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On eigenvalue problems governed by the (p, q)-Laplacian

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Dedicated to the memory of Professor Csaba Varga

Abstract. This is a survey on recent results, mostly of the authors, regarding eigenvalue problems governed by the $(p, q)-$Laplacian and related open problems.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N,\ N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. For $\theta \in (1, \infty)$, consider in Ω the θ-Laplace operator $\Delta_\theta u =$ div $(\mid \nabla u \mid^{\theta-2} \nabla u)$. Obviously, Δ_2 is the classic Laplacian Δ. There are many applications involving such kind of operators, including the so called two phase problems. For example, the operator $\big(\Delta + c\Delta_\theta\big),\ c > 0,\ \theta \in (1, \infty)$, has applications in Born-Infeld theory for electrostatic fields (see Bonheure, Colasuonno & Fortunato [16], Fortunato, Orsina & Pisani [26]). We also refer to Benci et al. [14] and Benci, Fortunato & Pisani [15] for more general applications to quantum physics. Two phase equations arise also in other parts of mathematical physics as reaction diffusion equations (see Cherfils & Il'yasov [18]) and nonlinear elasticity theory (see Marcellini [35] and Zhikov [45]). In fact, the literature related to this subject is vast and daily increasing.

For $p,\ q \in (1, \infty)$, define $\mathcal{A}_{pq} := \Delta_p + \Delta_q$, which is usually called $(p, q)-$Laplacian. We assumes that $p \neq q$, because for $p = q$ $\mathcal{A}_{pq} = 2\Delta_p$ and this case is not relevant for our discussion here. Notice that the operator introduced above $\big(\Delta + c\Delta_\theta\big)$ with $c = 1$ is a $(2, \theta)-$Laplacian. The restriction to the case $c = 1$ does not affect the generality.

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In what follows we recall some facts concerning the classic eigenvalue problem for $-\Delta_p$, $p \in (1, \infty)$, under the Dirichlet boundary condition

$$\begin{cases} -\Delta_p u = \lambda \mid u \mid^{p-2} u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

A real number λ is called an *eigenvalue* of problem (1.1) if this problem admits a nontrivial weak solution, i.e. there exists $u_\lambda \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \mid \nabla u_\lambda \mid^{p-2} \nabla u_\lambda \cdot \nabla w \, dx = \lambda \int_\Omega \mid u_\lambda \mid^{p-2} u_\lambda w \, dx \ \forall \ w \in W_0^{1,p}(\Omega). \tag{1.2}$$

The nontrivial solutions u_λ of problem (1.1) are called *eigenfunctions* corresponding to the eigenvalue λ, and (λ, u_λ) are called *eigenpairs* of problem (1.1).

A standard method to show the existence of an increasing sequence of eigenvalues for problem (1.1),

$$0 < \lambda_1^D < \lambda_2^D \leq \lambda_3^D \leq \cdots \to \infty, \tag{1.3}$$

relies on the Ljusternik-Schnirelmann principle and on the concept of Krasnosel'skĭi genus. There are also other methods to prove the existence of such a sequence (see García-Azorero & Peral [28], Drábek & Robinson [23]). It is still not known whether this sequence includes all eigenvalues of problem (1.1), except for the well-known particular case $p = 2$.

On the other hand, it is well-known that $-\Delta_p$ with the Dirichlet boundary condition admits a lowest positive eigenvalue λ_1 (called *principal eigenvalue*), which is simple, and there exists a corresponding eigenfunction which is positive in Ω (see Lindqvist [34], Lê [33] and the references therein). Note also that the properties of the next lowest eigenvalue λ_2 have been investigated by Anane & Tsouli in [2], who proved that λ_2 has a variational characterization similar to that corresponding to the linear case $p = 2$.

Similar situations can be reported in the case of Neumann, Robin or Steklov boundary conditions.

2. Eigenvalue problems governed by the $(p, q)-$Laplacian

In this section we shall present some recent results on eigenvalue problems involving the $(p, q)-$Laplacian with various boundary conditions. More precisely, these results contain information regarding the corresponding eigenvalue sets. As seen below, the fact that the differential operator $\mathcal{A}_{pq}$ is *non-homogeneous* (i.e., $p \neq q$) implies that the eigenvalue sets are intervals or contain intervals. Throughout this section we will assume that $p, q \in (1, \infty)$, $p \neq q$, and introduce the following notations:

$$\begin{aligned} W &:= W^{1,\max\{p,q\}}(\Omega), \\ \frac{\partial u}{\partial \nu_{pq}} &:= \big(\mid \nabla u \mid^{p-2} + \mid \nabla u \mid^{q-2} \big) \frac{\partial u}{\partial \nu}, \end{aligned} \tag{2.1}$$

where ν is the outward unit normal to $\partial\Omega$.

2.1. The case of Dirichlet, Neumann, Robin or Steklov boundary conditions

Let us begin with the case of the *Dirichlet boundary condition.* Specifically, we consider the problem

$$\begin{cases} -\mathcal{A}_{pq}u = \lambda \mid u \mid^{p-2} u \ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

The definitions of eigenvalues, eigenfunctions and eigenpairs for problem (2.2) are similar to those corresponding to problem (1.1), the only differences being the following: the left hand side of equation (1.2) is replaced by

$$\int_\Omega \Big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} \Big) \nabla u_\lambda \cdot \nabla w \, dx,$$

and the Sobolev space in which the weak solution is sought is now $W_0^{1,\max\{p,q\}}(\Omega)$.

The existence of eigenvalues for this problem in the case when the right hand side of equation $(2.2)_1$ is of the form $\lambda m_p(x) \mid u \mid^{p-2} u$ in Ω, where $m_p \in L^\infty(\Omega)$ such that the Lebesgue measure of $\{x \in \Omega;\ m_p(x) > 0\}$ is positive, was studied by Tanaka in [42]. Using the Mountain Pass Theorem, Tanaka was able to obtain the full eigenvalue set ([42, Theorem 1, Theorem 2]). In the particular case $m_p \equiv 1$, Tanaka's result is the following:

Theorem 2.1. *If $p, q \in (1, \infty)$, $p \neq q$, then the set of eigenvalues of problem* (2.2) *is precisely (λ_1^D, ∞), where λ_1^D denotes the first eigenvalue of the negative Dirichlet $p-$Laplacian, more exactly*

$$\lambda_1^D := \inf\left\{ \frac{\int_\Omega \mid \nabla u \mid^p dx}{\int_\Omega \mid u \mid^p dx},\ u \in W_0^{1,p}(\Omega) \right\}. \tag{2.3}$$

Notice that the eigenvalue set of $-\mathcal{A}_{pq}$ with Dirichlet boundary condition has been completely determined, being an interval independent of q.

Next, let us consider the case of a generalized *Neumann boundary condition.* More precisely, consider the eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq}u = \lambda \mid u \mid^{q-2} u \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.4}$$

The solution u of problem (2.4) is understood in a weak sense, as an element of the Sobolev space W satisfying equation $(2.4)_1$ in the sense of distributions and $(2.4)_2$ in the sense of traces. The scalar $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2.4) if there exists $u_\lambda \in W \setminus \{0\}$ such that for all $w \in W$ we have

$$\int_\Omega \Big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} \Big) \nabla u_\lambda \cdot \nabla w \, dx = \lambda \int_\Omega \mid u_\lambda \mid^{q-2} u_\lambda w \, dx. \tag{2.5}$$

Problem (2.4) was investigated by Mihăilescu [36, Theorem 1.1] (for $q = 2$, $p \in (2, \infty)$), Fărcăşeanu, Mihăilescu & Stancu-Dumitru [24, Theorem 1.1] (for $q = 2$, $p \in (1, 2)$), Mihăilescu & Moroşanu [37, Theorem 1.1] (for $q \in (2, \infty), p \in (1, \infty)$, $p \neq q$) and Barbu & Moroşanu [7, Theorem 1] (for $q \in (1, 2)$, $p \in (1, \infty)$, $p \neq q$).

To investigate such a problem, one can use techniques based on minimization arguments, which will be briefly described in what follows.

To begin with, let us choose $w = u_\lambda$ in (2.5). Clearly, we see that the eigenvalues of problem (2.4) cannot be negative. It is also obvious that $\lambda_0 = 0$ is an eigenvalue of this problem with the corresponding eigenfunctions given by the nonzero constant functions.

Now, if we assume that $\lambda > 0$ is an eigenvalue of problem (2.4) and choose $w \equiv 1$ in (2.5) we obtain that every eigenfunction u_λ corresponding to λ necessarily belong to the set

$$\mathcal{C}_{Ne} := \Big\{ u \in W;\ \int_\Omega |\, u\,|^{q-2}\, u\ dx = 0 \Big\}. \tag{2.6}$$

This is a symmetric cone. Moreover, $\mathcal{C}_{Ne}$ is a weakly closed subset of W and $\mathcal{C}_{Ne}\setminus\{0\} \neq \emptyset$ (see [6, Section 2]).

Next, we shall briefly describe the method we can use to solve the eigenvalue problem (2.4).

For $\lambda > 0$ consider the C^1 functional $\mathcal{J}_\lambda : W \to \mathbb{R}$, defined as

$$\mathcal{J}_\lambda(u) = \frac{1}{p}\int_\Omega |\, \nabla u\,|^p\ \ dx + \frac{1}{q}\int_\Omega |\, \nabla u\,|^q\ \ dx - \frac{\lambda}{q}\int_\Omega |\, u\,|^q\ \ dx. \tag{2.7}$$

This functional is often called the *energy functional* associated to problem (2.4). Clearly, λ is an eigenvalue of problem (2.4) if and only if there exists a critical point $u_\lambda \in W \setminus \{0\}$ of $\mathcal{J}_\lambda$, i. e. $\mathcal{J}_\lambda'(u_\lambda) = 0$.

Define

$$\widetilde{\lambda}^{Ne} := \inf_{w \in \mathcal{C}_{Ne}\setminus\{0\}} \frac{\int_\Omega |\, \nabla w\,|^q\ \ dx}{\int_\Omega |\, w\,|^q\ \ dx}. \tag{2.8}$$

Since $\widetilde{\lambda}^{Ne} = \lambda_1^{Ne_q}$ for $q > p$ and $\widetilde{\lambda}^{Ne} \geq \lambda_1^{Ne_q}$ for $q < p$, it follows that $\widetilde{\lambda}^{Ne} > 0$ (we have denoted by $\lambda_1^{Ne_q}$ the first positive eigenvalue of the negative Neumann $q-$Laplace operator).

Also, one can easily check that there is no eigenvalue of problem (2.4) in the set $(-\infty, \widetilde{\lambda}^{Ne}] \setminus \{0\}$. So, from now on we shall consider that λ is arbitrary but fixed in the interval $(\widetilde{\lambda}^{Ne}, \infty)$.

We distinguish two cases related to p and q:

Case 1: $1 < q < p$. In this case, as $\lambda > \widetilde{\lambda}^{Ne}$, the functional $\mathcal{J}_\lambda$ is coercive on $\mathcal{C}_{Ne} \subset W = W^{1,p}(\Omega)$, i.e.,

$$\lim_{\|u\|_{W^{1,p}(\Omega)} \to \infty, u \in \mathcal{C}_{Ne}} \mathcal{J}_\lambda(u) = \infty.$$

In particular, there exists $u_* \in \mathcal{C}_{Ne} \setminus \{0\}$ where $\mathcal{J}_\lambda$ attains its minimal value over $\mathcal{C}_{Ne}$,

$$J_\lambda(u_*) = \inf_{w \in \mathcal{C}_{Ne}\setminus\{0\}} \mathcal{J}_\lambda(w) \neq 0$$

(see [7, Lemma 6]).

Case 2: $1 < p < q$. Under this assumption, the functional $\mathcal{J}_\lambda$ is no longer coercive and may be unbounded below on $W = W^{1,q}(\Omega)$. So, we consider the restriction of functional $\mathcal{J}_\lambda$ to the Nehari type manifold (see [41]):

$$\mathcal{N}_\lambda = \{v \in \mathcal{C}_{Ne} \setminus \{0\}; \langle \mathcal{J}'_\lambda(v), v\rangle = 0\}.$$

We observe that

$$\mathcal{J}_\lambda(u) = \frac{q-p}{qp}\int_\Omega |\nabla u|^p \; dx > 0 \;\forall\; u \in \mathcal{N}_\lambda.$$

Moreover, any possible eigenfunction corresponding to λ belongs to $\mathcal{N}_\lambda$.

In addition, since $\lambda > \widetilde{\lambda}^{Ne}$, we can easily check that $\mathcal{N}_\lambda \neq \emptyset$.

In this case we have the following result (see [6, Case 2, Steps 1-4] and [7, Lemma 6]):

If $1 < p < q$ and $\lambda > \widetilde{\lambda}^{Ne}$, then there exists $u_* \in \mathcal{N}_\lambda$ where $\mathcal{J}_\lambda$ attains its minimal value over $\mathcal{N}_\lambda$,

$$m_\lambda := \inf_{w\in\mathcal{N}_\lambda} \mathcal{J}_\lambda(w) > 0.$$

Using the above preliminary results and applying the Lagrange Multipliers Rule in the case $q \geq 2$ and, respectively, an approximation technique in the case $1 < q < 2$, one can show that in fact the minimizer u_* of functional $\mathcal{J}_\lambda$ over $\mathcal{C}_{Ne}$ if $q < p$ and, respectively, over $\mathcal{N}_\lambda$ if $q > p$, is a global minimizer of $\mathcal{J}_\lambda$ over the whole W, i.e. u_* is an eigenfunction of problem (2.4) corresponding to the eigenvalue $\lambda > \widetilde{\lambda}^{Ne}$.

Thus, we have the following important result which provides the full spectrum of the eigenvalue problem (2.4):

Theorem 2.2. *Assume that* $p, q \in (1,\infty)$, $p \neq q$. *Then the set of eigenvalues of problem* (2.4) *is precisely* $\{0\} \cup (\widetilde{\lambda}^{Ne}, \infty)$, *where* $\widetilde{\lambda}^{Ne}$ *is the positive constant defined by* (2.8).

Now, consider the eigenvalue problem for the *Steklov* $(p,q)-$*Laplacian*, namely

$$\begin{cases} \mathcal{A}_{pq}u = 0 \;\text{ in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} = \lambda \mid u \mid^{q-2} u \text{ on } \partial\Omega. \end{cases} \tag{2.9}$$

Using an approach similar to that used before for the Neumann $(p,q)-$Laplacian, one can determine the full spectrum of the eigenvalue problem (2.9). More exactly, if we denote

$$\mathcal{C}_S := \Big\{u \in W;\; \int_{\partial\Omega} |u_\lambda|^{q-2} u_\lambda \; d\sigma = 0\Big\}, \tag{2.10}$$

$$\widetilde{\lambda}^S := \inf_{w\in\mathcal{C}_S\setminus\{0\}} \frac{\int_\Omega |\nabla w|^q \; dx}{\int_{\partial\Omega} |w|^q \; d\sigma}, \tag{2.11}$$

we have the following result

Theorem 2.3. *Assume that* $p, q \in (1,\infty)$, $p \neq q$. *Then the set of eigenvalues of problem* (2.9) *is precisely* $\{0\} \cup (\widetilde{\lambda}_S, \infty)$, *where* $\widetilde{\lambda}_S$ *is the positive constant defined by* (2.11).

This theorem was proved by Costea & Moroşanu [19, Theorem 3.1] in the case $p \in (1,\infty)$, $q \in [2,\infty)$, $p \neq q$ and later by Barbu & Moroşanu [7, Theorem 1] in the case $p \in (1,\infty)$, $q \in (1,2)$, $p \neq q$.

Next, we pay attention to equation $(2.4)_1$ with a generalized *Robin boundary condition.* More precisely, we consider the following eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq}u = \lambda \mid u \mid^{q-2} u \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} + \beta \mid u \mid^{q-2} u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.12}$$

where β is a positive constant.

The eigenvalue problem (2.12) was studied by Gyulov & Moroşanu [30], who found an interval of eigenvalues for this problem. In order to state the main result in [30], we define

$$\begin{aligned} \widetilde{\lambda}^R &:= \inf_{w\in W\setminus\{0\}} \frac{\int_\Omega \mid \nabla w \mid^q \ dx + \beta \int_{\partial\Omega} \mid \nabla w \mid^q \ d\sigma}{\int_\Omega \mid w \mid^q \ dx}, \\ \lambda_0 &:= \beta \frac{\mid \partial\Omega \mid_{N-1}}{\mid \Omega \mid_N}, \end{aligned} \tag{2.13}$$

where $| \cdot |_N$ and $| \cdot |_{N-1}$ denote the Lebesgue measures of the two sets. Obviously, the constant $\widetilde{\lambda}_R$ coincides with the first eigenvalue of the Robin $q-$Laplace operator (see Lê [33]) in the case $q > p$ and is greater than or equal to that if $q < p$, so it is positive.

The results concerning the spectrum of problem (2.12) can be summarized as follows:

Theorem 2.4. *Assume that* $p, q \in (1, \infty)$, $p \neq q$ *and* β *is a positive constant. Then* $\widetilde{\lambda}^R < \lambda_0$ *and any* $\lambda \in (\widetilde{\lambda}_R, \lambda_0)$ *is an eigenvalue of problem* (2.12). *Moreover, the problem* (2.12) *has no nontrivial solution for* $\lambda \in (-\infty, \widetilde{\lambda}^R]$.

Note that this theorem does not say whether there are eigenvalues of problem (2.12) in the interval $[\lambda_0, \infty)$. On the other hand, we know that there exists a sequence of eigenvalues of problem (2.12) which converges to ∞ (see [5]). However, the full spectrum of problem (2.12) is still not completely known.

We also mention the paper by Papageorgiou, Vetro & Vetro [38] where an eigenvalue problem more general than (2.12) is considered in the case $1 < p < q$. Here the operator $\mathcal{A}_{pq}$ is perturbed with an indefinite and unbounded potential, $\zeta \in L^s(\Omega)$, $s < N/q$ if $q \leq N$ and $s = 1$ if $q > N$. The constant β is replaced by a function $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta \geq 0$, $\beta \not\equiv 0$ such that

$$\int_\Omega \zeta \ dx + \int_{\partial\Omega} \beta \ d\sigma > 0. \tag{2.14}$$

By arguing as in [30], the authors obtain a result similar to Theorem 2.4 (see [38, Theorem 1]).

Finally, let us consider the Steklov like eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq}u + \rho_1(x) \mid u \mid^{p-2} u + \rho_2(x) \mid u \mid^{q-2} u = 0, \ \ x \in \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} + \gamma_1(x) \mid u \mid^{p-2} u + \gamma_2(x) \mid u \mid^{q-2} u = \lambda \mid u \mid^{q-2} u, \ \ x \in \partial\Omega. \end{cases} \tag{2.15}$$

Assume that the following hypotheses are fulfilled:

$(h_{\rho_1\gamma_1})$ $\rho_1 \in L^\infty(\Omega)$ and $\gamma_1 \in L^\infty(\partial\Omega)$, $\rho_1,\ \gamma_1$ are nonnegative functions such that

$$\int_\Omega \rho_1\ dx + \int_{\partial\Omega} \gamma_1\ d\sigma > 0; \tag{2.16}$$

$(h_{\rho_2\gamma_2})$ $\rho_2 \in L^\infty(\Omega)$, $\gamma_2 \in L^\infty(\partial\Omega)$ and ρ_2 is a nonnegative function.

It is worth pointing out that the potential function γ_2 is allowed to be sign changing.

As usual, a scalar $\lambda \in \mathbb{R}$ is said to be an eigenvalue of the problem (2.15) if there exists $u_\lambda \in W \setminus \{0\}$ such that for all $w \in W$

$$\begin{aligned}
&\int_\Omega \big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2}\big)\nabla u_\lambda \cdot \nabla w\ dx \\
&\quad + \int_\Omega \big(\rho_1 \mid u_\lambda \mid^{p-2} + \rho_2 \mid u_\lambda \mid^{q-2}\big) u_\lambda w\ dx \\
&\quad + \int_{\partial\Omega} \big(\gamma_1 \mid u_\lambda \mid^{p-2} + \gamma_2 \mid u_\lambda \mid^{q-2}\big) u_\lambda w\ d\sigma = \lambda \int_{\partial\Omega} \mid u_\lambda \mid^{q-2} u_\lambda w\ d\sigma.
\end{aligned} \tag{2.17}$$

The function u_λ is called an eigenfunction of the problem (2.15) (corresponding to the eigenvalue λ).

Define

$$\widetilde{\lambda}^{SR} := \inf_{w \in W\setminus\{0\}} \frac{\int_\Omega \big(\mid \nabla w \mid^q + \rho_2 \mid w \mid^q\big) dx + \int_{\partial\Omega} \gamma_2 \mid w \mid^q d\sigma}{\int_{\partial\Omega} \mid w \mid^q d\sigma}. \tag{2.18}$$

Problem (2.15) was studied by Barbu & Moroşanu [11]. Let us recall the main result on its eigenvalue set:

Theorem 2.5. ([11, Theorem 1]) *Assume that $p, q \in (1,\infty)$, $p \neq q$ and assumptions $(h_{\rho_i\gamma_i})$, $i = 1,2$, are fulfilled. Then the set of eigenvalues of problem* (2.15) *is precisely* $(\widetilde{\lambda}^{SR}, \infty)$.

Note that if $\gamma_1 \equiv 0$ and $\gamma_2 \equiv$ const. > 0, then we have a Steklov-Robin boundary condition. The arguments we have used in the mentioned paper can easily be adapted to the following eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq}u + \rho_1(x) \mid u \mid^{p-2} u + \rho_2(x) \mid u \mid^{q-2} u = \lambda \mid u \mid^{q-2} u, & x \in \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} + \gamma_1(x) \mid u \mid^{p-2} u + \gamma_2(x) \mid u \mid^{q-2} u = 0, & x \in \partial\Omega, \end{cases} \tag{2.19}$$

under similar assumptions for the functions $\rho_i,\ \gamma_i,\ i = 1,2$. While in the previous works [30] and [38] only subsets of the corresponding spectra were found, in this case the presence of the potential functions $\rho_i,\ \gamma_i$ satisfying assumptions $(h_{\rho_i\gamma_i})$, $i = 1,2$, allows the full description of the spectrum.

2.2. The case of parametric boundary conditions

Consider the following eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq}u = \lambda\alpha(x) \mid u \mid^{r-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} = \lambda\beta(x) \mid u \mid^{r-2} u & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

under the following hypotheses

(h_{pqr}) $p, q, r \in (1, \infty),\ p \neq q$;

$(h_{\alpha\beta})$ $\alpha \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$ are given nonnegative functions satisfying

$$\int_\Omega \alpha\ dx + \int_{\partial\Omega} \beta\ d\sigma > 0. \tag{2.21}$$

Such eigenvalue problems were discussed for the first time by Von Below & François [43] (see also François [27]) who considered the linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda \beta u & \text{on } \partial\Omega. \end{cases}$$

They call it a *dynamical eigenvalue problem* since it can be derived from the study of the heat equation with dynamical boundary conditions. Also, the motivation behind problem (2.20) comes from the study of a double phase parabolic equation (see Arora & Shmarev [3], Huang [31], Marcellini [35] and the references therein) under a dynamical boundary condition. The existence theory for such parabolic problems relies on the spectral theory of associated elliptic problems with the parameter λ both in the equation and the boundary condition.

The eigenvalues and eigenfunctions of problem (2.20) can be defined as before. All eigenfunctions of problem (2.20) belong to the set

$$\mathcal{C}_r := \Big\{u \in W;\ \int_\Omega \alpha \mid u \mid^{r-2} u\ dx + \int_{\partial\Omega} \beta \mid u \mid^{r-2} u\ d\sigma = 0\Big\}. \tag{2.22}$$

In the case $r = q$, define

$$\widetilde{\lambda} := \inf_{w \in \mathcal{C}_q \setminus \{0\}} \frac{\int_\Omega \mid \nabla w \mid^q\ dx}{\int_\Omega \alpha \mid w \mid^q\ dx + \int_{\partial\Omega} \beta \mid w \mid^q\ d\sigma}. \tag{2.23}$$

If $r \neq q$ we assume, without any loss of generality, that $1 < p < q$ and for $r \in (p, q)$ define

$$\lambda_* := \inf_{v \in \mathcal{C}_r \setminus \mathcal{Z}_r} \Gamma \frac{K_q(v)^{1-\gamma} K_p(v)^\gamma}{\mathcal{K}_r(v)},\ \lambda^* := \frac{r}{q^{1-\gamma} p^\gamma} \lambda_*, \tag{2.24}$$

where

$$\begin{aligned} \mathcal{Z}_r &:= \{v \in W;\ \int_\Omega \alpha \mid v \mid^r dx + \int_{\partial\Omega} \beta \mid v \mid^r d\sigma = 0\}, \\ K_p(u) &:= \int_\Omega \mid \nabla u \mid^p dx,\ K_q(u) := \int_\Omega \mid \nabla u \mid^q dx, \\ \mathcal{K}_r(u) &:= \int_\Omega \alpha \mid u \mid^r dx + \int_{\partial\Omega} \beta \mid u \mid^r d\sigma\ \forall\ u \in W = W^{1,q}(\Omega), \\ \gamma &:= \frac{q - r}{q - p},\ \Gamma := \frac{q - p}{(r - p)^{1-\gamma}(q - r)^\gamma}. \end{aligned} \tag{2.25}$$

In the case $r = q$ we have obtained the following result:

Theorem 2.6. ([7, Theorem 1]) *Assume that $p, q \in (1, \infty),\ p \neq q,\ r = q$ and $(h_{\alpha\beta})$ holds. Then $\widetilde{\lambda} > 0$ and the set of eigenvalues of problem* (2.20) *(with $r = q$) is precisely $\{0\} \cup (\widetilde{\lambda}, \infty)$, where $\widetilde{\lambda}$ is the constant defined by* (2.23).

Note that problem (2.20) in the case $q = 2$ and $p \in (1,\infty)$, $p \neq 2$, has been previously studied by Abreu & Madeira[1].

In the case $r \notin \{p,q\}$, we have the following result:

Theorem 2.7. ([8, Theorem 1.1], [10, Theorem 1]) *Suppose that assumption* $(h_{\alpha\beta})$ *holds.*
(a) If either $(1 < r < p < q < \infty)$ *or* $(1 < q < p < r < \infty$ *and* $r \in \Big(1, \frac{q(N-1)}{N-q}\Big)$ *if* $1 < q < N)$, *then the set of eigenvalues of problem* (2.20) *is* $[0,\infty)$.
(b) If $1 < p < r < q < \infty$, *with* $r < \frac{q(N-1)}{N-q}$ *if* $q < N$, *then* $0 < \lambda_* < \lambda^*$ *and for* $\lambda \in \{0\} \cup [\lambda^*,\infty)$ *there exists a weak solution* $u_\lambda \in W^{1,p}(\Omega) \setminus \{0\}$ *to problem* (2.20). *For any* $\lambda \in (-\infty,\lambda_*) \setminus \{0\}$ *problem* (2.20) *has only the trivial solution. Moreover, the constants* λ_*, λ^* *can be expressed as follows*

$$\lambda_* = \inf_{v \in \mathcal{C}_r \setminus \mathcal{Z}_r} \frac{K_p(v) + K_q(v)}{\mathcal{K}_r(v)}, \ \lambda^* = \inf_{v \in \mathcal{C}_r \setminus \mathcal{Z}} \frac{\frac{1}{p}K_p(v) + \frac{1}{q}K_q(v)}{\frac{1}{r}\mathcal{K}_r(v)}. \tag{2.26}$$

Thus, we were able to find the full eigenvalue sets in two of the three possible cases. The difficult case is $r \in (p,q)$, for which the eigenvalue set is not completely known.

Now, let us pay attention to the following eigenvalue problem governed by the $(p,q,r)-$Laplacian, which is defined by $\mathcal{A}_{pqr}u := \Delta_p u + \Delta_q u + \Delta_r u$,

$$\begin{cases} -\mathcal{A}_{pqr} = \lambda\alpha(x) \mid u \mid^{r-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{pqr}} = \lambda\beta(x) \mid u \mid^{r-2} u & \text{on } \partial\Omega, \end{cases} \tag{2.27}$$

under the assumption $(h_{\alpha\beta})$ above and

$(h_{pqr})'$ $p,q,r \in (1,+\infty)$, $q < p$, $r \notin \{p,q\}$.

In the boundary condition $(2.27)_2$, $\frac{\partial u}{\partial \nu_{pqr}}$ denotes the conormal derivative corresponding to the differential operator $\mathcal{A}_{pqr}$, i.e.,

$$\frac{\partial u}{\partial \nu_{pqr}} := \Big(\sum_{\alpha \in \{p,q,r\}} \mid \nabla u \mid^{\alpha-2} \Big) \frac{\partial u}{\partial \nu}.$$

where ν is the outward unit normal to $\partial\Omega$.

Such a triple-phase eigenvalue problem is motivated by some models arising in mathematical physics. More exactly, let us consider the operator

$$Qu := -\text{div}\Big(\frac{\nabla u}{\sqrt{1- \mid \nabla u \mid^2}} \Big).$$

This operator occurs in the electrostatic Born-Infeld equation (see [16]), in string theory, in particular in the study of D-branes (see, e.g., [29]), and in classical relativity, where Q represents the mean curvature operator in Lorent-Minkowski space (see, e.g., [12] and [17]). A second order approximation of Q is $\mathcal{B} := -\triangle u - \triangle_4 u - \frac{3}{2}\triangle_6 u$, which is a negative $(2,4,6)$-Laplacian (see [40]), with the coefficient $-3/2$ instead of -1.

In fact, one can consider a more general eigenvalue problem, with

$$\mathcal{B}u := \Delta_p u + \rho_q \Delta_q u + \rho_r \Delta_r u, \ \ \rho_q, \, \rho_r > 0,$$

instead of $\mathcal{A}_{pqr}$, and with

$$\frac{\partial u}{\partial \nu_{\mathcal{B}}} := \Big(\sum_{\alpha \in \{p,q,r\}} \rho_\alpha \mid \nabla u \mid^{\alpha - 2} \Big) \frac{\partial u}{\partial \nu}, \ \rho_p = 1,$$

instead of $\frac{\partial u}{\partial \nu_{pqr}}$ (see [9, Section 4]).

Under assumption $(h_{pqr})'$, the appropriate Sobolev space for problem (2.27) is $\widetilde{W} := W^{1,\max\{p,r\}}(\Omega)$. One can define the eigenvalues of problem (2.27) as follows: $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2.27) if there exists $u_\lambda \in \widetilde{W} \setminus \{0\}$ such that

$$\begin{aligned} &\int_\Omega \Big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} + \mid \nabla u_\lambda \mid^{r-2} \Big) \nabla u_\lambda \cdot \nabla w \, dx \\ &\quad = \lambda \Big(\int_\Omega a \mid u_\lambda \mid^{r-2} u_\lambda w \, dx + \int_{\partial\Omega} b \mid u_\lambda \mid^{r-2} u_\lambda w \, d\sigma \Big) \ \forall \ w \in \widetilde{W}. \end{aligned} \tag{2.28}$$

If u_λ is an eigenfunction corresponding to a positive eigenvalue λ then necessarily u_λ belongs to the set

$$\mathcal{C} := \Big\{ u \in \widetilde{W}; \ \int_\Omega \alpha \mid u \mid^{r-2} u \, dx + \int_{\partial\Omega} \beta \mid u \mid^{r-2} u \, d\sigma = 0 \Big\}. \tag{2.29}$$

Let us introduce the notations

$$\begin{aligned} K_\alpha(u) &:= \int_\Omega \mid \nabla u \mid^\alpha dx, \ \alpha \in \{p, q, r\}, \\ k_r(u) &:= \int_\Omega \alpha \mid u \mid^r dx + \int_{\partial\Omega} \beta \mid u \mid^r d\sigma \ \forall \ u \in W, \\ \mathcal{Z} &:= \{ v \in W; \ k_r(v) = 0 \}. \end{aligned} \tag{2.30}$$

Define

$$\Lambda_r := \inf_{v \in \mathcal{C} \setminus \mathcal{Z}} \frac{K_r(v)}{k_r(v)}. \tag{2.31}$$

For $r \in (q, p)$ denote

$$\begin{aligned} \Lambda_* &:= \inf_{v \in \mathcal{C} \setminus \mathcal{Z}} \left(\Gamma \frac{K_p(v)^{1-\gamma} K_q(v)^\gamma}{k_r(v)} + \frac{K_r(v)}{k_r(v)} \right), \\ \Lambda^* &:= \inf_{v \in \mathcal{C} \setminus \mathcal{Z}} \left(\Gamma \frac{r}{p^{1-\gamma} q^\gamma} \frac{K_p(v)^{1-\gamma} K_q(v)^\gamma}{k_r(v)} + \frac{K_r(v)}{k_r(v)} \right), \\ \gamma &:= \frac{p-r}{p-q}, \ \Gamma := \frac{p-q}{(r-q)^{1-\gamma} (p-r)^\gamma}. \end{aligned} \tag{2.32}$$

The main result concerning problem (2.27) is the following:

Theorem 2.8. (see [9, Theorems 1.1 and 1.2]) *Assume that (h'_{pqr}) and $(h_{\alpha\beta})$ above are fulfilled. If $r \notin (q, p)$, then $\Lambda_r > 0$ and the set of eigenvalues of problem (2.27) is precisely $\{0\} \cup (\Lambda_r, \infty)$, where Λ_r is the constant defined by (2.31). Otherwise, if $r \in (q, p)$, and $r < q(N-1)/(N-q)$ if $q < N$, then $0 < \Lambda_* < \Lambda^*$, every $\lambda \in \{0\} \cup [\Lambda^*, \infty)$*

is an eigenvalue of problem (2.27)*, and for any* $\lambda \in (-\infty, \Lambda_*) \setminus \{0\}$ *problem* (2.27) *has only the trivial solution.*

It would be nice to see whether some of the above result could be extended to the case in which operator $\mathcal{A}_{pq}$ is replaced by the operator $\mathcal{Q}_{pq} := \mathcal{Q}_p + \mathcal{Q}_q$, where for $\theta \in (1,\infty)$ we have denoted by $\mathcal{Q}_\theta$ the operator defined as follows

$$\mathcal{Q}_\theta u := \operatorname{div}\Big(F^{\theta-1}(\nabla u)F_\xi(\nabla u)\Big), \tag{2.33}$$

where F is a positive, one-homogeneous, convex function on $\mathbb{R}^N$ and F_ξ denotes the gradient of F.

If we assume that $F \in C^2(\mathbb{R}^N \setminus \{0\})$ and the Hessian matrix of F^p, $\big(F^p_{\xi_i\xi_j}(\xi)\big)_{i,j}$, is positive definite on $\mathbb{R}^N \setminus \{0\}$, then operator $\mathcal{Q}_\theta$ is elliptic. This operator is a natural generalization of Δ_θ which can be obtained from $\mathcal{Q}_\theta$ if F is the Euclidean norm. A typical example of F satisfying the above conditions is the l_r-norm (denoted by $\|\cdot\|_r$),

$$F(\xi) := \Big(\sum_{i=1}^N |\, \xi_i \,|^r\Big)^{1/r}, \ r \in (1,\infty),$$

for which the operator $\mathcal{Q}_\theta$ has the form

$$\Delta_{r\theta}(u) := \operatorname{div}\big(\, \|\, \nabla u \,\|_r^{\theta-r}\, \nabla^r u\big),$$

where

$$\nabla^r u := \Bigg(\Big|\frac{\partial u}{\partial x_1}\Big|^{r-2}\frac{\partial u}{\partial x_1}, \cdots, \Big|\frac{\partial u}{\partial x_N}\Big|^{r-2}\frac{\partial u}{\partial x_N}\Bigg).$$

Note that $\Delta_{r\theta}$ is a nonlinear operator unless $\theta = r = 2$ when it reduces to the usual Laplacian. An important special case is $r = \theta$, when $\Delta_{\theta\theta}$ is the so-called pseudo $\theta-$Laplacian.

The operator defined in (2.33) is often called anisotropic p-Laplacian or Finsler p-Laplacian. There exist many papers dedicated to the study of its eigenvalues, for different boundary conditions (Dirichlet, Neumann, Robin or Steklov). See, e.g., [13], [20], [21], [22], [25], [32], [44] and references therein.

As an example, let us consider the eigenvalue problem

$$\begin{cases} -\mathcal{Q}_p u = \lambda\alpha(x) \mid u \mid^{q-2} u \ \text{ in } \Omega, \\ F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)\cdot\nu = \lambda\beta(x) \mid u \mid^{q-2} u \text{ on } \partial\Omega. \end{cases} \tag{2.34}$$

As usual, a real number λ is an eigenvalue of problem (2.34) if there exists $u_\lambda \in W^{1,p} \setminus \{0\}$ such that for all $w \in W^{1,p}(\Omega)$

$$\begin{aligned} &\int_\Omega F(\nabla u_\lambda)^{p-1}\nabla_\xi F(\nabla u_\lambda)\cdot\nabla w\; dx \\ &= \lambda\Big(\int_\Omega \alpha \mid u_\lambda \mid^{q-2} u_\lambda w\; dx + \int_{\partial\Omega} \beta \mid u_\lambda \mid^{q-2} u_\lambda w\; d\sigma\Big). \end{aligned} \tag{2.35}$$

The following result holds for problem (2.34).

Theorem 2.9. ([4, Theorem 1.2]) *Assume that* $q \in (1, \infty)$, $p \in \left(\frac{Nq}{N+q-1}, \infty\right)$, $p \neq q$, *and* $(h_{\alpha\beta})$ *are fulfilled. Then the set of eigenvalues of problem* (2.34) *is* $[0, \infty)$.

We expect that many of the above results will be extended to eigenvalue problems governed by the operator $\mathcal{Q}_{pq}$.

References

[1] Abreu, J., Madeira, G., *Generalized eigenvalues of the* $(p, 2)$*-Laplacian under a parametric boundary condition*, Proc. Edinb. Math. Soc, **63**(2020), no. 1, 287-303.

[2] Anane, A., Tsouli, N., *On the second eigenvalue of the p-Laplacian*, in "Nonlinear Partial Differential Equations (From a Conference in Fes, Maroc, 1994)" (Benkirane, A., Gossez, J.-P., Eds.), Pitman Research Notes in Math. 343, Longman, 1996.

[3] Arora, R., Shmarev, S., *Double-phase parabolic equations with variable growth and nonlinear sources*, Adv. Nonlinear Anal., **12**(2023), no. 1, 304-335.

[4] Barbu, L., *Eigenvalues for anisotropic p-Laplacian under a Steklov-like boundary condition*, Stud. Univ. Babeş-Bolyai Math., **66**(2021), no. 1, 85-94.

[5] Barbu, L., Burlacu, A., Moroşanu, G., *On a bulk-boundary eigenvalue problem involving the* (p, q)*-Laplacian* (in preparation).

[6] Barbu, L., Moroşanu, G., *Eigenvalues of the negative* $(p, q)-$*Laplacian under a Steklov-like boundary condition*, Complex Var. Elliptic Equ., **64**(2019), no. 4, 685-700.

[7] Barbu, L., Moroşanu, G., *Full description of the eigenvalue set of the* (p, q)*-Laplacian with a Steklov-like boundary condition*, J. Differential Equations, **290**(2021), 1-16.

[8] Barbu, L., Moroşanu, G., *On a Steklov eigenvalue problem associated with the* $(p, q)-$*Laplacian*, Carpathian J. Math., **37**(2021), 161-171.

[9] Barbu, L., Moroşanu, G., *Eigenvalues of the* $(p, q, r)-$*Laplacian with a parametric boundary condition*, Carpathian J. Math., **38**(2022), no. 3, 547-561.

[10] Barbu, L., Moroşanu, G., *On the eigenvalue set of the* $(p, q)-$*Laplacian with a Neumann-Steklov boundary condition*, Differential Integral Equations, **36**(2023), no. 5-6, 437-452.

[11] Barbu, L., Moroşanu, G., *Full description of the spectrum of a Steklov-like eigenvalue problem involving the* (p, q)*-Laplacian*, Ann. Acad. Rom. Sci., Ser. Math. Appl. (in press).

[12] Bartnik, R. and Simon, L., *Space-like hypersurfaces with prescribed boundary values and mean curvature*, Comm. Math. Phys., **87**(1982), 131-152.

[13] Belloni M., Kawohl B., Juutinen P., *The p-Laplace eigenvalue problem as* $p \to \infty$ *in a Finsler metric*, J. Eur. Math. Soc., **8**(2006), 123-138.

[14] Benci, V., D'Avenia, P., Fortunato, D., et al., *Solitons in several space dimensions: Derrick's problem and infinitely many solutions*, Arch. Ration. Mech. Anal., **154**(2000), 297-324.

[15] Benci, V., Fortunato, D., Pisani, L., *Soliton like solutions of a Lorentz invariant equation in dimension 3*, Rev. Math. Phys., **10**(1998), 315-344.

[16] Bonheure, D., Colasuonno, F., Földes, J., *On the Born-Infeld equation for electrostatic fields with a superposition of point charges*, Ann. Mat. Pura Appl., **198**(2019), 749-772.

[17] Cheng, S.-Y., Yau, S.-T., *Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces*, Ann. of Math., **104**(1976), 407-419.

[18] Cherfils, L., Il'yasov, Y., *On the stationary solutions of generalized reaction diffusion equations with p&q−Laplacian*, Commun. Pure Appl. Anal., **4**(2005), 9-22.

[19] Costea, N., Moroşanu, G., *Steklov-type eigenvalues of* $\Delta p + \Delta q$, Pure Appl. Funct. Anal., **3**(2018), no. 1, 75-89.

[20] Della Pietra, F., Gavitone, N., *Faber-Krahn inequality for anisotropic eigenvalue problems with Robin boundary conditions*, Potential Anal., **41**(2014), 1147-1166.

[21] Della Pietra F., Gavitone N., *Sharp bounds for the first eigenvalue and the torsional rigidity related to some anisotropic operators*, Math. Nachr., **287**(2014), 194-209.

[22] Della Pietra F., Gavitone N., Piscitelli G., *On the second Dirichlet eigenvalue of some nonlinear anisotropic elliptic operators*, Bull. Sci. Math., **155**(2019), 10-32.

[23] Drábek P., Robinson, S., *Resonance problems for the p−Laplacian*, J. Funct. Anal., **169**(1999), 189-200.

[24] Fărcăşeanu, M., Mihăilescu, M., Stancu-Dumitru, D., *On the set of eigenvalues of some PDEs with homogeneous Neumann boundary condition*, Nonlinear Anal., **116**(2015), 19-25.

[25] Ferone, V., Kawohl, B., *Remarks on Finsler-Laplacian*, Proc. Amer. Math. Soc., **137**(2008), no. 1, 247-253.

[26] Fortunato, D., Orsina, L., Pisani, L., *Born-Infeld type equations for electrostatic fields*, J. Math. Phys., **43**(2002), 5698-5706.

[27] François, F., *Spectral asymptotics stemming from parabolic equations under dynamical boundary conditions*, Asymptot. Anal., **46**(2006), no. 1, 43-52.

[28] García-Azorero, J.P., Peral, I., *Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues*, Comm. Partial Differential Equations, **12**(1987), 1389-1430.

[29] Gibbons, G.W., *Born-Infeld particles and Dirichlet p-branes*, Nuclear Phys. B, **514**(1998), 603-639.

[30] Gyulov, T., Moroşanu, G., *Eigenvalues of* $-(\Delta_p + \Delta_q)$ *under a Robin-like boundary condition*, Ann. Acad. Rom. Sci. Ser. Math. Appl., **8**(2016), 114-131.

[31] Huang, Z., *The weak solutions of a nonlinear parabolic equation from two-phase problem*, J. Inequal. Appl., **1**(2021), 1-19.

[32] Kawohl B., Novaga M., *The p-Laplace eigenvalue problem as* $p \to 1$ *and Cheeger sets in a Finsler metric*, J. Convex Anal., **15**(2008), 623-634.

[33] Lê, A., *Eigenvalue problems for p-Laplacian*, Nonlinear Anal., **64**(2006), 1057-1099.

[34] Lindqvist, P., *On the equation* $-div\ (\mid \nabla u \mid^{p-2} \nabla u) - \lambda \mid u \mid^{p-2} u = 0$, Proc. Amer. Math. Soc., **109**(1990), 157-164.

[35] Marcellini, P., *A variational approach to parabolic equations under general and p, q-growth conditions*, Nonlinear Anal., **194**(2020), 111-456.

[36] Mihăilescu, M., *An eigenvalue problem possesing a continuous family of eigenvalues plus an isolated eigenvale*, Commun. Pure Appl. Anal., **10**(2011), 701-708.

[37] Mihăilescu, M., Moroşanu, G., *Eigenvalues of* $-\triangle_p - \triangle_q$ *under Neumann boundary condition*, Canad. Math. Bull., **59**(2016), no. 3, 606-616.

[38] Papageorgiou, N.S., Vetro, C., Vetro, F., *Continuous spectrum for a two phase eigenvalue problem with an indefinite and unbounded potential*, J. Differential Equations, **268**(2020), no. 8, 4102-4118.

[39] Pohozaev, S.I., *The fibering method and its applications to nonlinear boundary value problem*, Rend. Istit. Mat. Univ. Trieste, **31**(1999), no. 1-2, 235-305.

[40] Pomponio, A., Watanabe, T., *Some quasilinear elliptic equations involving multiple P-Laplacians*, Indiana Univ. Math. J., **67**(2018), no. 6, 2199-2224.

[41] Szulkin, A., Weth, T., *The Method of Nehary Manifold, Handbook of Nonconvex Analysis and Applications*, Int. Press, Somerville, MA, 597-632, 2010.

[42] Tanaka, M., *Generalized eigenvalue problems for* $(p, q)-$*Laplacian with indefinite weight*, J. Math. Anal. Appl., **419**(2014), 1181-1192.

[43] Von Below, J., François, G., *Spectral asymptotics for the Laplacian under an eigenvalue dependent boundary condition*, Bull. Belg. Math. Soc. Simon Stevin, **12**(2005), no. 4, 505-519.

[44] Wang G., Xia C., *An optimal anisotropic Poincaré inequality for convex domains*. Pacific J. Math., **258**(2012), 305-326.

[45] Zhikov, V.V., *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat., **50**(1986), 675-710; English translation in Math. USSR-Izv., **29**(1987), 33-66.

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