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Ostrowski-type fractional integral inequalities for mappings whose derivatives are h-convex via Katugampola fractional integrals

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Abstract. In this paper we generalize some Riemann-Liouville fractional integral inequalities of Ostrowski-type for h-convex functions via Katugampola fractional integrals, generalizations of the Riemann-Liouville and the Hadamard fractional integrals. Also we deduce some known results by using p-functions, convex functions and s-convex functions.

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1. Introduction

The following inequality is known as Ostrowski inequality [17] (see also, [16, page 468]) which gives an upper bound for approximation of the integral average by the value $f(x)$ at a point $x \in [a, b]$. It is proved by Ostrowski in 1938.

Theorem 1.1. *Let $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a differentiable mapping in I°, the interior of I and $a, b \in I^\circ$, $a < b$. If $|f'(t)| \le M$ for all $t \in [a, b]$, then we have*

$$\left| f(x) - \frac{1}{(b-a)} \int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M,$$

where $x \in [a, b]$.

Ostrowski and Ostrowski-type inequalities have great importance in numerical analysis as they provide the bounds of different quadrature rules [1]. Over the years researchers are working to obtain Ostrowski-type inequalities for different kinds of functions. Recently Ostrowski-type inequalities via Riemann-Liouville fractional integrals are in focus (see [3, 4, 5, 6, 14, 15] and references therein).

Definition 1.2. A function f is called convex function on the interval $[a, b]$ if for any two points $x, y \in [a, b]$ and any t, where $0 \leq t \leq 1$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Definition 1.3. [2] A non-negative function $f : I \to \mathbb{R}$ is said to be p-function, if for any two points $x, y \in I$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Definition 1.4. [7] A function $f : I \to \mathbb{R}$ is said to be Godunova-Levin function, if for any two points $x, y \in I$ and $t \in (0, 1)$,

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

s-convex functions in the second sense have been introduced by Hudzik and Maligranda in [10] as follows.

Definition 1.5. [10] A function $f : [0, \infty) \to \mathbb{R}$ is called $s-$convex in the second sense on the interval $[0, \infty)$ if for any two points $x, y \in [0, \infty)$ and any t where $0 \leq t \leq 1$ and for some fixed $s \in (0, 1]$,

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y).$$

Definition 1.6. [19] Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a positive function. We say $f : I \to \mathbb{R}$ is a h-convex function, if f is non-negative and

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in (0, 1)$. If above inequality is reversed, then f is called h-concave.

It is easy to see that
(i) If $h(t) = t$, then (1.1) gives non-negative convex function.
(ii) If $h(t) = \frac{1}{t}$, then (1.1) gives Godunova-levin function.
(iii) If $h(t) = 1$, then (1.1) gives p-function.
(iv) If $h(t) = t^s$ where $s \in (0, 1)$, then (1.1) gives s-convex function in the second sense.

In a paper by Sonin in 1869 [18], he used the Cauchy's integral formula as a starting point to reach the differentiation with arbitrary index. Letnikov [13] extended the idea of Sonin a short time later in 1872. Both tried to define fractional derivatives by utilizing a closed contour. Finally, Laurent in [12] used a contour given as an open circuit instead of a closed circuit, led to the definition of the Riemann-Liouville fractional integral, which is due to a little known paper published by Holmgren in 1865 [9].

Definition 1.7. [12] Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, x<b,$$

where

$$\Gamma(\alpha)=\int_0^{\infty} e^{-u}u^{\alpha-1}du$$

is the integral representation of Euler gamma function. Here

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x).$$

In case of $\alpha=1$, the Riemann-Liouville fractional integrals reduces to the classical integral.

Definition 1.8. J. Hadamard introduced the Hadamard fractional integral in [8], and is given by

$$I_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x \left(log\frac{x}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau},$$

for $Re(\alpha)>0$, $x>a\geq 0$.

Recently Katugampola generalized Riemann-Liouville and Hadamard fractional integrals into a unique form as follows.

Definition 1.9. [11] Let $[a,b]$ be a finite interval in $\mathbb{R}$. Then the Katugampola fractional integrals of order $\alpha>0$ for a real valued function f are defined by

$${}^{\rho}I_{a+}^{\alpha}f\left(x\right)=\frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_a^x t^{\rho-1}\left(x^{\rho}-t^{\rho}\right)^{\alpha-1}f\left(t\right)dt$$

and

$${}^{\rho}I_{b-}^{\alpha}f\left(x\right)=\frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_x^b t^{\rho-1}\left(t^{\rho}-x^{\rho}\right)^{\alpha-1}f\left(t\right)dt$$

with $a<x<b$ and $\rho>0$, if the integrals exist, where $\Gamma\left(\alpha\right)$ is the Euler gamma function. For $\rho=1$, Katugampola fractional integrals give Riemann-Liouville fractional integrals, while $\rho\to 0^{+}$ produces the Hadamard fractional integral. For its proof one can refer [11].

We organize the paper as follows:

In this paper we prove some Ostrowski-type inequalities for mappings whose derivatives are h-convex via Katugampola fractional integrals. We deduce some known results by using p-functions, convex functions and s-convex functions. In particular we find Ostrowski-type inequalities for Riemann-Liouville fractional integrals.

2. Ostrowski-type fractional inequalities for h-convex functions via Katugampola fractional integral

In this section we present some Ostrowski-type inequalities for h-convex functions via Katugampola fractional integrals. The following lemma is very useful to obtain our results.

Lemma 2.1. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $a < b$ such that $f' \in L_1[a, b]$. Then we have the following equality*

$$\begin{aligned}
&f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho-a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right] \\
&= \frac{\rho(x^\rho-a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)a^\rho)dt \\
&- \frac{(b^\rho-x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)b^\rho)dt;\ x \in (a,b), \qquad (2.1)
\end{aligned}$$

with $\alpha, \rho > 0$.

Proof. It is easy to see that

$$\begin{aligned}
&\int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)a^\rho)dt \\
&= \left.\frac{t^{\alpha\rho+\rho-1} f(t^\rho x^\rho + (1-t^\rho)a^\rho)}{\rho t^{\rho-1}(x^\rho-a^\rho)}\right|_0^1 \\
&- \frac{\alpha\rho+\rho-1}{\rho(x^\rho-a^\rho)}\int_0^1 t^{\alpha\rho-1} f(t^\rho x^\rho + (1-t^\rho)a^\rho)dt \\
&= \frac{f(x^\rho)}{\rho(x^\rho-a^\rho)} - \frac{\alpha\rho+\rho-1}{\rho(x^\rho-a^\rho)}\int_a^x \left(\frac{y^\rho-a^\rho}{x^\rho-a^\rho}\right)^{\alpha-1}\frac{y^{\rho-1}f(y^\rho)}{x^\rho-a^\rho}dy \\
&= \frac{f(x^\rho)}{\rho(x^\rho-a^\rho)} - \frac{{}^\rho I^\alpha_{x-} f(a^\rho)(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{2-\alpha}(x^\rho-a^\rho)^{\alpha+1}} \qquad (2.2)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)b^\rho)dt \\
&= \left.\frac{t^{\alpha\rho+\rho-1} f(t^\rho x^\rho + (1-t^\rho)b^\rho)}{\rho t^{\rho-1}(x^\rho-b^\rho)}\right|_0^1 \\
&- \frac{\alpha\rho+\rho-1}{\rho(x^\rho-b^\rho)}\int_0^1 t^{\alpha\rho-1} f(t^\rho x^\rho + (1-t^\rho)b^\rho)dt \\
&= \frac{-f(x^\rho)}{\rho(b^\rho-x^\rho)} + \frac{\alpha\rho+\rho-1}{\rho(b^\rho-x^\rho)}\int_x^b \left(\frac{y^\rho-b^\rho}{x^\rho-b^\rho}\right)^{\alpha-1}\frac{y^{\rho-1}f(y^\rho)}{x^\rho-b^\rho}dy \\
&= \frac{-f(x^\rho)}{\rho(b^\rho-x^\rho)} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{2-\alpha}(b^\rho-x^\rho)^{\alpha+1}}. \qquad (2.3)
\end{aligned}$$

Multiplying (2.2) by $\frac{\rho(x^\rho-a^\rho)}{2}$ and (2.3) by $\frac{\rho(b^\rho-x^\rho)}{2}$, then adding resulting equations we get (2.1). □

Theorem 2.2. *Let $f : [a^\rho, b^\rho] \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $a < b$ such that $f' \in L_1[a, b]$. If $|f'|$ is h-convex on $[a^\rho, b^\rho]$ and $|f'(x^\rho)| \leq M$,*

$x \in [a,b]$, then the following inequality for Katugampola fractional integrals holds

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$
$$\leq \frac{M\rho(b^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho) + h(1-t^\rho) \right] dt; x \in (a,b), \tag{2.4}$$

with $\alpha, \rho > 0$.

Proof. Using Lemma 2.1, h-convexity of $\left|f'\right|$, and upper bound of $\left|f'(x^\rho)\right|$ we have

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$
$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1-t^\rho)a^\rho) \right| dt$$
$$+ \frac{\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1-t^\rho)b^\rho) \right| dt$$
$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho)\left|f'(x^\rho)\right| + h(1-t^\rho)\left|f'(a^\rho)\right| \right] dt$$
$$+ \frac{\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho)\left|f'(x^\rho)\right| + h(1-t^\rho)\left|f'(b^\rho)\right| \right] dt$$
$$\leq \frac{M\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho) + h(1-t^\rho) \right] dt$$
$$+ \frac{M\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho) + h(1-t^\rho) \right] dt$$
$$= \frac{M\rho(b^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho) + h(1-t^\rho) \right] dt.$$

This completes the proof. □

Corollary 2.3. *In Theorem 2.2, if we take $h(t) = 1$, which means that $\left|f'\right|$ is p-function, then (2.4) becomes the following inequality*

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$
$$\leq \frac{M(b^\rho - a^\rho)}{\alpha+1}; x \in [a,b], \tag{2.5}$$

with $\alpha, \rho > 0$.

Remark 2.4. (i) If we put $\rho = 1$ in (2.4) we get [15, Theorem 1].

(ii) If we put $\rho = 1$, $\alpha = 1$ and $x = \dfrac{a+b}{2}$ in (2.4) we get [15, Corollary 3].

(iii) If we put $\rho = 1$ and $h(t) = t$, which means that $\left|f'\right|$ is convex function in (2.4), then we get [15, Corollary 1].

(iv) If we put $\rho = 1$ and $h(t) = t^s$, which means that $|f'|$ is h-convex function in (2.4), then we get [15, Corollary 2].

Theorem 2.5. *Let $f : [a^\rho, b^\rho] \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $a < b$ such that $f' \in L_1[a, b]$. If $|f'|^q, q > 1$, is h-convex on $[a^\rho, b^\rho]$ and $|f'(x^\rho)| \leq M$, $x \in [a, b]$, then the following inequality for Katugampola fractional integrals holds*

$$\left| f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$
$$\leq \frac{M\rho(b^\rho - a^\rho)}{2\left(p(\alpha\rho + \rho - 1) + 1\right)^{\frac{1}{p}}} \left(\int_0^1 [h(t^\rho) + h(1 - t^\rho)]\, dt \right)^{\frac{1}{q}};\ x \in (a, b), \tag{2.6}$$

with $\alpha, \rho > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 2.1 and Holder's inequality we have

$$\left| f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$
$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1 - t^\rho)a^\rho) \right| dt$$
$$+ \frac{\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1 - t^\rho)b^\rho) \right| dt$$
$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(t^\rho x^\rho + (1 - t^\rho)a^\rho) \right|^q dt \right)^{\frac{1}{q}}$$
$$+ \frac{\rho(b^\rho - x^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(t^\rho x^\rho + (1 - t^\rho)a^\rho) \right|^q dt \right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is h-convex and $|f'(x^\rho)| \leq M$, $x \in [a, b]$, therefore we have for $x \in (a, b)$

$$\left| f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$

$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left[h(t^\rho) \left| f'(x) \right|^q + h(1 - t^\rho) \left| f'(a^\rho) \right|^q \right] dt \right)^{\frac{1}{q}}$$
$$+ \frac{\rho(b^\rho - x^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left[h(t^\rho) \left| f'(x) \right|^q + h(1 - t^\rho) \left| f'(b^\rho) \right|^q \right] dt \right)^{\frac{1}{q}}$$
$$\leq \frac{M\rho(x^\rho - a^\rho)}{2p(\alpha\rho + \rho - 1) + 1)^{\frac{1}{p}}} \left(\int_0^1 [h(t^\rho) + h(1 - t^\rho)]\, dt \right)^{\frac{1}{q}}$$
$$+ \frac{M\rho(b^\rho - x^\rho)}{2(p(\alpha\rho + \rho - 1) + 1)^{\frac{1}{p}}} \left(\int_0^1 [h(t^\rho) + h(1 - t^\rho)]\, dt \right)^{\frac{1}{q}}$$

$$= \frac{M\rho(b^\rho - a^\rho)}{2\left(p(\alpha\rho+\rho-1)+1\right)^{\frac{1}{p}}} \left(\int_0^1 \left[h(t^\rho) + h(1-t^\rho)\right] dt\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 2.6. *In Theorem* 2.5*, if we take* $h(t) = 1$*, which means that* $|f'|^q$ *is* p*-function, then* (2.6) *becomes the following inequality*

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$

$$\leq \frac{(2)^{\frac{1}{q}-1} M\rho(b^\rho - a^\rho)}{\left(p(\alpha\rho+\rho-1)+1\right)^{\frac{1}{p}}};\ x \in [a,b], \tag{2.7}$$

with $\alpha, \rho > 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2.7. (i) If we put $\rho = 1$ in (2.6) we get [15, Theorem 2].
(ii) If we put $\rho = 1$, $\alpha = 1$ and $x = \dfrac{a+b}{2}$ in (2.6) we get [15, Corollary 6].
(iii) If we put $\rho = 1$ and $h(t) = t$ in (2.6), which means that $|f'|$ is convex function, then we get [15, Corollary 4].
(iv) If we put $\rho = 1$ and $h(t) = t^s$, which means that $|f'|$ is h-convex function in (2.6), then we get [15, Corollary 5].

Theorem 2.8. *Let* $f : [a^\rho, b^\rho] \subseteq [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* (a^ρ, b^ρ) *such that* $f' \in L_1[a,b]$*, where* $a < b$*. If* $|f'|^q, q > 1$ *is* h*-convex on* $[a^\rho, b^\rho]$ *and* $|f'(x^\rho)| \leq M$*,* $x \in [a,b]$*, then the following inequality for Katugampola fractional integrals holds for* $x \in (a,b)$

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$

$$\leq \frac{M\rho(b^\rho - a^\rho)}{2} \left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho) + h(1-t^\rho)\right] dt\right)^{\frac{1}{q}}, \tag{2.8}$$

with $\alpha, \rho > 0$.

Proof. Using Lemma 2.1 and power mean inequality we have

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$

$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1-t^\rho)a^\rho) \right| dt$$

$$+ \frac{\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1-t^\rho)b^\rho) \right| dt$$

$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \left(\int_0^1 t^{\alpha\rho+\rho-1} dt\right)^{1-\frac{1}{q}} \left(t^{\alpha\rho+\rho-1} \int_0^1 \left| f'(t^\rho x^\rho + (1-t^\rho)a^\rho) \right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\rho(b^\rho - x^\rho)}{2}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho + (1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f'\right|^q$ is h-convex and $\left|f'(x^\rho)\right| \leq M$, $x \in [a,b]$, there for we have for $x \in (a,b)$

$$\begin{aligned}
&\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha}\right]\right| \\
&\leq \frac{\rho(x^\rho - a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}} \times \\
&\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho)\left|f'(x)\right|^q + h(1-t^\rho)\left|f'(a^\rho)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+ \frac{\rho(b^\rho - x^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}} \times \\
&\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho)\left|f'(x)\right|^q + h(1-t^\rho)\left|f'(b^\rho)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\leq \frac{M\rho(x^\rho - a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho) + h(1-t^\rho)\right]dt\right)^{\frac{1}{q}} \\
&+ \frac{M\rho(b^\rho - x^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho) + h(1-t^\rho)\right]dt\right)^{\frac{1}{q}} \\
&= \frac{M\rho(b^\rho - a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho) + h(1-t^\rho)\right]dt\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Remark 2.9. (i) If we put $\rho = 1$ in (2.8) we get [15, Theorem 3].
(ii) If we put $\rho = 1$, $\alpha = 1$ and $x = \frac{a+b}{2}$ in (2.8), we get [15, Corollary 9].
(iii) If we put $\rho = 1$ and $h(t) = t$ in (2.8) which means that $\left|f'\right|$ is convex function, then we get [15, Corollary 7].
(iv) If we put $\rho = 1$ and $h(t) = t^s$, which means that $\left|f'\right|$ is h-convex function in (2.8), then we get [15, Corollary 8].

Corollary 2.10. *In Theorem 2.8, if we take $h(t) = 1$, which means that $\left|f'\right|^q$ is p-function, then (2.8) becomes the following inequality*

$$\begin{aligned}
&\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha}\right]\right| \\
&\leq \frac{(2)^{\frac{1}{q}-1} M(b^\rho - a^\rho)}{\rho(\alpha+1)};\ x \in (a,b),
\end{aligned} \tag{2.9}$$

with $\alpha, \rho > 0$.

Conclusion. Due to the fact that the Katugampola fractional integrals are the generalizations of both the Riemann-Liouville fractional integrals and Hadamard fractional integrals, so in our paper by taking $\rho = 1$ we have deduced the known results for Riemann-Liouville fractional integrals. All results proved in this research paper can also be deduced for the Hadamard fractional integrals by taking limits when parameter $\rho \to 0^+$.

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