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Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative

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Abstract. The purpose of this paper is to establish some types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for a class of implicit Hadamard fractional-order differential equation.

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1. Introduction

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. See, for example, the books [1, 2, 3, 5, 17, 34] and references therein. Fractional differential equations arise naturally in various fields such as viscoelastic materials, polymer science, fractals, chaotic dynamics, nonlinear control, signal processing, bioengineering and chemical engineering, etc. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. We refer the reader, for example, to the books such as [6, 18, 20, 22, 30], and references therein.

On the other hand, the stability problem of functional equations (of group homomorphisms) was formulated in 1940 by Ulam, in a talk given at Wisconsin University [31, 32]. In 1941, Hyers [13] gave the partial answer to the question of Ulam (for the approximately additive mappings) in the case Banach spaces. Hyers's theorem was generalized by Aoki [4] (for additive mappings). Between 1978 and 1998, Th. M. Rassias established the Hyers-Ulam stability of linear and nonlinear mappings [21, 23, 24]. In 1997, Obloza is the first author who has investigated the Hyers-Ulam stability of linear differential equations [19]. During the last two decades, many papers

[7, 12, 14, 15, 16, 33] and books [11, 25, 26, 27] on this subject have been published in order to generalize the results of Hyers in many directions. Recently in [8, 9, 10] Benchohra and Lazreg considered some existence and Ulam stability results for various classes of implicit differential equations involving the Caputo fractional derivative.

The purpose of this paper is to establish existence, uniqueness and stability results of solutions for the following initial value problem for implicit fractional-order differential equation

$$^{H}D^{\alpha}y(t) = f(t, y(t), {}^{H}D^{\alpha}y(t)), \text{ for each } t \in J,\ 0 < \alpha \leq 1, \tag{1.1}$$

$$y(1) = y_1, \tag{1.2}$$

where $^{H}D^{\alpha}$ is the Hadamard fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function space, $y_1 \in \mathbb{R}$ and $J = [1, T]$, $T > 1$.

The paper is organized as follows. In Section 2 we introduce some definitions, notations, and lemmas which are used throughout the paper. In Section 3, we will prove an existence and uniqueness results concerning the problem (1.1)-(1.2). Section 4 is devoted to Ulam-Hyers stabilities for the problem (1.1)-(1.2). Finally, in the last section, we give two examples to illustrate our main results.

This paper initiates the existence and Ulam stability of implicit differential equations involving the Hadamard fractional derivative.

2. Preliminaries

Definition 2.1. ([17]) The Hadamard fractional integral of order α for a continuous function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$_{H}I^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds,\ \alpha > 0,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt,\ \alpha > 0.$

Definition 2.2. ([17]) The Hadamard derivative of fractional order α for a continuous function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$^{H}D^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \frac{g(s)}{s} ds, \quad n-1 < \alpha < n,\ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.3. ([17]) The Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad \Re(\alpha) > 0.$$

The general Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad \Re(\alpha) > 0,\ \Re(\beta) > 0.$$

Thus,

$$E_\alpha(z) = E_{\alpha,1}(z),$$
$$E_1(z) = E_{1,1}(z) = e^z,$$
$$E_2(z) = \cosh\sqrt{z},$$
$$E_{1,2}(z) = \frac{e^z - 1}{z}$$

and

$$E_{2,2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}.$$

We state the following generalization of Gronwall's inequality.

Lemma 2.4. *([29]) For any $t \in [1,T]$,*

$$u(t) \le a(t) + b(t)\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{u(s)}{s}ds,$$

where all the functions are not negative and continuous. The constant $\alpha > 0$, b is a bounded and monotonic increasing function on $[1,T]$, then,

$$u(t) \le a(t) + \int_1^t \left[\sum_{n=1}^{\infty}\frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\log\frac{t}{s}\right)^{n\alpha-1}a(s)\right]\frac{ds}{s}, \quad t \in [1,T].$$

We adopt the definitions in Rus [28]: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the equation, for the implicit fractional-order differential equation (1.1).

Definition 2.5. The equation (1.1) is **Ulam-Hyers stable** if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J,\mathbb{R})$ of the inequality

$$|{}^H D^\alpha z(t) - f(t, z(t), {}^H D^\alpha z(t))| \le \epsilon, \ t \in J, \tag{2.1}$$

there exists a solution $y \in C^1(J,\mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \le c_f\epsilon, \ t \in J.$$

Definition 2.6. The equation (1.1) is **generalized Ulam-Hyers stable** if there exists $\psi_f \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\psi_f(0) = 0$, such that for each solution $z \in C^1(J,\mathbb{R})$ of the inequality (2.1) there exists a solution $y \in C^1(J,\mathbb{R})$ of the equation (1.1) with

$$|z(t) - y(t)| \le \psi_f(\epsilon), \ t \in J.$$

Definition 2.7. The equation (1.1) is **Ulam-Hyers-Rassias stable** with respect to $\varphi \in C(J,\mathbb{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J,\mathbb{R})$ of the inequality

$$|{}^H D^\alpha z(t) - f(t, z(t), {}^H D^\alpha z(t))| \le \epsilon\varphi(t), \ t \in J, \tag{2.2}$$

there exists a solution $y \in C^1(J,\mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \le c_f\epsilon\varphi(t), \ t \in J.$$

Definition 2.8. The equation (1.1) is **generalized Ulam-Hyers-Rassias stable** with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$|^H D^\alpha z(t) - f(t, z(t), ^H D^\alpha z(t))| \leq \varphi(t), \ t \in J, \tag{2.3}$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \leq c_{f,\varphi}\varphi(t), \ t \in J.$$

Remark 2.9. A function $z \in C^1(J, \mathbb{R})$ is a solution of of the inequality (2.1) if and only if there exists a function $g \in C(J, \mathbb{R})$ (which depend on y) such that

(i) $|g(t)| \leq \epsilon$, $\forall t \in J$.
(ii) $^H D^\alpha z(t) = f(t, z(t), ^H D^\alpha z(t)) + g(t)$, $t \in J$.

Remark 2.10. Clearly,

(i) Definition 2.5 $\Longrightarrow$ Definition 2.6.
(ii) Definition 2.7 $\Longrightarrow$ Definition 2.8.

Remark 2.11. A solution of the implicit differential inequality (2.1) is called an fractional ϵ-solution of the implicit fractional differential equation (1.1).

So, the Ulam stabilities of the implicit differential equations with fractional order are some special types of data dependence of the solutions of fractional implicit differential equations.

3. Existence and uniqueness of solutions

By a solution of the problem (1.1) $-$ (1.2) we mean a function $u \in C^1(J, \mathbb{R})$ satisfying equation (1.1) on J and condition (1.2).

Lemma 3.1. *Let a function $f(t, u, v) : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Then the problem* (1.1) $-$ (1.2) *is equivalent to the problem*

$$y(t) = y_1 +_H I^\alpha g(t), \tag{3.1}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y_1 +_H I^\alpha g(t), g(t)).$$

Proof. If $^H D^\alpha y(t) = g(t)$ then $_H I^\alpha \ ^H D^\alpha y(t) =_H I^\alpha g(t)$. So we obtain

$$y(t) = y_1 +_H I^\alpha g(t).$$

□

Theorem 3.2. *Assume*

(H1) *The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(H2) *There exist constants $k > 0$ and $l > 0$ such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq k|u - \bar{u}| + l|v - \bar{v}| \text{ for any } u, v, \bar{u}, \bar{v} \in \mathbb{R} \text{ and } t \in J.$$

If

$$\frac{k(\log T)^{\alpha}}{\Gamma(\alpha+1)}+l<1, \tag{3.2}$$

then there exists a unique solution for the IVP (1.1) – (1.2) *on* J.

Proof. Define the operator $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by:

$$N(z)(t)=f\left(t, y_1+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}z(s)\frac{ds}{s}, z(t)\right), \text{ for each } t\in J \tag{3.3}$$

Let $u, w \in C(J,\mathbb{R})$. Then for $t \in J$, we have

$$\begin{aligned}|(Nu)(t)-(Nw)(t)| &\leq \frac{k}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|u(s)-w(s)|\frac{ds}{s}\\ &\quad +l|u(t)-w(t)|\\ &\leq \left(\frac{k}{\Gamma(\alpha)}\int_1^t(\log t)^{\alpha-1}\frac{ds}{s}+l\right)||u-w||_{\infty}\\ &\leq \left(\frac{k(\log T)^{\alpha}}{\Gamma(\alpha+1)}+l\right)||u-w||_{\infty}.\end{aligned}$$

Then

$$||Nu-Nw||_{\infty}\leq\left(\frac{k(\log T)^{\alpha}}{\Gamma(\alpha+1)}+l\right)||u-w||_{\infty}. \tag{3.4}$$

By (3.2), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point $z \in C(J,\mathbb{R})$, i.e $z = N(z)$.
Therefore

$$z(t)=f\left(t, y_1+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}z(s)\frac{ds}{s}, z(t)\right), \text{ for each } t\in J$$

Set

$$y(t)=y_1+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}z(s)\frac{ds}{s}.$$

This implies that ${}^{H}D^{\alpha}y(t) = z(t)$ and hence

$${}^{H}D^{\alpha}y(t)=f(t, y(t), {}^{H}D^{\alpha}y(t)), \text{ for each } t\in J.$$

□

4. Ulam-Hyers stability

Theorem 4.1. *Assume that the assumptions (H1), (H2) and* (3.2) *hold. Then the equation* (1.1) *is Ulam-Hyers stable.*

Proof. Let $z \in C(J,\mathbb{R})$ be a solution of the inequality (2.1), i.e.

$$|{}^{H}D^{\alpha}z(t)-f(t, z(t), {}^{H}D^{\alpha}z(t))|\leq\epsilon, \ t\in J. \tag{4.1}$$

Let us denote by $y \in C(J,\mathbb{R})$ the unique solution of the Cauchy problem

$${}^{H}D^{\alpha}y(t)=f(t, y(t), {}^{H}D^{\alpha}y(t)), \text{ for each } t\in J, \ 0<\alpha\leq 1,$$

$$y(1) = z(1).$$

By using Lemma 3.1, we have

$$y(t) = z(1) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_y(s) \frac{ds}{s},$$

where $g_y \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_y(t) = f(t, y(1) +_H I^\alpha g_y(t), g_y(t)).$$

By integration of (4.1) we obtain

$$\begin{aligned} \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_z(s) \frac{ds}{s} \right| &\leq \frac{\epsilon (\log t)^\alpha}{\Gamma(\alpha+1)} \\ &\leq \frac{\epsilon (\log T)^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{4.2}$$

where $g_z \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_z(t) = f(t, z(1) +_H I^\alpha g_z(t), g_z(t)).$$

On the other hand, we have, for each $t \in J$

$$\begin{aligned} |z(t) - y(t)| &= \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_y(s) \frac{ds}{s} \right| \\ &= \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_z(s) \frac{ds}{s} \right. \\ &+ \left. \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (g_z(s) - g_y(s)) \frac{ds}{s} \right| \\ &\leq \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_z(s) \frac{ds}{s} \right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |g_z(s) - g_y(s)| \frac{ds}{s}, \end{aligned} \tag{4.3}$$

where

$$g_y(t) = f(t, y(t), g_y(t)),$$

and

$$g_z(t) = f(t, z(t), g_z(t)).$$

By $(H2)$, we have, for each $t \in J$

$$\begin{aligned} |g_z(t) - g_y(t)| &= |f(t, z(t), g_z(t)) - f(t, y(t), g_y(t))| \\ &\leq k|z(t) - y(t)| + l|g_z(t) - g_y(t)|. \end{aligned}$$

Then

$$|g_z(t) - g_y(t)| \leq \frac{k}{1-l} |z(t) - y(t)|. \tag{4.4}$$

Thus, by (4.2), (4.3), (4.4), and Lemma 2.4 we get

$$\begin{aligned}
|z(t)-y(t)| &\leq \frac{\epsilon(\log T)^\alpha}{\Gamma(\alpha+1)}+\frac{k}{(1-l)\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|z(s)-y(s)|\frac{ds}{s}\\
&\leq \frac{\epsilon(\log T)^\alpha}{\Gamma(\alpha+1)}\\
&\quad+\int_1^t\left[\sum_{n=1}^{\infty}\left(\frac{k}{(1-l)}\right)^n\frac{1}{\Gamma(n\alpha)}\left(\log\frac{t}{s}\right)^{n\alpha-1}\frac{\epsilon(\log T)^\alpha}{\Gamma(\alpha+1)}\right]\frac{ds}{s}\\
&\leq \frac{\epsilon(\log T)^\alpha}{\Gamma(\alpha+1)}\left[1+\sum_{n=1}^{\infty}\left(\frac{k}{1-l}\right)^n\frac{1}{\Gamma(n\alpha)}\frac{(\log T)^{n\alpha}}{n\alpha}\right]\\
&= \frac{\epsilon(\log T)^\alpha}{\Gamma(\alpha+1)}\left[1+\sum_{n=1}^{\infty}\left(\frac{k}{1-l}\right)^n\frac{(\log T)^{n\alpha}}{\Gamma(n\alpha+1)}\right]\\
&= \frac{\epsilon(\log T)^\alpha}{\Gamma(\alpha+1)}\left[1+\sum_{n=1}^{\infty}\frac{\left(\frac{k}{1-l}(\log T)^\alpha\right)^n}{\Gamma(n\alpha+1)}\right]\\
&= \frac{\epsilon(\log T)^\alpha}{\Gamma(\alpha+1)}E_\alpha\left(\frac{k}{1-l}(\log T)^\alpha\right).
\end{aligned}$$

Then, for each $t\in J$

$$|z(t)-y(t)| \leq \frac{\epsilon(\log T)^\alpha}{\Gamma(\alpha+1)}E_\alpha\left(\frac{k}{1-l}(\log T)^\alpha\right) := c_f\epsilon. \tag{4.5}$$

So, the equation (1.1) is Ulam-Hyers stable. This completes the proof. By putting $\psi(\epsilon)=c\epsilon$, $\psi(0)=0$ yields that the equation (1.1) is generalized Ulam-Hyers stable. □

5. Ulam-Hyers-Rassias stability

Theorem 5.1. *Assume (H1), (H2), (3.2) and*

(H3) *The function $\varphi\in C(J,\mathbb{R}_+)$ is increasing and there exists $\lambda_\varphi>0$ such that, for each $t\in J$, we have*

$$_HI^\alpha\varphi(t)\leq\lambda_\varphi\varphi(t).$$

Then the equation (1.1) is Ulam-Hyers-Rassias stable with respect to φ.

Proof. Let $z\in C(J,\mathbb{R})$ be a solution of the inequality (2.2), i.e.

$$|^HD^\alpha z(t)-f(t,z(t),^HD^\alpha z(t))|\leq\epsilon\varphi(t),\ t\in J,\ \epsilon>0. \tag{5.1}$$

Let us denote by $y\in C(J,\mathbb{R})$ the unique solution of the Cauchy problem

$$^HD^\alpha y(t)=f(t,y(t),^HD^\alpha y(t)),\ \text{for each},\ t\in J,\ 0<\alpha\leq1,$$

$$y(1)=z(1).$$

By using Lemma 3.1, we have

$$y(t)=z(1)+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_y(s)\frac{ds}{s},$$

where $g_y\in C(J,\mathbb{R})$ satisfies the functional equation

$$g_y(t)=f(t,y(1)+_H I^{\alpha}g_y(t),g_y(t)).$$

By integration of (5.1) and from (H3), we obtain

$$\begin{aligned}\left|z(t)-z(1)-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_z(s)\frac{ds}{s}\right| &\leq \frac{\epsilon}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\varphi(s)\frac{ds}{s}\\ &\leq \epsilon\lambda_{\varphi}\varphi(t), \qquad (5.2)\end{aligned}$$

where $g_z\in C(J,\mathbb{R})$ satisfies the functional equation

$$g_z(t)=f(t,z(1)+I^{\alpha}g_z(t),g_z(t)).$$

On the other hand, we have, for each $t\in J$

$$\begin{aligned}|z(t)-y(t)| &= \left|z(t)-z(1)-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_y(s)\frac{ds}{s}\right|\\ &= \left|z(t)-z(1)-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_z(s)\frac{ds}{s}\right.\\ &+ \left.\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(g_z(s)-g_y(s))\frac{ds}{s}\right|\\ &\leq \left|z(t)-z(1)-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_z(s)\frac{ds}{s}\right|\\ &+ \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|g_z(s)-g_y(s)|\frac{ds}{s}, \qquad (5.3)\end{aligned}$$

where

$$g_y(t)=f(t,y(t),g_y(t)),$$

and

$$g_z(t)=f(t,z(t),g_z(t)).$$

Then, by (4.4), (5.2), and (5.3)

$$\begin{aligned}|z(t)-y(t)| &\leq \epsilon\lambda_{\varphi}\varphi(t)+\frac{k}{(1-l)\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|z(s)-y(s)|\frac{ds}{s}\\ &\leq \epsilon\lambda_{\varphi}\varphi(t)+\frac{k\|z-y\|_{\infty}}{(1-l)\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\ &\leq \epsilon\lambda_{\varphi}\varphi(t)+\frac{k\|z-y\|_{\infty}}{(1-l)\Gamma(\alpha)}\frac{(\log T)^{\alpha}}{\alpha}.\end{aligned}$$

Thus,

$$||z-y||_\infty\left[1-\frac{k(\log T)^\alpha}{(1-l)\Gamma(\alpha+1)}\right] \quad \leq \quad \epsilon\lambda_\varphi\varphi(t).$$

We obtain, by (3.2)

$$||z-y||_\infty \quad \leq \quad \frac{\epsilon\lambda_\varphi\varphi(t)}{\left[1-\frac{k(\log T)^\alpha}{(1-l)\Gamma(\alpha+1)}\right]}.$$

Then, for each $t\in J$

$$|z(t)-y(t)| \quad \leq \quad \left[1-\frac{k(\log T)^\alpha}{(1-l)\Gamma(\alpha+1)}\right]^{-1}\lambda_\varphi\epsilon\varphi(t) := c_f\epsilon\varphi(t). \tag{5.4}$$

So, the equation (1.1) is Ulam-Hyers-Rassias stable. □

6. Examples

Example 6.1. Consider the following Cauchy problem

$${}^H D^{\frac{1}{2}}y(t)=\frac{1}{200}(t\sin y(t)-y(t)\cos(t))+\frac{1}{100}\sin {}^H D^{\frac{1}{2}}y(t),\ \text{for each } t\in[1,e], \tag{6.1}$$

$$y(1)=1. \tag{6.2}$$

Set

$$f(t,u,v)=\frac{1}{200}(t\sin u-u\cos(t))+\frac{1}{100}\sin v,\quad t\in[1,e],\ u,v\in\mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in[1,e]$:

$$\begin{aligned}|f(t,u,v)-f(t,\bar{u},\bar{v})| &\leq \frac{1}{200}|t||\sin u-\sin\bar{u}|+\frac{1}{200}|\cos t||u-\bar{u}|\\ &\quad+\frac{1}{100}|\sin v-\sin\bar{v}|\\ &\leq \frac{e}{200}|u-\bar{u}|+\frac{1}{200}|u-\bar{u}|+\frac{1}{100}|v-\bar{v}|\\ &= \frac{e+1}{200}|u-\bar{u}|+\frac{1}{100}|v-\bar{v}|.\end{aligned}$$

Hence condition (H2) is satisfied with $k=\frac{e+1}{200}$ and $l=\frac{1}{100}$.
Thus condition

$$\frac{k(\log T)^\alpha}{(1-l)\Gamma(\alpha+1)}=\frac{\frac{e+1}{200}}{(1-\frac{1}{100})\Gamma(\frac{3}{2})}=\frac{e+1}{99\sqrt{\pi}}<1,$$

is satisfied. It follows from Theorem 3.2 that the problem (6.1)-(6.2) as a unique solution, and from Theorem 4.1 the equation (6.1) is Ulam-Hyers stable.

Example 6.2. Consider the following Cauchy problem

$$^{H}D^{\frac{1}{2}}y(t) = \frac{2+|y(t)|+|^{H}D^{\frac{1}{2}}y(t)|}{120e^{t+10}(1+|y(t)|+|^{H}D^{\frac{1}{2}}y(t)|)}, \text{ for each } t\in[1,e], \tag{6.3}$$

$$y(1)=1. \tag{6.4}$$

Set

$$f(t,u,v)=\frac{2+|u|+|v|}{120e^{t+10}(1+|u|+|v|)}, \quad t\in[1,e], \ u,v\in\mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in[1,e]$

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\le\frac{1}{120e^{10}}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K=L=\frac{1}{120e^{10}}$.
Let $\varphi(t)=(\log t)^{\frac{1}{2}}$. We have

$$\begin{aligned} {}_{H}I^{\alpha}\varphi(t) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^t\left(\log\frac{t}{s}\right)^{\frac{1}{2}-1}(\log t)^{\frac{1}{2}}\frac{ds}{s} \\ &\le \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^t\left(\log\frac{t}{s}\right)^{\frac{1}{2}-1}\frac{ds}{s} \\ &= \frac{2\omega(t)}{\sqrt{\pi}}. \end{aligned}$$

Thus

$${}_{H}I^{\alpha}\varphi(t)\le\frac{2}{\sqrt{\pi}}(\log t)^{\frac{1}{2}}:=\lambda_{\varphi}\varphi(t).$$

Thus condition (H3) is satisfied with $\varphi(t)=(\log t)^{\frac{1}{2}}$ and $\lambda_{\varphi}=\frac{2}{\sqrt{\pi}}$ It follows from Theorem 3.2 that the problem (6.3)-(6.4) as a unique solution on J, and from Theorem 5.1 the equation (6.3) is Ulam-Hyers-Rassias stable.

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