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Ulam-Hyers stability of Black-Scholes equation

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Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. The goal of this paper is to give a Ulam-Hyers stability result for Black-Scholes equation, in which the unknown function is the price of a derivative financial product. Our approach is based on Green function.

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1. Introduction

The Black-Scholes equation was introduced as a model for the financial mathematics ([1]). We will consider the following equation ([10], [11]):

$$\frac{\partial V(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) = F(s,t) \tag{1.1}$$

$$\Omega = \{(s,t) \mid s \in (s_1,s_2),\ t \in (T_1,T)\},\ V \in C^2(\Omega),$$

where $V(s,t)$ represents the price of the derivative financial product. The independent variables (s,t) are the share price of the underlying assets and time, respectively. The constants σ and r are the volatility of the underlying asset and the risk-free interest rate, respectively. This equation is of the parabolic type and it can be considered as a diffusion equation. In what follows, we refer to this equation as BS equation. In this case we consider the conditions ([11]):

(i) Cauchy problem:

$$V(s,T) = \varphi(s), \tag{1.2}$$

$\varphi(s)$ is the pay-off function of a given derivative problem at $t = T$.

(ii) The boundary conditions (Darboux):

$$V(s_1,t) = b_1(t), \quad V(s_2,t) = b_2(t). \tag{1.3}$$

By an appropriate substitution ([11]), we obtain the equation:

$$\frac{\partial v(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 v(s,t)}{\partial s^2} + rs\frac{\partial v(s,t)}{\partial s} - rv(s,t) = h(s,t), \tag{1.4}$$

with

$$v(s,T)=f(s) \tag{1.5}$$

and homogeneous conditions:

$$v(s_1,t)=v(s_2,t)=0, \tag{1.6}$$

$$\Omega=\{(s,t)\mid s\in(s_1,s_2),\ t\in(T_1,T)\},\ h\in C(\Omega,\mathbb{R}).$$

In what follows we consider the Cauchy-Darboux problem (1.4)+(1.5)+(1.6).
Here ([11])

$$\begin{aligned} h(s,t)=F(s,t)+\frac{s-s_1}{s_2-s_1}[r(b_2(t)-b_1(t))+b_1'(t)-b_2'(t)]\\ -b_1'(t)+rb_1(t)+rs\frac{b_1(t)-b_2(t)}{s_2-s_1} \end{aligned} \tag{1.7}$$

and

$$v(s,t)=\int_{s_1}^{s_2}G(s,t;\eta)f(\eta)d\eta+\int_t^T\int_{s_1}^{s_2}G(s,t-\tau;\eta)h(\eta,\tau)d\eta d\tau. \tag{1.8}$$

Further we study the problem of Ulam-Hyers stability of this equation, where the unknown function appears here as the price of financial derivatives.

We recall that this equation can be called Black-Merton-Scholes equation and it was a subject of the Nobel Prize in Economics in 1997.

2. Notions and definitions

In this section we will present some types of Ulam stability for the Black-Scholes equation.

In 1940, on a talk given at Wisconsin University, S.M. Ulam formulated the following problem: "*Under what conditions does there exist near every approximately homomorphism of a given metric group an homomorphism of the group?*" ([4], [8], [9], [12], [13], [20]). Generally, we say that a differential equation is stable in Ulam sense if for every approximate solution of the differential equation, there exists an exact solution near it. The goal of this paper is to give a stability result for Black-Scholes equation ([1], [11]).

It seems that the first paper on the Ulam-Hyers stability of partial differential equations was written by Prástaro and Rassias ([15]). For other results on the stability of differential equations and partial differential equations we refer to ([2], [3], [5], [6], [7], [14], [17], [19]).

Let $\varepsilon>0$, $\varphi\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $\varphi(0)=0$. We consider the following inequations:

$$\left|\frac{\partial u(s,t)}{\partial t}+\frac{\sigma^2s^2}{2}\frac{\partial^2u(s,t)}{\partial s^2}+rs\frac{\partial u(s,t)}{\partial s}-ru(s,t)-h(s,t)\right|\le\varepsilon,\ \forall\ (s,t)\in\Omega \tag{2.1}$$

$$\left|\frac{\partial u(s,t)}{\partial t}+\frac{\sigma^2s^2}{2}\frac{\partial^2u(s,t)}{\partial s^2}+rs\frac{\partial u(s,t)}{\partial s}-ru(s,t)-h(s,t,u)\right|\le\varepsilon,\ \forall\ (s,t)\in\Omega. \tag{2.2}$$

Definition 2.1. ([17], [18]) *The equation* (1.4) *is Ulam-Hyers stable if there exists a real number* c_1 *such that for each solution* u *of* (2.1) *there exists a solution* v *of* (1.4) *with*

$$|u(s,t)-v(s,t)| \leq c_1 \cdot \varepsilon, \ \forall \ (s,t) \in \Omega. \tag{2.3}$$

Definition 2.2. *The equation* (1.4) *is generalized Ulam-Hyers stable if there exists* $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\varphi(0) = 0$, *continuous, such that for each solution* u *of* (2.2) *there exists a solution* v *of* (1.4) *with*

$$|u(s,t)-v(s,t)| \leq \varphi(\varepsilon), \ \forall \ (s,t) \in \Omega. \tag{2.4}$$

Remark 2.3. A function u is a solution of (2.1) if and only if there exists a function $g \in C(\Omega)$ such that

(i) $|g(s,t)| \leq \varepsilon, \ \forall \ (s,t) \in \Omega$;

(ii) $\dfrac{\partial u(s,t)}{\partial t} + \dfrac{\sigma^2 s^2}{2} \dfrac{\partial^2 u(s,t)}{\partial s^2} + rs \dfrac{\partial u(s,t)}{\partial s} - ru(s,t) = h(s,t) + g(s,t)$.

Remark 2.4. A function u is a solution of (2.2) if and only if there exists a function $g \in C(\Omega)$ such that

(i) $|g(s,t)| \leq \varepsilon, \ \forall \ (s,t) \in \Omega$;

(ii) $\dfrac{\partial u(s,t)}{\partial t} + \dfrac{\sigma^2 s^2}{2} \dfrac{\partial^2 u}{\partial s^2} + rs \dfrac{\partial u(s,t)}{\partial s} - ru(s,t) = h(s,t) + g(s,t)$.

3. Ulam-Hyers stability of equation BS

Here we will present some results of Ulam-Hyers stability for the equation BS.

Theorem 3.1. *We suppose that:*

(i) Ω *is a bounded domain and* G *is the Green function for the BS equation;*

(ii) $h \in C(\overline{\Omega})$, $f \in C(s_1, s_2)$;

(iii) $\displaystyle\int_t^T \int_{s_1}^{s_2} |G(s, t-\tau; \eta)| d\eta d\tau \leq q < 1, \ \forall \ (s,t) \in \Omega.$

Then:

(a) the problem (1.5) + (1.6) *has a unique solution;*

(b) the equation BS, (1.5), *is Ulam-Hyers stable.*

Proof. (a) This is a well known result, consequence of Banach principle ([16]).

(b) Let u be a solution of the inequation (2.1). Let v be the unique solution of the problem (1.5)+(1.6). From Remark 2.3 and the condition (iii) we have that

$$|u(s,t)-v(s,t)| \leq \left| \int_{s_1}^{s_2} G(s,t;\eta) f(\eta) d\eta + \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(\eta,\tau) d\eta d\tau \right.$$
$$+ \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) g(\eta,\tau) d\eta d\tau - \int_{s_1}^{s_2} G(s,t;\eta) f(\eta) d\tau$$
$$\left. - \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(\eta,\tau) d\eta d\tau \right|$$

$$\leq \int_t^T \int_{s_1}^{s_2} |G(s,t-\tau;\eta)| \cdot |g(\eta,\tau)| d\eta d\tau \leq q \cdot \varepsilon.$$

So, the equation (1.5) is Ulam-Hyers stable.

4. Generalized Ulam-Hyers stability of nonlinear BS equation

In this paragraph we will consider the nonlinear BS equation. Let Ω be the domain considered above.

In what follows, we consider the mixed problem (Cauchy-Darboux) ([11]):

$$\frac{\partial v(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2} \frac{\partial^2 v(s,t)}{\partial s^2} + rs\frac{\partial v(s,t)}{\partial s} - rv(s,t) = h(s,t,v), \tag{4.1}$$

$$\begin{aligned} v(s,T) &= f(s), \\ v(s_1,t) &= v(s_2,t) = 0. \end{aligned} \tag{4.2}$$

Theorem 4.1. *Let us consider the equation* (4.1) *and the inequation* (2.2). *Let G be the Green function corresponding to BS equation.*

We suppose that:

(i) $h \in C(\Omega)$ and there exists $l_h > 0$ with

$$l_h \int_t^T \int_{s_1}^{s_2} |G(s,t-\tau;\eta)| d\eta d\tau \leq q < 1$$

and a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|h(s,t,u) - h(s,t,v)| \leq l_h \varphi(|u-v|).$$

Then

(a) the Cauchy-Darboux problem (4.1) + (4.2) *has a unique solution v;*

(b) if the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(z) = z - \varphi(z)$, is strictly increasing and onto or bijective, the problem (4.1) + (4.2) *is generalized Ulam-Hyers stable.*

Proof. (a) This result is a consequence of Banach theorem.

(b) Let u be a solution of the inequality (2.2) and v the unique solution of the problem (4.1)+(4.2). From the above conditions we have

$$|u(s,t) - v(s,t)| = \left| \int_{s_1}^{s_2} G(s,t;\eta) f(\eta) d\eta + \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(s,t,u) d\eta d\tau \right.$$

$$+ \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) g(\eta,\tau) d\eta d\tau - \int_{s_1}^{s_2} G(s,t;\eta) f(\eta) d\eta$$

$$\left. - \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(\eta,\tau,v) d\eta d\tau \right|$$

$$= \left| \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(s,t,u) d\eta d\tau + \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) g(\eta,\tau) d\eta d\tau \right.$$

$$- \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(s,t,v) d\eta d\tau \Big|$$

$$\leq \int_t^T \int_{s_1}^{s_2} |G(s,t-\tau;\eta)| \cdot |h(s,t,u) - h(s,t,v)| d\eta d\tau$$

$$+ \int_t^T \int_{s_1}^{s_2} |G(s,t-\tau;\eta)| \cdot |g(\eta,\tau)| d\eta d\tau$$

$$\leq \int_t^T \int_{s_1}^{s_2} |G(s,t-\tau;\eta)| l_h \varphi(|u-v|) d\eta d\tau + \int_t^T \int_{s_1}^{s_2} |G(s,t-\tau;\eta)| \cdot |g(\eta,\tau)| d\eta d\tau,$$

then we have

$$|u(s,t) - v(s,t)| \leq \varphi(|u(s,t) - v(s,t)|) + \frac{\varepsilon}{l_h}$$

and

$$\psi(|u(s,t) - v(s,t)|) \leq \frac{\varepsilon}{l_h},$$

therefore we have

$$|u(s,t) - v(s,t)| \leq \psi^{-1}\left(\frac{\varepsilon}{l_h}\right).$$

So the equation (4.1) is generalized Ulam-Hyers stable.

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