= s \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))]$$

$$+s \cdot [d(T(x_n, y_n), T(x^*, y^*)) + d(T(y_n, x_n), T(y^*, x^*))]$$

$$\leq s \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))]$$

$$+s \cdot \frac{\alpha \cdot [d(x^*, T(x^*, y^*)) + d(y^*, T(y^*, x^*))][1 + d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))]}{1 + d(x_n, x^*) + d(y_n, y^*)}$$

$$+s \cdot \beta \cdot [1 + d(x_n, x^*) + d(y_n, y^*)]$$

$$\leq s \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))] + s \cdot \beta \cdot \widetilde{d}((x_n, y_n),(x^*, y^*)).$$

We obtain that

$$(1 - s\beta) \cdot \widetilde{d}((x_n, y_n),(x^*, y^*)) \leq s \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))]$$

$$\widetilde{d}((x_n, y_n),(x^*, y^*)) \leq \frac{s}{1 - s\beta} \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))] \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$.

Definition 4.4. Let (X, d) be a b-metric space with constant $s \geq 1$ and $T : X \times X \to X$ be an operator. Let $\tilde{d}$ any *b*-metric on $X \times X$ generated by d By definition, the coupled fixed point problem (P_1) is said to be Ulam-Hyers stable if there exists a function

$\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $(\overline{x}, \overline{y}) \in X \times X$ of the inequality

$$\tilde{d}((x, y), (T(x, y), T(y, x))) \leq \varepsilon,$$

there exists a solution $(x^*, y^*) \in X \times X$ of the coupled fixed point problem (P_1) such that

$$\tilde{d}((\overline{x}, \overline{y}), (x^*, y^*)) \leq \psi(\varepsilon).$$

Theorem 4.5. Assume that all the hypotheses of Theorem 3.3. take place. Then the coupled fixed point problem (P_1) is Ulam-Hyers stable.

Proof. Let $Z = X \times X$. By Theorem 3.3., we have $CFix(T) = \{(x^*, y^*)\}$. Let any $\varepsilon > 0$ and let $(\overline{x}, \overline{y}) \in Z$ such that $d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x})) \leq \varepsilon$.

Consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$ given by

$$\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v),\ \forall (x, y), (u, v) \in Z$$

and $F : Z \to Z$ an operator defined by $F(x, y) = (T(x, y), T(y, x))$ for all $(x, y) \in Z$.

We have

$$\widetilde{d}((\overline{x}, \overline{y}), (x^*, y^*)) = d(\overline{x}, x^*) + d(\overline{y}, y^*) = d(\overline{x}, T(x^*, y^*)) + d(\overline{y}, T(y^*, x^*))$$
$$\leq s \cdot [d(\overline{x}, T(\overline{x}, \overline{y})) + d(T(\overline{x}, \overline{y}), T(x^*, y^*))] + s \cdot [d(\overline{y}, T(\overline{y}, \overline{x})) + d(T(\overline{y}, \overline{x}), T(y^*, x^*))]$$
$$\leq s \cdot [d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x}))]$$
$$+s \cdot \frac{\alpha \cdot [d(x^*, T(x^*, y^*)) + d(y^*, T(y^*, x^*))][1 + d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x}))]}{1 + d(\overline{x}, x^*) + d(\overline{y}, y^*)}$$
$$+s \cdot \beta \cdot [d(\overline{x}, x^*) + d(\overline{y}, y^*)].$$

Thus

$$\widetilde{d}((\overline{x}, \overline{y}), (x^*, y^*)) \leq \frac{s}{1 - s\beta} \cdot [d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x}))] \leq \frac{s}{1 - s\beta} \cdot \varepsilon.$$

Therefore the coupled fixed point problem (P_1) is Ulam-Hyers stable, with a mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := ct$, where $c = \frac{s}{1-s\beta} > 0$.

Definition 4.6. Let (X, d) be a b-metric space with constant $s \geq 1$ and $T : X \times X \to X$ be an operator. By definition, the coupled fixed point problem (P_1) has the limit shadowing property, if for any sequence $(x_n, y_n)_{n \in \mathbb{N}} \in X \times X$ for which $d(x_{n+1}, T(x_n, y_n)) \to 0$ and respectively $d(y_{n+1}, T(y_n, x_n)) \to 0$ as $n \to \infty$, there exists a sequence $(T^n(x, y), T^n(y, x))_{n \in \mathbb{N}}$ such that $d(x_n, T^n(x, y)) \to 0$ and $d(y_n, T^n(y, x)) \to 0$ as $n \to \infty$.

Theorem 4.7. Assume that the hypotheses from Theorem 3.3. take place. Then the coupled fixed point problem (P_1) for T has the limit shadowing property.

Proof. By Theorem 3.3, we have $CFix(T) = \{(x^*, y^*)\}$ and for any initial point $(x, y) \in X \times X$ the sequence $z_{n+1} = (T^n(x, y), T^n(y, x)) \in X \times X$ converge to (x^*, y^*) as $n \to \infty$.

Let $(x_n, y_n)_{n \in \mathbb{N}}$ be a sequence on $Z = X \times X$ such that $d(x_{n+1}, T(x_n, y_n)) \to 0$ and $d(y_{n+1}, T(y_n, x_n)) \to 0$ as $n \to \infty$.

We consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, defined by

$$\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v) \text{ for all } (x, y), (u, v) \in Z.$$

Let $F : Z \to Z$ be an operator defined by $F(u,v) = (T(u,v), T(v,u))$ for all $(u,v) \in Z$. We know that $F(x^*,y^*) = (x^*,y^*)$. Then for every $(x,y) \in Z$ we have:

$$\widetilde{d}((x_{n+1},y_{n+1}),(T^{n+1}(x,y),T^{n+1}(y,x)))$$
$$\leq s\cdot[\widetilde{d}((x_{n+1},y_{n+1}),(x^*,y^*))+\widetilde{d}((x^*,y^*),(T^{n+1}(x,y),T^{n+1}(y,x)))])$$

But

$$\widetilde{d}((x_{n+1},y_{n+1}),(x^*,y^*)) \leq s\cdot[\widetilde{d}((x_{n+1},y_{n+1}),F(x_n,y_n))+\widetilde{d}(F(x_n,y_n),F(x^*,y^*))]$$
$$\leq s\cdot\widetilde{d}((x_{n+1},y_{n+1}),F(x_n,y_n))+s\cdot\frac{\alpha\cdot\widetilde{d}((x^*,y^*),F(x^*,y^*))[1+\widetilde{d}((x_n,y_n),F(x_n,y_n))]}{1+\widetilde{d}((x_n,y_n),(x^*,y^*))}$$
$$+s\cdot\beta\cdot\widetilde{d}((x_n,y_n),(x^*,y^*))$$
$$= s\cdot[d(x_{n+1},T(x_n,y_n))+d(y_{n+1},T(y_n,x_n))]+s\cdot\beta\cdot\widetilde{d}((x_n,y_n),(x^*,y^*)).$$

This yields to

$$\widetilde{d}((x_{n+1},y_{n+1}),(x^*,y^*)) \leq s\cdot[d(x_{n+1},T(x_n,y_n))+d(y_{n+1},T(y_n,x_n))]$$
$$+s\cdot\beta\cdot\{s\cdot[d(x_n,T(x_{n-1},y_{n-1}))+d(y_n,T(y_{n-1},x_{n-1}))]+s\cdot\beta\cdot\widetilde{d}((x_{n-1},y_{n-1}),(x^*,y^*))\}.$$

Therefore,

$$\widetilde{d}((x_{n+1},y_{n+1}),(x^*,y^*)) \leq s\cdot[d(x_{n+1},T(x_n,y_n))+d(y_{n+1},T(y_n,x_n))]$$
$$+s\cdot(s\cdot\beta)\cdot[d(x_n,T(x_{n-1},y_{n-1}))+d(y_n,T(y_{n-1},x_{n-1}))]+(s\cdot\beta)^2\cdot\widetilde{d}((x_{n-1},y_{n-1}),(x^*,y^*))$$
$$\leq \ldots \leq (s\cdot\beta)^{n+1}\cdot\widetilde{d}((x_0,y_0),(x^*,y^*))+s\cdot\left[\sum_{p=0}^{n}(s\cdot\beta)^{n-p}\cdot\widetilde{d}((x_{p+1},y_{p+1}),F(x_p,y_p))\right].$$

From Cauchy's Lemma we have $\widetilde{d}((x_{n+1},y_{n+1}),(x^*,y^*)) \to 0$ as $n\to\infty$.

Thus $\widetilde{d}((x_{n+1},y_{n+1}),(T^{n+1}(x,y),T^{n+1}(y,x))) \to 0$ as $n\to\infty$, so there exists a sequence $(T^n(x,y),T^n(y,x)) \in Z$ with

$$\widetilde{d}((x_n,y_n),(T^n(x,y),T^n(y,x))) = d(x_n,T^n(x,y))+d(y_n,T^n(y,x)) \to 0 \quad \text{as} \quad n\to\infty.$$

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