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Power of a meromorphic function that share a set with its derivative

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Abstract. In this article, we deal with the problem of the uniqueness of the power of a meromorphic function with its derivative counterpart sharing a set and thus improve our recent result under some constraints.

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1. Introduction and Definitions

In this article, we assume that readers are familiar with basic Nevanlinna theory([6]). By $\mathbb{C}$ and $\mathbb{N}$, we mean the set of complex numbers and the set of natural numbers respectively.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a in counting multiplicities (in short, CM). Similarly, we say that f and g share the value a in ignoring multiplicities (in short, IM), provided that $f-a$ and $g-a$ have the same set of zeros, where the multiplicities are not taken into account.

Also, we say that f and g share ∞ CM (resp. IM), if $1/f$ and $1/g$ share 0 CM (resp. IM).

Next we shortly recall the notion of weighted sharing which appeared in the literature in 2001 ([7]) as scaling between IM sharing to CM sharing.

Definition 1.1. ([7]) Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$, the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.
If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly f and g share a value a IM (resp. CM) if and only if f and g share $(a,0)$ (resp. (a,∞)).

Definition 1.2. ([7]) Let $S\subset\mathbb{C}\cup\{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_f(S,k)$, the set $\cup_{a\in S}E_k(a;f)$.

If $E_f(S,k)=E_g(S,k)$, then we say f, g share the set S with weight k.

Definition 1.3. A set $S\subset\mathbb{C}\cup\{\infty\}$ is called a unique range set for meromorphic functions with weight k (in short, $URSM_k$), if for any two non-constant meromorphic functions f and g, $E_f(S,k)=E_g(S,k)$ implies $f\equiv g$.

Similarly, one can define unique range set for entire functions with weight k (in brief, $URSE_k$).

Next we recall following two definitions:

Definition 1.4. ([2]) Let z_0 be a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q.

i) We denote by $\overline{N}_L(r,a;f)$, the counting function of those a-points of f and g where $p>q\geq 1$,

ii) by $N_E^{1)}(r,a;f)$, the counting function of those a-points of f and g where $p=q=1$ and

iii) by $\overline{N}_E^{(2}(r,a;f)$, the counting function of those a-points of f and g where $p=q\geq 2$, each point in these counting functions is counted only once.

In the same way, we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$.

Definition 1.5. ([2]) Let f and g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$, the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly

$$\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)\quad\text{and}\quad\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g).$$

The subject of sharing values between entire functions and their derivatives was first studied by Rubel and Yang ([11]). In 1977, they proved that if non-constant entire functions f and $f^{(1)}$ share two distinct finite numbers a, b CM, then $f\equiv f^{(1)}$.

Later, in 1979, analogous result for IM sharing was obtained by Mues and Steinmetz ([10]) in the following manner:

Theorem A. ([10]) Let f be a non-constant entire function. If f and $f^{(1)}$ share two distinct values a, b IM, then $f\equiv f^{(1)}$.

In the direction of value sharing and uniqueness problem, Yang and Zhang ([12]) were the first authors to consider the uniqueness of a power of a meromorphic (resp. entire) function $F=f^m$ and its derivative $F^{(1)}$ as:

Theorem B. ([12]) Let f be a non-constant entire (resp. meromorphic) function and $m>7$ (resp. 12) be an integer. If F and $F^{(1)}$ share 1 CM, then $F=F^{(1)}$, and f assumes the form

$$f(z)=ce^{\frac{z}{m}},$$

where c is a nonzero constant.

In this direction, Zhang ([13]) further improved the above result in the following manner:

Theorem C. ([13]) Let f be a non-constant entire function, m, k be positive integers and $a(z)(\not\equiv 0,\infty)$ be a small function of f. Suppose $f^m - a$ and $(f^m)^{(k)} - a$ share the value 0 CM and $m > k+4$. Then $f^m \equiv (f^m)^{(k)}$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{m}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

Afterwards, there were many improvements and generalizations concerning the uniqueness of f^m and $(f^m)^{(k)}$. But all authors paid their attention on value sharing or small function sharing, not on set sharing problem. Thus the natural curiosity will be:

Question 1.1. Is it possible to change the *value sharing notion* into *set sharing notion* in *Theorem C* keeping the conclusions same?

In connection to Question 1.1, recently we considered the uniqueness of f and $f^{(k)}$ when they share a set S instead of a value $a(\neq 0,\infty)$. To discuss the results in ([3]), we first introduce the polynomial of Lin and Yi ([8]).

$$P(z) = az^n - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2, \tag{1.1}$$

where $n \geq 3$ is an integer and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. Clearly the polynomial $P(z)$ has only simple zeros.
In ([3]), we considered the uniqueness of f and $f^{(k)}$ when they share a set.

Theorem E. ([3]) Let $n(\geq 8)$, $k(\geq 1)$ be two positive integers and f be a non-constant meromorphic function. Suppose that $S = \{z : P(z) = 0\}$ where $P(z)$ is defined by (1.1). If $E_f(S,3) = E_{f^{(k)}}(S,3)$, then $f \equiv f^{(k)}$.

But in this paper, we will see that if we impose some restrictions on f, then the cardinality of the set S defined in Theorem E will be reduced remarkably.

Thus our main goal is *to reduce the cardinality of this particular set S and to establish the uniqueness of the power of a meromorphic function with its derivative counterpart sharing the set S.*

The method of proving of the main result of this paper is from ([3, 4]).

2. Main Result

Theorem 2.1. Let f be a non-constant meromorphic function and $n(\geq 6)$, $k(\geq 1)$ and $m(\geq k+1)$ be three positive integers. Suppose that $S = \{z : P(z) = 0\}$ where $P(z)$ is defined by (1.1). If $E_{f^m}(S,3) = E_{(f^m)^{(k)}}(S,3)$, then $f^m \equiv (f^m)^{(k)}$ and hence f takes the form

$$f(z) = ce^{\frac{\zeta}{m}z},$$

where c is a non-zero constant and $\zeta^k = 1$.

The following example shows that for a non-constant entire function the set S in Theorem 2.1 can not be replaced by an arbitrary set containing six distinct elements.

Example 2.1. ([3]) For a non-zero complex number a, let

$$S = \{a\omega, a\sqrt{\omega}, a, \frac{a}{\sqrt{\omega}}, \frac{a}{\omega}, \frac{a}{\omega\sqrt{\omega}}\},$$

where ω is the non-real cubic root of unity. Choosing

$$f(z) = e^{\frac{\omega^{\frac{1}{2k}}}{m} z}$$

(taking the principal branch when $m \geq 2$), it is easy to verify that f^m and $(f^m)^{(k)}$ share (S, ∞), but $f^m \not\equiv (f^m)^{(k)}$

Remark 2.1. However the following questions are still unknown to us:

i) Is it possible to omit the condition $m \geq k+1$ keeping the condition $n(\geq 6)$ same in Theorem 2.1?

ii) Under the same conditions of Theorem 2.1, is it possible to further reduce the cardinality of S?

3. Auxiliary Lemmas

Before going to discuss the necessary lemmas, we recall a well known auxiliary function as

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{3.1}$$

where $F := R(f^m)$, $G := R\left((f^m)^{(k)}\right)$ and the expression

$$\frac{a(*)^n}{n(n-1)(*-\alpha_1)(*-\alpha_2)}$$

is denoted by $R(*)$.

In addition, in the expression of $R(*)$, we choose α_1 and α_2 as the distinct roots of the equation

$$n(n-1)z^2 - 2n(n-2)bz + (n-1)(n-2)b^2 = 0.$$

Lemma 3.1. ([1]) Let $Q(z) = (n-1)^2(z^n - 1)(z^{n-2} - 1) - n(n-2)(z^{n-1} - 1)^2$. Then

$$Q(z) = (z-1)^4 \prod_{i=1}^{2n-6} (z - \beta_i),$$

where $\beta_i \in \mathbb{C} \setminus \{0, 1\} (i = 1, 2, ..., 2n-6)$, which are distinct.

Lemma 3.2. Let F and G share $(1, l)$ where F and G defined as earlier, then

i) $\overline{N}_L(r, 1; F) \leq \mu\left(\overline{N}(r, 0; f) + \overline{N}(r, \infty; f)\right) + S(r, f)$,

ii) $\overline{N}_L(r, 1; G) \leq \mu\left(\overline{N}(r, 0; (f^m)^k) + \overline{N}(r, \infty; f)\right) + S(r, f)$,

where $\mu = \min\{\frac{1}{l}, 1\}$.

Proof. The proofs are similar to the proof of Lemma 2.2 of ([3]). So we omit the details. □

Lemma 3.3. Suppose that F and G share $(1,l)$ and $F \not\equiv G$. If $m \geq k+1$, then

$$\overline{N}(r,0;f) \leq \overline{N}(r,0,(f^m)^{(k)}) \leq \frac{2\mu+1}{\eta-2\mu}\overline{N}(r,\infty;f) + \frac{2}{\eta-2\mu}T(r) + S(r),$$

where $T(r) = T(r,f^m) + T\left(r,(f^m)^{(k)}\right)$, $S(r) = S(r,f)$ and $\eta = (m-k)n-1$.

Proof. For the proof, we define $U := \frac{F'}{(F-1)} - \frac{G'}{(G-1)}$ and consider two cases:
Case 1. Assume $U \equiv 0$. Then by integration, we get

$$F - 1 = B(G-1).$$

If z_0 is a zero of f, then $B = 1$ which is impossible, thus $\overline{N}(r,0;f) = S(r,f)$. Hence the result holds.
Case 2. Next we assume that $U \not\equiv 0$.

If z_0 is a zero of f of order t, then it is a zero of F of order mtn and that of G is of order $(mt-k)n$. Hence z_0 is a zero of U of order at least $\eta = (m-k)n-1$. Thus

$$\begin{aligned}
\overline{N}(r,0;f) &\leq \overline{N}\left(r,0,(f^m)^{(k)}\right) \\
&\leq \frac{1}{\eta}N(r,0;U) \leq \frac{1}{\eta}N(r,\infty;U) + S(r,f) \\
&\leq \frac{1}{\eta}\{\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_L(r,\infty;F) + \overline{N}_L(r,\infty;G) \\
&\quad + \overline{N}(r,\infty;G|F\neq\infty) + \overline{N}(r,\infty;F|G\neq\infty)\} + S(r,f) \\
&\leq \frac{1}{\eta}\{\mu\left(\overline{N}(r,0;f) + \overline{N}(r,0;(f^m)^{(k)}) + 2\overline{N}(r,\infty;f)\right) \\
&\quad + \overline{N}(r,\infty;f) + \overline{N}(r,\alpha_1;f^m) + \overline{N}(r,\alpha_2;f^m) \\
&\quad + \overline{N}(r,\alpha_1;(f^m)^{(k)}) + \overline{N}(r,\alpha_2;(f^m)^{(k)}\} + S(r,f) \\
&\leq \frac{1}{\eta}\{2\mu\overline{N}(r,0;(f^m)^{(k)}) + (2\mu+1)\overline{N}(r,\infty;f) + 2T(r)\} + S(r).
\end{aligned}$$

Thus

$$\overline{N}(r,0;f) \leq \overline{N}(r,0,(f^m)^{(k)}) \leq \frac{2\mu+1}{\eta-2\mu}\overline{N}(r,\infty;f) + \frac{2}{\eta-2\mu}T(r) + S(r).$$

Hence the proof of the lemma is completed. □

Lemma 3.4. Let F and G share $(1,l)$ and $F \not\equiv G$. If $m \geq k+1$, then

$$\overline{N}(r,\infty;f) \leq \frac{\mu(\eta-2\mu)+2}{(\lambda-2\mu)(\eta-2\mu)-(2\mu+1)}T(r) + S(r), \tag{3.2}$$

where $T(r) = T(r,f^m) + T\left(r,(f^m)^{(k)}\right)$ and $S(r) = S(r,f)$, $\lambda = m(n-2)-1$ and $\mu = \min\{\frac{1}{l},1\}$.

Proof. For the proof, we define $V := \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}$ and consider two cases:
Case 1. Assume that $V \equiv 0$. Then by integration, we get

$$\left(1 - \frac{1}{F}\right) = A\left(1 - \frac{1}{G}\right).$$

As f^m and $(f^m)^{(k)}$ share $(\infty, 0)$, so if $\overline{N}(r, \infty; f) \neq S(r, f)$, then $A = 1$, i.e., $F = G$, which is not possible. So $\overline{N}(r, \infty; f) = S(r, f)$. Thus the lemma holds.
Case 2. Next we assume that $V \not\equiv 0$.

If z_0 is a pole of f of order p, then it is a pole of $(f^m)^{(k)}$ of order $(pm + k)$ and that of F and G are $pm(n-2)$ and $(pm + k)(n-2)$ respectively.

Hence z_0 is a zero of $(\frac{F'}{F-1} - \frac{F'}{F})$ of order at least $pm(n-2) - 1$ and a zero of V of order at least λ where $\lambda = m(n-2) - 1$.

Since the zeros of F comes from zeros of f^m and that of G comes from zeros of $(f^m)^{(k)}$, so for the points where $f = 0$, each zero of F will be of larger multiplicities than that of G. Consequently

$$\begin{aligned} & \overline{N}_*(r, 0; F, G) + \overline{N}(r, 0; G \mid F \neq 0) \\ \leq\ & \overline{N}_L(r, 0; F) + \overline{N}_L(r, 0; G) + \overline{N}(r, 0; G \mid F \neq 0) \\ \leq\ & \overline{N}_L(r, 0; F) + \overline{N}(r, 0; G \mid F \neq 0) \leq \overline{N}(r, 0; G). \end{aligned}$$

Thus

$$\begin{aligned} \overline{N}(r, \infty; f) &\leq \frac{1}{\lambda} N(r, 0; V) \\ &\leq \frac{1}{\lambda} N(r, \infty; V) + S(r, f) \\ &\leq \frac{1}{\lambda}\{\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_L(r, 0; F) \\ &\quad + \overline{N}_L(r, 0; G) + \overline{N}(r, 0; G | F \neq 0\} + S(r, f) \\ &\leq \frac{1}{\lambda}\{\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}(r, 0; G)\} + S(r, f) \\ &\leq \frac{1}{\lambda}\{\mu\left(\overline{N}(r, 0; f) + \overline{N}(r, 0; (f^m)^{(k)}) + 2\overline{N}(r, \infty; f)\right) \\ &\quad + \overline{N}(r, 0; (f^m)^{(k)})\} + S(r, f). \end{aligned}$$

Now using Lemma 3.3, we get

$$\begin{aligned} (\lambda - 2\mu)\overline{N}(r, \infty; f) &\leq \mu T(r) + \overline{N}(r, 0; (f^m)^{(k)}) + S(r) \\ &\leq \mu T(r) + \frac{2\mu + 1}{\eta - 2\mu}\overline{N}(r, \infty; f) + \frac{2}{\eta - 2\mu}T(r) + S(r). \end{aligned}$$

Thus

$$\overline{N}(r, \infty; f) \leq \frac{\mu(\eta - 2\mu) + 2}{(\lambda - 2\mu)(\eta - 2\mu) - (2\mu + 1)}T(r) + S(r).$$

Hence the proof is completed. □

Lemma 3.5. If $H \not\equiv 0$ and F and G share $(1,l)$ then

$$\begin{aligned} & N(r,\infty;H) \qquad (3.3)\\ \leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,0;(f^m)^{(k)})+\overline{N}(r,b;f^m)+\overline{N}(r,b;(f^m)^{(k)})\\ & +\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_0(r,0;(f^m)')+\overline{N}_0(r,0;(f^m)^{(k+1)}), \end{aligned}$$

where $\overline{N}_0(r,0;(f^m)')$ denotes the counting function of the zeros of $(f^m)'$ which are not the zeros of $f(f^m-b)$ and $F-1$. Similarly $\overline{N}_0(r,0;(f^m)^{(k+1)})$ is defined.

Proof. The proof is obvious if we are keeping the followings in our mind:

As zeros of F come from the zeros of f^m and that of G come from the zeros of $(f^m)^{(k)}$, so $\overline{N}(r,0;F)\leq\overline{N}(r,0;G)$ when $m\geq k+1$. Also

$$\overline{N}(r,\infty;F)\leq\overline{N}(r,\infty;f^m)+\overline{N}(r,\alpha_1;f^m)+\overline{N}(r,\alpha_2;f^m).$$

Again simple zeros of $f^m-\alpha_i$ are not poles of H and multiple zeros of $f^m-\alpha_i$ are zeros of $(f^m)'$.

Similar explanation for G also holds. □

Lemma 3.6. If $F\equiv G$ and $n>5$, then $f^m=(f^m)^{(k)}$, i.e., f takes the form

$$f(z)=ce^{\frac{\zeta}{m}z},$$

where c is a non zero constant and $\zeta^k=1$.

Proof. Given $F\equiv G$, that is,

$$\begin{aligned} & n(n-1)f^{2m}\big((f^m)^{(k)}\big)^2\big\{f^{(n-2)m}-((f^m)^{(k)})^{n-2}\big\}\\ & -2n(n-2)bf^m(f^m)^{(k)}\big\{f^{(n-1)m}-\big((f^m)^{(k)}\big)^{n-1}\big\}\\ & +(n-1)(n-2)b^2\big\{(f^m)^n-\big((f^m)^{(k)}\big)^n\big\}=0. \end{aligned}$$

By substituting $h=\frac{(f^m)^{(k)}}{f^m}$ in above, we get

$$\begin{aligned} n(n-1)h^2f^{2m}(h^{n-2}-1) &- 2n(n-2)bhf^m(h^{n-1}-1) \qquad (3.4)\\ &+ (n-1)(n-2)b^2(h^n-1)=0. \end{aligned}$$

If h is a non-constant meromorphic function, then by Lemma 3.1, we get

$$\begin{aligned} & \{n(n-1)hf^m(h^{n-2}-1)-n(n-2)b(h^{n-1}-1)\}^2\\ & =-n(n-2)b^2(h-1)^4\prod_{i=1}^{2n-6}(h-\beta_i). \end{aligned}$$

Then by the second fundamental theorem, we get

$$\begin{aligned} & (2n-6)T(r,h)\\ \leq\ & \overline{N}(r,\infty;h)+\overline{N}(r,0;h)+\sum_{i=1}^{2n-6}\overline{N}(r,0;h-\beta_i)+S(r,h)\\ \leq\ & \overline{N}(r,\infty;h+\overline{N}(r,0;h)+\frac{1}{2}\sum_{i=1}^{2n-6}N(r,0;h-\beta_i)+S(r,h)\\ \leq\ & (n-1)T(r,h)+S(r,h), \end{aligned}$$

which is a contradiction as $n > 5$.
Thus h is a constant. Hence as f is non-constant and $b \neq 0$, we get from (3.4) that

$$(h^{n-2} - 1) = 0, \ (h^{n-1} - 1) = 0 \text{ and } (h^n - 1) = 0.$$

That is, $h = 1$. Consequently $f^m = (f^m)^{(k)}$.

If $f^m = (f^m)^{(k)}$, then we claim that 0 and ∞ are the Picard exceptional value of f.

For the proof, if z_0 is a zero of f of order t, then it is a zero of f^m and $(f^m)^{(k)}$ of order mt and $(mt - k)$ respectively, which is impossible.

Again if z_0 is a pole of f of order s, then it is a pole of f^m and $(f^m)^{(k)}$ of order ms and $(ms + k)$ respectively, which is impossible.

Thus our claim is true and hence f takes the form of

$$f(z) = ce^{\frac{\zeta}{m}z},$$

where c is a non zero constant and $\zeta^k = 1$. □

Lemma 3.7. If $H \equiv 0$ and $n > 5$, then $f^m = (f^m)^{(k)}$.

Proof. Since $H \equiv 0$, on integration, we have

$$F = \frac{AG + B}{CG + D}, \tag{3.5}$$

where A, B, C, D are constant satisfying $AD - BC \neq 0$, and F and G share $(1, \infty)$.

Thus applying Mokhon'ko's Lemma ([9]) in (3.5), we get

$$T(r, f^m) = T(r, (f^m)^{(k)}) + S(r, f). \tag{3.6}$$

Again from (3.5), we get $\overline{N}(r, \infty; f) = S(r, f)$ if $C \neq 0$, otherwise f^m and $(f^m)^{(k)}$ share (∞, ∞) if $C = 0$.

As $AD - BC \neq 0$, so $A = C = 0$ never occurs. Thus we consider the following cases:
Case 1. If $AC \neq 0$, then

$$F - \frac{A}{C} = \frac{BC - AD}{C(CG + D)}. \tag{3.7}$$

So,

$$\overline{N}\left(r, \frac{A}{C}; F\right) = \overline{N}(r, \infty; G).$$

Now by using the second fundamental theorem, we get

$$\begin{aligned}
& T(r, F) \\
\leq\ & \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, \frac{A}{C}; F) + S(r, F) \\
\leq\ & \overline{N}(r, \infty; f) + \overline{N}(r, \alpha_1; f^m) + \overline{N}(r, \alpha_2; f^m) + \overline{N}(r, 0; f^m) \\
& + \overline{N}(r, \infty; (f^m)^{(k)}) + \overline{N}(r, \alpha_1; (f^m)^{(k)}) + \overline{N}(r, \alpha_2; (f^m)^{(k)}) + S(r, f) \\
\leq\ & \frac{5}{n} T(r, F) + S(r, F),
\end{aligned}$$

which is a contradiction as $n > 5$.
Case 2. Next we consider $AC = 0$. Then obviously $A = 0$ and $C = 0$ can't occur. Thus we consider the following two subcases:
Subcase 2.1. If $A = 0$ and $C \neq 0$, then obviously $B \neq 0$ and

$$F = \frac{1}{\gamma G + \delta},$$

where $\gamma = \frac{C}{B}$ and $\delta = \frac{D}{B}$.
If F has no 1-point, then by using the second fundamental theorem, we get

$$\begin{aligned}
& T(r,F) \\
\leq\ & \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) + S(r,F) \\
\leq\ & \overline{N}(r,\infty;f^m) + \overline{N}(r,\alpha_1;f^m) + \overline{N}(r,\alpha_2;f^m) + \overline{N}(r,0;f^m) + S(r,f) \\
\leq\ & \frac{3}{n}T(r,F) + S(r,F),
\end{aligned}$$

which is a contradiction as $n > 5$. Thus $\gamma + \delta = 1$ and $\gamma \neq 0$. So,

$$F = \frac{1}{\gamma G + 1 - \gamma}.$$

Consequently, $\overline{N}(r,0;G + \frac{1-\gamma}{\gamma}) = \overline{N}(r,\infty;F)$.

Now if $\gamma \neq 1$, then applying the second fundamental theorem and equation (3.6), we get

$$\begin{aligned}
& T(r,G) \\
\leq\ & \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,0;G + \frac{1-\gamma}{\gamma}) + S(r,G) \\
\leq\ & \overline{N}(r,\infty;(f^m)^{(k)}) + \overline{N}(r,\alpha_1;(f^m)^{(k)}) + \overline{N}(r,\alpha_2;(f^m)^{(k)}) \\
+\ & \overline{N}(r,0;(f^m)^{(k)}) + \overline{N}(r,\infty;(f^m)) + \overline{N}(r,\alpha_1;(f^m)) \\
+\ & \overline{N}(r,\alpha_2;(f^m)) + S(r,f) \\
\leq\ & \frac{5}{n}T(r,G) + S(r,G),
\end{aligned}$$

which is a contradiction as $n > 5$. Thus $\gamma = 1$ and hence $FG \equiv 1$ which give

$$f^{mn}\left((f^m)^{(k)}\right)^n = \frac{n^2(n-1)^2}{a^2}(f^m - \alpha_1)(f^m - \alpha_2)((f^m)^{(k)} - \alpha_1)((f^m)^{(k)} - \alpha_2).$$

It is clear from the above equation that f has no pole, because $n > 5$. Now let z_0 be a α_{1i} point of f of order s, where $(\alpha_{1i})^m = \alpha_1$, then it can't be a pole of $(f^m)^{(k)}$ as f has no pole, so z_0 is a zero of $(f^m)^{(k)}$ of order q satisfying $n \leq nq = s$. Thus

$$\begin{aligned}
\overline{N}(r,\infty;f) &= S(r,f), \\
\overline{N}(f,\alpha_{1i};f) &\leq \frac{1}{n}N(f,\alpha_{1i};f) \text{ and} \\
\overline{N}(f,\alpha_{2j};f) &\leq \frac{1}{n}N(f,\alpha_{2j};f).
\end{aligned}$$

Thus by the second fundamental theorem, we get

$$\begin{aligned} & (2m-1)T(r,f) \\ \leq\ & \overline{N}(r,\infty;f) + \sum_{i=1}^{m}\overline{N}(r,\alpha_{1i};f) + \sum_{j=1}^{m}\overline{N}(r,\alpha_{2j};f) + S(r,f) \\ \leq\ & \frac{2m}{n}T(r,f) + S(r,f), \end{aligned}$$

which is not possible as $n > 5$.
Subcase 2.2. If $A \neq 0$ and $C = 0$, then obviously $D \neq 0$ and

$$F = \lambda G + \mu,$$

where $\lambda = \frac{A}{D}$ and $\mu = \frac{B}{D}$. If F has no 1-point, then we arrive at a contradiction as the previous case. Thus $\lambda + \mu = 1$ with $\lambda \neq 0$. Also

$$\overline{N}\left(r,0;G+\frac{1-\lambda}{\lambda}\right) = \overline{N}(r,0;F).$$

Now if $\lambda \neq 1$, then by using the second fundamental theorem and equation (3.6), we get

$$\begin{aligned} & T(r,G) \\ \leq\ & \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,0;G+\frac{1-\lambda}{\lambda}) + S(r,G) \\ \leq\ & \overline{N}(r,\infty;(f^m)^{(k)}) + \overline{N}(r,\alpha_1;(f^m)^{(k)}) + \overline{N}(r,\alpha_2;(f^m)^{(k)}) \\ +\ & \overline{N}(r,0;(f^m)^{(k)}) + \overline{N}(r,0;(f^m)) + S(r,f) \\ \leq\ & \frac{5}{n}T(r,G) + S(r,G), \end{aligned}$$

which is a contradiction as $n > 5$. Thus $\lambda = 1$ and hence $F \equiv G$.

Now in view of Lemma 3.6, we get $f^m = (f^m)^{(k)}$ as $n > 5$, i.e. f takes the form

$$f(z) = ce^{\frac{\zeta}{m}z},$$

where c is a non zero constant and $\zeta^k = 1$. □

4. Proof of Main Result

Proof of Theorem 2.1 . Let H be defined by equation (3.1). Now we consider two cases:
Case 1. First we assume $H \not\equiv 0$. Then clearly $F \not\equiv G$ and

$$\overline{N}(r,1;F|=1) = \overline{N}(r,1;G|=1) \leq N(r,\infty;H).$$

So by the second fundamental theorem and Lemma 3.5, we get

$$\begin{aligned} & (n+1)T(r,f^m) \quad (4.1)\\ \leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}(r,b;f^m)\\ & +\overline{N}(r,1;F)-N_0(r,0,(f^m)')+S(r,f)\\ \leq\ & 2\{\overline{N}(r,\infty;f)+\overline{N}(r,b;f^m)\}+\overline{N}(r,0;(f^m)^{(k)})\\ & +\overline{N}(r,0;f)+\overline{N}(r,b;(f^m)^{(k)})+\overline{N}(r,1;F|\geq 2)\\ & +\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_0\left(r,0;(f^m)^{(k+1)}\right)+S(r,f). \end{aligned}$$

Now

$$\begin{aligned} & \overline{N}(r,1;F|\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0\left(r,0;(f^m)^{(k+1)}\right) \quad (4.2)\\ \leq\ & \overline{N}(r,1;G|\geq 2)+\overline{N}(r,1;G|\geq 3)+\overline{N}_0\left(r,0;(f^m)^{(k+1)}\right)\\ \leq\ & N\left(r,0;(f^m)^{(k+1)}\mid (f^m)^{(k)}\neq 0\right)+S(r,f)\\ \leq\ & \overline{N}\left(r,0;(f^m)^{(k)}\right)+\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

Thus

$$\begin{aligned} & (n+1)T(r,f^m) \quad (4.3)\\ \leq\ & 2\{\overline{N}(r,\infty;f)+\overline{N}(r,b;f^m)\}+\overline{N}(r,0;f)\\ & +2\overline{N}(r,0;(f^m)^{(k)})+\overline{N}(r,b;(f^m)^{(k)})+\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

Similarly for $(f^m)^{(k)}$, we get

$$\begin{aligned} & (n+1)T(r,(f^m)^{(k)}) \quad (4.4)\\ \leq\ & 2\{\overline{N}(r,\infty;f)+\overline{N}(r,b;(f^m)^{(k)})\}+\overline{N}(r,0;(f^m)^{(k)}) \quad (4.5)\\ & +2\overline{N}(r,0;f)+\overline{N}(r,b;f^m)+\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

Adding (4.3) and (4.4), we get

$$\begin{aligned} (n+1)T(r) \leq\ & 6\overline{N}(r,\infty;f)+3\{\overline{N}(r,0;f)+\overline{N}(r,0;(f^m)^{(k)})\} \quad (4.6)\\ & +3\{\overline{N}(r,b;f^m)+\overline{N}(r,b;(f^m)^{(k)})\}+S(r,f) \end{aligned}$$

and

$$(n-5)T(r) \leq 6\overline{N}(r,\infty;f)+S(r). \quad (4.7)$$

Thus using Lemma 3.4, we get

$$(n-5)T(r) \leq \frac{6\mu(\eta-2\mu)+12}{(\lambda-2\mu)(\eta-2\mu)-(2\mu+1)}T(r)+S(r),$$

which is a contradiction as $n\geq 6$ and $l\geq 3$.

Case 2. Next we assume $H\equiv 0$. Then for $n>5$, applying Lemma 3.7, we have $f^m=(f^m)^{(k)}$. Thus by the same arguments using in Lemma 3.6, we see that f takes the form

$$f(z)=ce^{\frac{\zeta}{m}z},$$

where c is a non zero constant and $\zeta^k = 1$. Thus the proof is completed. □

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