Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 87–94

Fixed point theorems for a system of operator equations with applications

Cristina Urs

Abstract. The purpose of this paper is to present some existence and uniqueness theorems for a general system of operator equations. The abstract result generalizes some existence results obtained in [V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 7347-7355] for the case of coupled fixed point problem. We also provide an application to a system of integro-differential equations.

Mathematics Subject Classification (2010): 47H10, 54H25, 34B15.

Keywords: Fixed point, contractive condition, metric space, coupled fixed point, integral equation.

1. Introduction

The classical Banach contraction principle is remarkable in its simplicity and it is perhaps the most widely applied fixed point theorem in all analysis. This is because the contractive condition on the operator is easy to test and it requires only the structure of a complete metric space for its setting (see S. Banach [1]). This principle is also a very useful tool in nonlinear analysis with many applications to operatorial equations, fractal theory, optimization theory and other topics. Several authors have been dedicated to the improvement and generalization of this principle (see [3], [6], [4], [5], etc.)

The purpose of this paper is to present some existence and uniqueness results which will extend and generalize some theorems obtained by V. Berinde in [2] for the case of coupled fixed point problems. We also provide an application to an integral equation system. For related results see also [7].

This work was possible due to the financial support of the Sectorial Operational Program for Human Resources Development 2007 – 2013, co-financed by the European Social Fund, under the project number POSDRU/159/1.5/S/132400 with the title ,,Young successful researchers- professional development in a international and interdisciplinary enviroment".

2. Main results

The first result is an existence and uniqueness result which generalizes Theorem 3 presented by V. Berinde in [2].

Theorem 2.1. *Let X be a nonempty set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let T_1, $T_2 : X \times X \to X$ be two operators for which there exists a constant $k \in [0, 1)$ such that*

$$d(T_1(x, y), T_1(u, v)) + d(T_2(x, y), T_2(u, v)) \leq k(d(x, u) + d(y, v)),$$

for all (x, y), $(u, v) \in X \times X$.
Then we have the following conclusions:
(i) there exists a unique element $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_2(x^*, y^*) \end{cases}$$

(ii) the sequence $(T_1^n(x, y), T_2^n(x, y))_{n\in\mathbb{N}}$ converges to (x^*, y^*) as $n \to \infty$

$$\begin{aligned} T_1^{n+1}(x, y) \quad &: \quad = T_1^n(T_1(x, y), T_2(x, y)) \\ T_2^{n+1}(x, y) \quad &: \quad = T_2^n(T_1(x, y), T_2(x, y)) \end{aligned}$$

for all $n \in \mathbb{N}$.
(iii) we have the following estimation

$$\begin{aligned} d(T_1^n(x_0, y_0), x^*) \quad &\leq \quad \frac{k^n}{1-k} d(x_0, T_1(x_0, y_0)) \\ d(T_2^n(x_0, y_0), y^*) \quad &\leq \quad \frac{k^n}{1-k} d(y_0, T_2(x_0, y_0)) \end{aligned}$$

(iv) let $F_1, F_2 : X \times X \to X$ be two operators such that, there exist $\epsilon_1, \epsilon_2 > 0$ with

$$\begin{aligned} d(T_1(x, y), F_1(x, y)) \quad &\leq \quad \epsilon_1 \\ d(T_2(x, y), F_2(x, y)) \quad &\leq \quad \epsilon_2 \end{aligned}$$

for all $(x, y) \in X \times X$. If $(a^*, b^*) \in X \times X$ is such that

$$\begin{cases} a^* = F_1(a^*, b^*) \\ b^* = F_2(a^*, b^*) \end{cases}$$

then

$$d(x^*, a^*) + d(y^*, b^*) \leq \frac{\epsilon_1 + \epsilon_2}{1-k}$$

Proof. (i)- (ii)
We define $T : X \times X \to X \times X$ by

$$T(x, y) = (T_1(x, y), T_2(x, y)).$$

Lets denote $Z := X \times X$ and $d^* : Z \times Z \to \mathbb{R}_+$

$$d^*((x, y), (u, v)) := \frac{1}{2}(d(x, u) + d(y, v))$$

for all $(x, y), (u, v) \in X \times X$.

Then we have

$$d^*(T(x,y),T(u,v)) = \frac{d(T_1(x,y),T_1(u,v)) + d(T_2(x,y),T_2(u,v))}{2}.$$

Then we denote $(x,y) := z$, $(u,v) := w$ we get that

$$d^*(T(z),T(w)) \le k \cdot d^*(z,w)$$

for every $z, w \in X \times X$.
Hence we obtained Banach's contraction condition. Applying Banach's contraction fixed point theorem we get that there exists a unique element $(x^*, y^*) := z^* \in X \times X$ such that

$$z^* = T(z^*)$$

and it is equivalent with

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_2(x^*, y^*) \end{cases}.$$

For each $z \in X \times X$, we have that $T^n(z) \to z^*$ as $n \to \infty$ where

$$\begin{aligned} T^0(z) &: = z, T^1(z) = T(x,y) = (T_1(x,y), T_2(x,y)) \\ T^2(z) &= T(T_1(x,y), T_2(x,y)) = (T_1^2(x,y), T_2^2(x,y)) \end{aligned}$$

and in generally

$$\begin{aligned} T_1^{n+1}(x,y) &: = T_1^n(T_1(x,y), T_2(x,y)) \\ T_2^{n+1}(x,y) &: = T_2^n(T_1(x,y), T_2(x,y)). \end{aligned}$$

We get that $T^n(z) = (T_1^n(z), T_2^n(z)) \to z^* = (x^*, y^*)$ as n$\to \infty$, for all $z = (x,y) \in X \times X$.
So for all $(x,y) \in X \times X$ we have that

$$\begin{aligned} T_1^n(x,y) &\to x^* \text{ as } n \to \infty \\ T_2^n(x,y) &\to y^* \text{ as } n \to \infty. \end{aligned}$$

(iii) We apply Banach's contraction principle and we have successively

$$\begin{aligned} d(T_1^n(x_0,y_0), x^*) &\le \frac{k^n}{1-k} d(x_0, T_1(x_0,y_0)) \\ d(T_2^n(x_0,y_0), y^*) &\le \frac{k^n}{1-k} d(x_0, T_2(x_0,y_0)) \end{aligned}$$

(iv) Let us consider $F : X \times X \to X \times X$ given by $F(x,y) = (F_1(x,y), F_2(x,y))$ and

$$\begin{aligned} d^*(T(x,y), F(x,y)) &= d^*((T_1(x,y), T_2(x,y)), (F_1(x,y), F_2(x,y))) \\ &= \frac{d(T_1(x,y), F_1(x,y)) + d(T_2(x,y), F_2(x,y))}{2} \le \epsilon, \end{aligned}$$

where $\epsilon := \frac{\epsilon_1 + \epsilon_2}{2}$.
Then, by the data dependence theorem related to Banachs contraction principle we get that

$$d(x^*, a^*) + d(y^*, b^*) \le \frac{\epsilon_1 + \epsilon_2}{1-k}.$$

Hence we get that

$$d^*((x^*,y^*),(a^*,b^*)) \le \frac{\epsilon}{1-k}. \qquad \square$$

An existence and uniqueness result, similar to Theorem 2.1, is the following theorem.

Theorem 2.2. *Let X be a nonempty set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $T_1, T_2 : X \times X \to X$ be two operators for which there exists a constant $k \in [0,1)$ such that, for each $(x,y),(u,v) \in X \times X$, we have*

$$\max\{d(T_1(x,y),T_2(u,v)), d(T_2(x,y),T_2(u,v))\} \le k \cdot \max\{d(x,u), d(y,v)\}.$$

Then we have the following conclusions:
(i) there exists a unique element $(x^*,y^*) \in X \times X$ such that

$$\begin{cases} x^* = T_1(x^*,y^*) \\ y^* = T_2(x^*,y^*) \end{cases}$$

(ii) the sequence $(T_1^n(x,y), T_2^n(x,y))_{n\in\mathbb{N}}$ converges to (x^*,y^*) as $n \to \infty$

$$\begin{aligned} T_1^{n+1}(x,y) \quad &: \quad = T_1^n(T_1(x,y),T_2(x,y)) \\ T_2^{n+1}(x,y) \quad &: \quad = T_2^n(T_1(x,y),T_2(x,y)) \end{aligned}$$

for all $n \in \mathbb{N}$.
(iii) we have the following estimation

$$\begin{aligned} d(T_1^n(x_0,y_0),x^*) &\le \frac{k^n}{1-k} d(x_0, T_1(x_0,y_0)) \\ d(T_2^n(x_0,y_0),y^*) &\le \frac{k^n}{1-k} d(y_0, T_2(x_0,y_0)) \end{aligned}$$

(iv) let $F_1, F_2 : X \times X \to X$ be two operators such that, there exist $\epsilon_1, \epsilon_2 > 0$ with

$$\begin{aligned} d(T_1(x,y),F_1(x,y)) &\le \epsilon_1 \\ d(T_2(x,y),F_2(x,y)) &\le \epsilon_2 \end{aligned}$$

for all $(x,y) \in X \times X$. If $(a^*,b^*) \in X \times X$ is such that

$$\begin{cases} a^* = F_1(a^*,b^*) \\ b^* = F_2(a^*,b^*) \end{cases}$$

then

$$\max\{d(x^*,a^*), d(y^*,b^*)\} \le \frac{\max\{\epsilon_1,\epsilon_2\}}{1-k}.$$

Proof. (i)- (ii)
We define $T : X \times X \to X \times X$ by

$$T(x,y) = (T_1(x,y), T_2(x,y)).$$

Lets denote $Z := X \times X$ and $d_* : Z \times Z \to \mathbb{R}_+$

$$d_*((x,y),(u,v)) := \frac{1}{2}\max\{d(x,u), d(y,v)\}$$

for all $(x,y),(u,v) \in X \times X$.

Then we have

$$d_*(T(x,y),T(u,v)) = \frac{1}{2}\max\{d(T_1(x,y),T_1(u,v)),d(T_2(x,y),T_2(x,y),T_2(u,v))\}$$

If we denote $(x,y) := z$, $(u,v) := w$ we get that

$$d_*(T(z),T(w)) \le k \cdot \max\{d(x,u),d(y,v)\} = k \cdot d_*(z,w)$$

for every $z,w \in X \times X$.

Hence we obtained Banach's type contraction condition. By Banach's contraction fixed point theorem we get that there exists a unique element $(x^*,y^*) := z^* \in X \times X$ such that

$$z^* = T(z^*)$$

and it is equivalent with

$$\begin{cases} x^* = T_1(x^*,y^*) \\ y^* = T_2(x^*,y^*) \end{cases}.$$

For each $z \in X \times X$, we have that $T^n(z) \to z^*$ as $n \to \infty$ where

$$\begin{aligned} T^0(z) \quad &: \quad = z, T^1(z) = T(x,y) = (T_1(x,y),T_2(x,y)) \\ T^2(z) \quad &= \quad T(T_1(x,y),T_2(x,y)) = (T_1^2(x,y),T_2^2(x,y)) \end{aligned}$$

and in generally

$$\begin{aligned} T_1^{n+1}(x,y) \quad &: \quad = T_1^n(T_1(x,y),T_2(x,y)) \\ T_2^{n+1}(x,y) \quad &: \quad = T_2^n(T_1(x,y),T_2(x,y)). \end{aligned}$$

We get that $T^n(z) = (T_1^n(z),T_2^n(z)) \to z^* = (x^*,y^*)$ as n$\to \infty$, for all $z = (x,y) \in X \times X$.

So, for all $(x,y) \in X \times X$ we have that

$$\begin{aligned} T_1^n(x,y) \quad &\to \quad x^* \text{ as } n \to \infty \\ T_2^n(x,y) \quad &\to \quad y^* \text{ as } n \to \infty. \end{aligned}$$

(iii) We apply Banach's contraction principle and we have successively

$$\begin{aligned} d(T_1^n(x_0,y_0),x^*) \quad &\le \quad \frac{k^n}{1-k}d(x_0,T_1(x_0,y_0)) \\ d(T_2^n(x_0,y_0),y^*) \quad &\le \quad \frac{k^n}{1-k}d(x_0,T_2(x_0,y_0)) \end{aligned}$$

(iv) Let us consider $F : X \times X \to X \times X$ given by

$$F(x,y) = (F_1(x,y),F_2(x,y))$$

and

$$\begin{aligned} d_*(T(x,y),F(x,y)) \quad &= \quad d_*((T_1(x,y),T_2(x,y)),(F_1(x,y),F_2(x,y))) \\ &= \quad \frac{1}{2}\max\{d(T_1(x,y),F_1(x,y)),d(T_2(x,y),F_2(x,y))\} \le \epsilon \end{aligned}$$

where $\epsilon := \frac{\max\{\epsilon_1,\epsilon_2\}}{2}$.

Then, applying the abstract data dependence theorem related to Banachs contraction principle we get that

$$\max\{d(x^*,a^*),d(y^*,b^*)\} \leq \frac{\max\{\epsilon_1,\epsilon_2\}}{1-k}.$$

We obtain that

$$d_*((x^*,y^*),(a^*,b^*)) \leq \frac{\epsilon}{1-k}. \qquad \square$$

3. An application

In this section, we will consider the following problem:

$$\begin{cases} x(t) = \int\limits_a^b K(s,t,x(s),y(s))ds \\ y''(t) = f(s,x(s),y(s)),\ \ y(a)=0\ y(b)=0 \end{cases} \tag{3.1}$$

This problem is equivalent to

$$\begin{cases} x(t) = \int\limits_a^b K(s,t,x(s),y(s))ds \\ y(t) = -\int\limits_a^b G(t,s)f(s,x(s),y(s))ds, \end{cases}$$

where $G:[a,b]\times[a,b]\to\mathbb{R}$ is given by

$$G(t,s) := \begin{cases} \frac{(s-a)(b-t)}{b-a}, & \text{if } s\leq t \\ \frac{(t-a)(b-s)}{b-a}, & \text{if } s\geq t \end{cases}.$$

Assumption (*) Suppose that $K:[a,b]^2\times\mathbb{R}^2\to\mathbb{R}$ and $f:[a,b]^2\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions and satisfy the following Lipschitz conditions

$$\begin{aligned} |K(t,s,u_1,v_1)-K(t,s,u_2,v_2)| &\leq \alpha|u_1-u_2|+\beta|v_1-v_2| \\ |f(s,u_1,v_1)-f(s,u_2,v_2)| &\leq \gamma|u_1-u_2|+\delta|v_1-v_2|, \end{aligned}$$

for every $t,s\in[a,b]$ and $u_1,v_1,u_2,v_2\in\mathbb{R}$, where $\alpha,\beta,\gamma,\delta>0$ such that

$$\max\left\{(\alpha(b-a)+\gamma\frac{(b-a)^2}{8}),\left(\beta(b-a)+\delta\frac{(b-a)^2}{8}\right)\right\}<1.$$

Let $X=(C[a,b],\|\cdot\|_C)$ be the Banach space of continuous functions endowed with the norm

$$\|x\|_c := \max_{t\in[a,b]}|x(t)|.$$

We define the following operators

$$T_1,T_2: X\times X\to X,\ \ (x,y)\to T_1(x,y) \text{ and } (x,y)\to T_2(x,y),$$

where

$$T_1(x,y)(t) = \int\limits_a^b K(s,t,x(s),y(s))ds$$

$$T_2(x,y)(t) = -\int_a^b G(t,s)f(s,x(s),y(s))ds.$$

An existence and uniqueness result for the system (3.1) is the following theorem.

Theorem 3.1. *Consider the problem (*3.1*) with* $K, f : [a,b]^2 \times \mathbb{R}^2 \to \mathbb{R}$ *and suppose that Assumption (*) is satisfied. Then there exists a unique solution* (x^*, y^*) *of the problem (*3.1*).*

Proof. We verify that T_1 and T_2 satisfy the hypotheses of Theorem 2.1. Indeed, for every $t \in [a,b]$, we have

$$\begin{aligned}
|T_1(x,y)(t) - T_1(u,v)(t)| &= \left| \int_a^b K(s,t,x(s),y(s))ds - \int_a^b K(s,t,u(s),v(s))ds \right| \\
&\le \int_a^b |K(s,t,x(s),y(s)) - K(s,t,u(s),v(s))|\, ds \\
&\le \alpha \int_a^b |x(s)-u(s)|\, ds + \beta \int_a^b |y(s)-v(s)|\, ds \\
&\le \alpha \|x-u\|_C (b-a) + \beta \|y-v\|_C (b-a).
\end{aligned}$$

Taking the $\max\limits_{t\in[a,b]}$ in the above relation we get that

$$\|T_1(x,y) - T_1(u,v)\|_C \le \alpha(b-a)\|x-u\|_C + \beta(b-a)\|y-v\|_C.$$

On the other hand, for every $t \in [a,b]$, we have

$$\begin{aligned}
|T_2(x,y)(t) - T_2(u,v)(t)| &= \left| -\int_a^b G(t,s)f(s,x(s),y(s))ds + \int_a^b G(t,s)f(s,u(s),v(s))ds \right| \\
&\le \int_a^b G(t,s)\,|f(s,u(s),v(s)) - f(s,x(s),y(s))|\, ds \\
&\le \gamma \int_a^b G(t,s)\,|u(s)-x(s)|\, ds + \delta \int_a^b G(t,s)\,|v(s)-y(s)|\, ds \\
&\le \gamma \|u-x\|_C \int_a^b G(t,s)ds + \delta \|v-y\|_C \int_a^b G(t,s)ds.
\end{aligned}$$

Taking the $\max\limits_{t\in[a,b]}$ in the above relation we obtain

$$\|T_2(x,y) - T_2(u,v)\|_C \le \gamma \frac{(b-a)^2}{8} \|u-x\|_C + \delta \frac{(b-a)^2}{8} \|v-y\|_C.$$

Hence we get that

$$\|T_1(x,y)-T_1(u,v)\|_C+\|T_2(x,y)-T_2(u,v)\|_C$$
$$\leq [\alpha(b-a)+\gamma\frac{(b-a)^2}{8}]\,\|x-u\|_C+[\beta(b-a)+\delta\frac{(b-a)^2}{8}]\,\|y-v\|_C$$
$$\leq \max\left\{(\alpha(b-a)+\gamma\frac{(b-a)^2}{8}),(\beta(b-a)+\delta\frac{(b-a)^2}{8})\right\}(\|x-u\|_C+\|y-v\|_C).$$

Since the hypothesis of Theorem 2.1 is satisfied we get that the problem (3.1) has a unique solution on I. □

References

[1] Banach, S., *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math., **3**(1922), 133-181.

[2] Berinde, V., *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces,* Nonlinear Anal., **74**(2011) 7347-7355.

[3] Krasnoselskii, M.A., Zabreiko, P.P., *Geometrical Methods of Nonlinear Analysis*, Springer Verlag Berlin, 1984.

[4] Lakshmikantham, V., Ciric, L., *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70**(2009), 4341-4349.

[5] Rus, I.A., *Principles and Applications of Fixed Point Theory (in Romanian)*, Dacia, Cluj-Napoca, 1979.

[6] Rus, I.A., Petruşel, A., Petruşel G., *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008.

[7] Urs, C.L., *Contributions to the study of the coupled fixed point problem*, Ph. D. Thesis, 2013.

Cristina Urs
Department of Mathematics
Babeş-Bolyai University
Cluj-Napoca, Romania
e-mail: `cristina.urs@math.ubbcluj.ro`