New hybrid conjugate gradient method as a convex combination of PRP and RMIL+ methods

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Abstract. The Conjugate Gradient (CG) method is a powerful iterative approach for solving large-scale minimization problems, characterized by its simplicity, low computation cost and good convergence. In this paper, a new hybrid conjugate gradient HLB method (HLB: Hadji-Laskri-Bechouat) is proposed and analysed for unconstrained optimization. We compute the parameter $\beta_k^{HLB}$ as a convex combination of the Polak-Ribiére-Polyak ($\beta_k^{PRP}$) and the Mohd Rivaie-Mustafa Mamat and Abdelrhaman Abashar ($\beta_k^{RMIL+}$) i.e. $\beta_k^{HLB} = (1 - \theta_k) \beta_k^{PRP} + \theta_k \beta_k^{RMIL+}$. By comparing numerically CGHLB with PRP and RMIL+ and by using the Dolan and More CPU performance, we deduce that CGHLB is more efficient.

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1. Introduction

Consider the nonlinear unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

(1.1)

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, bounded from below. The gradient of $f$ is denoted by $g(x)$. To solve this problem, we start from an initial
point $x_0 \in \mathbb{R}^n$. Nonlinear conjugate gradient methods generate sequences $\{x_k\}$ of the following form:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, 2, \ldots, \quad (1.2)$$

where $x_k$ is the current iterate point and $\alpha_k > 0$ is the step size which is obtained by line search [7].

The iterative formula of the conjugate gradient method is given by (1.2), where $d_k$ is the search direction defined by

$$d_{k+1} = \begin{cases} -g_k & \text{si } k = 1 \\ -g_{k+1} + \beta_k d_k & \text{si } k \geq 2 \end{cases} \quad (1.3)$$

where $\beta_k$ is a scalar and $g(x)$ denotes $\nabla f(x)$ [10]. If $f$ is a strictly convex quadratic function, namely,

$$f(x) = \frac{1}{2} x^T H x + b^T x, \quad (1.3\text{bis})$$

where $H$ is a positive definite matrix and if $\alpha_k$ is the exact one-dimensional minimizer along the direction $d_k$, i.e.

$$\alpha_k = \arg \min_{\alpha > 0} \{ f(x + \alpha d_k) \} \quad (1.3\text{tris})$$

then (1.2), (1.3), (1.3bis), (1.3tris) is called the linear conjugate gradient method. Otherwise, (1.2), (1.3) is called the nonlinear conjugate gradient method. Conjugate gradient methods can broadly be classified based on the used strategies of the way in which the search direction is updated and the algorithms dealing with the step size minimization along a direction [6]. In [12], a convex combination of LS and FR ([1]) is proposed with a newton descent direction.

The line search in the non linear conjugate gradient methods is often based on the standard Wolfe conditions [23]:

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \rho \alpha_k g^T_k d_k \quad (1.4)$$
$$g^T_{k+1} d_k \geq \delta g^T_k d_k \quad (1.5)$$

where $0 < \rho \leq \delta < 1$.

Conjugate gradient methods differ in their way of defining the scalar parameter $\beta_k$. In the literature, there have been proposed several choices for $\beta_k$ which give rise to distinct conjugate gradient methods [16], [27]. The most well known conjugate gradient methods are the Hestenes–Stiefel (HS) method [17], the Fletcher-Reeves (FR) method [1], [13], the Polak-Ribière-Polyak (PRP) method [20], [19], the Conjugate Descent method (CD) [13], the Liu-Storey (LS) method [18], the Dai-Yuan (DY) method [08], [09], Hager and Zhang (HZ) method [15] and the RMIL+ method [21], [22]. The update parameters of these methods are respectively specified as follows:

$$\beta^H_k = \frac{g^T_{k+1} y_k}{d_k^T y_k}, \quad \beta^F_k = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \quad \beta^R_k = \frac{g^T_{k+1} y_k}{\|g_{k+1}\|^2}, \quad \beta^P_k = \frac{d_k^T y_k}{\|d_k\|^2}$$
$$\beta^C_k = - \frac{\|g_{k+1}\|^2}{d_k^T y_k}, \quad \beta^L_k = \frac{\|g_{k+1}\|^2}{d_k^T y_k}, \quad \beta^D_k = \frac{\|g_{k+1}\|^2}{d_k^T y_k}, \quad \beta^H_k = \left(y_k - 2d_k \frac{\|y_k\|^2}{d_k^T y_k}\right) T \frac{g_{k+1}}{d_k^T y_k}.$$
\[ \beta_k^{RMIL+} = \frac{g_{k+1}^T(g_{k+1} - g_k - d_k)}{\|d_k\|^2}. \]

Some of these methods, such as Fletcher and Reeves (FR) [13], Dai and Yuan (DY) [8] and Conjugate Descent (CD) [13] have strong convergence properties, but they may have modest practical performance due to jamming. On the other hand, the methods of Polak and Ribiére and Polyak (PRP) [20], Hestenes and Stiefel (HS) [17] or Liu and Story (LS) [18] may not generally be convergent, but they often have better computational performance.

In the process of obtaining more robust and efficient conjugate gradient methods, some researchers suggested the hybrid conjugate gradient algorithm which combined the good features of the methods involve in the hybridization. Even though conjugate gradient improvement using hybridization is a classic deeply investigated problem; it still an attractive topic for the research community due to its contemporary use in numerous prominent disciplines [25].

The first hybrid conjugate gradient method was given by Touati-Ahmed and Storey (1990) [24] to avoid jamming phenomenon.

The researchers were motivated by the works of Andrei [5], [4]; Dai and Yuan [9]; Zhang and Zhou [26]. Their parameter \( \beta_k^N \) is computed as a convex combination of \( \beta_k^{FR} \) and \( \beta_k^* \) other algorithms, i.e.
\[
\beta_k^N = (1 - \theta_k) \beta_k^{FR} + \theta_k \beta_k^*.
\]

The Wolfe line search was employed to determine the step length \( \alpha_k > 0 \) and the new method proved to be more robust numerical wise as compared to FR and other methods. The global convergence was established under some suitable conditions.

In [4] Andrei has proposed a new hybrid conjugate gradient algorithm where the parameter \( \beta_k^A \) is computed as a convex combination of the Polak-Ribiére-Polyak and the Dai-Yuan conjugate gradient algorithms i.e.
\[
\beta_k^A = (1 - \theta_k) \beta_k^{PRP} + \theta_k \beta_k^{DY}
\]

and \( \theta_k \) is presented to satisfy the conjugacy condition
\[
\theta_k = \theta_k^{CCOMB} = \frac{(y_k^T g_{k+1}) (y_k^T s_k) - (y_k^T g_k) (y_k^T g_{k+1})}{(y_k^T g_{k+1}) (y_k^T s_k) - \|g_{k+1}\|^2 \|g_k\|^2}
\]

where \( s_k = x_{k+1} - x_k \). To satisfy Newton direction he takes
\[
\theta_k = \theta_k^{NDOMB} = \frac{(y_k^T g_{k+1} - y_k^T s_k) \|g_k\|^2 - (y_k^T g_k) (y_k^T s_k)}{\|g_{k+1}\|^2 \|g_k\|^2 - (y_k^T g_{k+1}) (y_k^T s_k)}
\]

but in the combination of HS and DY from Newton direction, he puts
\[
\theta_k = -\frac{s_k^T g_{k+1}}{g_k^T g_{k+1}}.
\]

On the other hand, from Newton direction with modified secant condition (Hybrid M-Andrei), Andrei has proposed another method
\[
\beta_k^{HYBRIDM} = (1 - \theta_k) \beta_k^{HS} + \theta_k \beta_k^{DY}
\]
where
\[ \theta_k = \frac{\left( \frac{\delta g_k}{s_k^k} - 1 \right) s_k^k g_{k+1} + \frac{g_k^t y_{k+1}}{s_k^k} \delta \eta_k}{g_k^t g_{k+1} + \frac{g_k^t y_{k+1}}{s_k^k} \delta \eta_k} \]

\( \delta \) is parameter. In [14] Salah Gazi Shareef and Hussein Ageel Khatab have introduced a new hybrid CG method
\[ \beta_{k^{New}} = (1 - \theta_k) \beta_{k^{PRP}} + \theta_k \beta_{k^{BA}} \]
where \( \beta_{k^{BA}} \) is selected in [2].

Recently Delladji et al. [11] proposed a hybridization of PRP and HZ schemes using the conjugacy condition.

In this paper, we present another hybrid CG algorithm noted CGHLB (HLB is an abbreviation to Hadji; Laskri and Bechouat), which is a convex combination of the PRP ([20]) and RMIL+ ([21]) conjugate gradient algorithms. We are interested to combine these two methods in a hybrid CG algorithm because PRP has good computational properties and RMIL+ has strong convergence properties. In section 2, we introduce our hybrid CG method and prove that it generates descent directions. In Section 3 we present and prove global convergence results. Numerical results and a conclusion are presented in section 4. By comparing numerically CGHLB with PRP and RMIL+ and by using the Dolan and More CPU performance, we deduce that CGHLB is more efficient.

2. HLB conjugate gradient method

The iterates \( x_0, x_1, \ldots \) of the proposed HLB algorithm are computed by means of the recurrence (1.2) where the step size \( \alpha_k > 0 \) is determined according to the Wolfe line search conditions (1.4), (1.5). The directions \( d_k \) are generated by the rule:
\[ d_k = \begin{cases} -g_0 & \text{if } k = 0 \\ -g_k + \beta_{k^{HLB}}^H d_{k-1} & \text{if } k \geq 1 \end{cases} \]  

(2.1)
where
\[ \beta_{k^{HLB}}^H = (1 - \theta_k) \beta_{k^{PRP}} + \theta_k \beta_{k^{RMIL+}} \]
i.e.
\[ \beta_{k^{HLB}}^H = (1 - \theta_k) \frac{g_{k+1}^t y_k}{\|g_k\|^2} + \theta_k \frac{g_{k+1}^t (g_{k+1} - g_k - d_k)}{\|d_k\|^2} \]

(2.2)
HLB is an abbreviation to Hadji; Laskri and Bechouat; \( \theta_k \) is a scalar parameter which will be determined in a specific way to be described in the following section. Observe that if \( \theta_k = 0 \) then \( \beta_{k^{HLB}}^H = \beta_{k^{PRP}} \) and if \( \theta_k = 1 \), then \( \beta_{k^{HLB}}^H = \beta_{k^{RMIL+}} \). On the other hand if \( 0 < \theta_k < 1 \), then \( \beta_{k^{HLB}}^H \) is a convex combination of \( \beta_{k^{PRP}} \) and \( \beta_{k^{RMIL+}} \). The parameter \( \theta_k \) is selected in such a way that at every iteration the conjugacy condition is satisfied. It can be noted that,
\[ d_{k+1} = -g_{k+1} + (1 - \theta_k) \frac{g_{k+1}^t y_k}{\|g_k\|^2} d_k + \theta_k \frac{g_{k+1}^t (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k \]

(2.3)
so multiply both sides of above equation by \( y_k \) and by using the conjugacy condition \((g_{k+1}^ty_k = 0)\) we have:

\[
0 = -g_{k+1}^ty_k + (1 - \theta_k) \frac{g_{k+1}^ty_k}{\|y_k\|^2} g_{k+1}^ty_k + \theta_k \frac{g_{k+1}^t(g_k - d_k)}{\|d_k\|^2} d_{k}^ty_k
\]  

(2.4)

After a simple calculation we get

\[
\theta_k = \frac{g_{k+1}^ty_k \|y_k\|^2 \|d_k\|^2 - (g_{k+1}^ty_k) (d_k^ty_k) \|d_k\|^2}{(g_{k+1}^t(y_k - d_k)) \|y_k\|^2 - (g_{k+1}^ty_k) \|d_k\|^2} (d_k^ty_k)
\]  

(2.5)

So, to ensure the convergence of this method when the parameter \( \theta_k \) goes out of interval \([0,1]\), i.e. when \( \theta_k \leq 0 \) or \( \theta_k \geq 1 \), we prefer to take \( \beta_{k}^{HLB} \) as following:

\[
\beta_{k}^{HLB} = \begin{cases} 
(1 - \theta_k) \beta_{k}^{PRP} + \theta_k \beta_{k}^{RMIL+} & \text{if } 0 < \theta_k < 1 \\
\beta_{k}^{PRP} & \text{if } \theta_k \leq 0 \\
\beta_{k}^{RMIL+} & \text{if } \theta_k \geq 1
\end{cases}
\]

(2.5(bis))

We are now able to present our new algorithm, the Conjugate Gradient CGHLB Algorithm:

**CGHLB Algorithm**

**Step 1: Initialization:**

Set \( k = 0 \), select the initial point \( x_0 \in \mathbb{R}^n \), select the parameters \( 0 < \rho \leq \delta < 1 \), and \( \varepsilon > 0 \).

Compute \( f(x_0) \) and \( g_0 = \nabla f(x_0) \). Consider \( d_0 = -g_0 \).

**Step 2: Test for continuation of iterations:**

If \( \|g_k\| \leq \varepsilon \) then stop else set \( d_k = -g_k \).

**Step 3: Line search:**

Compute \( \alpha_k > 0 \) satisfying the Wolfe line search condition (1,4) and (1,5) and update the variables, \( x_{k+1} = x_k + \alpha_k d_k \); compute \( f(x_{k+1}) \), \( g_{k+1} \) and \( s_k = x_{k+1} - x_k \); \( y_k = g_{k+1} - g_k \).

**Step 4: \( \theta_k \) Parameter computation:**

If \( (g_{k+1}^t(y_k - d_k)) \|y_k\|^2 - (g_{k+1}^ty_k)(d_k^ty_k) = 0 \)

then set \( \theta_k = 0 \), otherwise, compute \( \theta_k \) as in (2.5).

**Step 5: \( \beta_{k}^{HLB} \) Conjugate gradient parameter computation:**

If \( 0 < \theta_k < 1 \), then compute \( \beta_{k}^{HLB} \) as in (2.2).

If \( \theta_k \geq 1 \), then set \( \beta_{k}^{HLB} = \beta_{k}^{RMIL+} \).

If \( \theta_k \leq 0 \), then set \( \beta_{k}^{HLB} = \beta_{k}^{PRP} \).

**Step 6: Direction computation:**

Compute \( d_{k+1} = -g_{k+1} + \beta_{k}^{HLB} d_k \).

Set \( k = k+1 \) and go to step 3.

The following theorem shows that our method assures the descent condition, when \( 0 < \theta_k < 1 \).

**Theorem 2.1.** In the algorithm (1.2), (1.3) and (2.5) assume that \( d_k \) is a descent direction \((g_k^td_k < 0)\), and \( \alpha_k \) is determined by the Wolfe line search (1.4); (1.5). If \( 0 < \theta_k < 1 \) then the direction \( d_{k+1} \) given by (2.3) is a descent direction.
Proof. Multiply both sides of (2.3) by $g_{k+1}$ we have:

$$g_{k+1}^T d_{k+1} = -\|g_{k+1}\|^2 + (1 - \theta_k) \frac{g_{k+1}^T y_k}{\|g_k\|^2} d_k^T g_{k+1} + \theta_k \frac{g_{k+1}^T (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^T g_{k+1}$$

$$g_{k+1}^T d_{k+1} = - (1 - \theta_k + \theta_k) \|g_{k+1}\|^2 + (1 - \theta_k) \frac{g_{k+1}^T y_k}{\|g_k\|^2} d_k^T g_{k+1} + \theta_k \frac{g_{k+1}^T (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^T g_{k+1}$$

$$g_{k+1}^T d_{k+1} = \left[ - (1 - \theta_k) \|g_{k+1}\|^2 + (1 - \theta_k) \frac{g_{k+1}^T y_k}{\|g_k\|^2} d_k^T g_{k+1} \right]$$

$$\quad + \left[ - (\theta_k) \|g_{k+1}\|^2 + \theta_k \frac{g_{k+1}^T (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^T g_{k+1} \right]$$

$$g_{k+1}^T d_{k+1} = (1 - \theta_k) \left[ - \|g_{k+1}\|^2 + \frac{g_{k+1}^T y_k}{\|g_k\|^2} d_k^T g_{k+1} \right]$$

$$\quad + (\theta_k) \left[ - \|g_{k+1}\|^2 + \frac{g_{k+1}^T (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^T g_{k+1} \right]$$

since $0 < \theta_k < 1$ then

$$g_{k+1}^T d_{k+1} \leq \left[ - \|g_{k+1}\|^2 + \frac{g_{k+1}^T y_k}{\|g_k\|^2} d_k^T g_{k+1} \right]$$

$$\quad + \left[ - \|g_{k+1}\|^2 + \frac{g_{k+1}^T (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^T g_{k+1} \right]$$

(2.6)

If the step length $\alpha_k$ is chosen by an exact line search. Then $g_{k+1}^T d_k = 0$.

If the step length $\alpha_k$ is chosen by an inexact line search ($g_{k+1}^T d_k \neq 0$) then we have:

$$g_{k+1}^T d_{k+1} < 0$$

because the algorithms of (PRP) and (RMIL+) satisfied the descent property.

The proof is completed. \qed

3. Global convergence properties

The following assumptions are often needed to prove the convergence of the nonlinear CG:
**Assumption 1**

The level set $\Omega = \{ x \in \mathbb{R}^n / f(x) \leq f(x_0) \}$ is bounded, where $x_0$ is the starting point.

**Assumption 2**

In some neighborhood $N$ of $\Omega$, the objective function is continuously differentiable and its gradient is Lipschitz continuous, namely, there exists a constant $l > 0$ such that:

$$\| g(x) - g(y) \| \leq l \| x - y \| \quad \text{for any } x, y \in N$$

Under these assumptions on $f$ there exists a constant $\mu$ such that $\| g(x) \| \leq \mu$, for all $x \in \Omega$.

**Lemma 3.1.** [28] Suppose Assumption 1 and 2 hold, and consider any conjugate gradient method (1.2) and (1.3), where $d_k$ is a descent direction and $\alpha_k$ is obtained by the strong Wolfe line search. If

$$\sum_{k=1}^{\infty} \frac{1}{\| d_k \|^2} = +\infty \quad (3.1)$$

then

$$\lim_{k \to \infty} \| g_k \| = 0 \quad (3.2)$$

Assume that the function $f$ is uniformly convex function, i.e. there exists a constant $\Gamma \geq 0$ such that,

$$\forall x, y \in \Omega : (\nabla f(x) - \nabla f(y))^t (x - y) \geq \Gamma \| x - y \|^2 \quad (3.3)$$

and the step length $\alpha_k$ is given by the strong Wolfe line search.

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \sigma_1 \alpha_k g_k^t d_k \quad (3.4)$$

$$\| g_{k+1} \| \leq -\sigma_2 g_k^t d_k \quad (3.5)$$

For uniformly convex function which satisfies the above assumptions, we can prove that the norm of $d_{k+1}$ given by (2.3) is bounded above.

Using the above lemma, we obtain the following theorem.

**Theorem 3.2.** Suppose that Assumption 1 and 2 hold. Consider the algorithm (1.2), (2.3) and (2.5), where $0 \leq \theta_k \leq 1$ and $\alpha_k$ is obtained by the strong Wolfe line search (3.4) and (3.5).

If $d_k$ tends to zero and there exists non negative constants $\eta_1$ and $\eta_2$ such that:

$$\| g_k \|^2 \geq \eta_1 \| s_k \|^2 \quad \text{and} \quad \| g_{k+1} \|^2 \leq \eta_2 \| s_k \| \quad (3.6)$$

and $f$ is uniformly convex function, then

$$\lim_{k \to \infty} g_k = 0 \quad (3.7)$$

**Proof.** From (3.3) it follows that

$$y_k^t s_k \geq \Gamma \| s_k \|^2$$
since $0 \leq \theta_k \leq 1$, from uniform convexity and (3.6) we have
\[
|\beta_k^{HLB}| \leq \left| \frac{g_{k+1}^T y_k}{\|g_k\|^2} \right| + \frac{|g_{k+1}^T (g_{k+1} - g_k - d_k)|}{\|d_k\|^2}
\leq \frac{|g_{k+1}^T y_k|}{\|g_k\|^2} + \frac{|g_{k+1}^T d_k|}{\|d_k\|^2}
\leq \frac{\|g_{k+1}\| \|y_k\|}{\|g_k\|^2} + \frac{\|g_{k+1}\| \|d_k\|}{\|d_k\|^2}
\]
from Lipschitz condition
\[
\|y_k\| \leq \nu \|s_k\|
\]
\[
|\beta_k^{HLB}| \leq \frac{\|g_{k+1}\| \|y_k\|}{\eta_1 \|s_k\|^2} + \frac{\|g_{k+1}\| \|y_k\|}{\|d_k\|^2} + \frac{\|g_{k+1}\| \|d_k\|}{\|d_k\|^2}
\leq \frac{\mu \|s_k\|}{\eta_1 \|s_k\|^2} + \frac{\mu \|s_k\| \alpha_k^2}{\|s_k\|^2} + \frac{\mu \alpha_k}{\|s_k\|}
\leq \frac{\mu \|s_k\|}{\eta_1 \|s_k\|^2} + \frac{\mu \alpha_k^2}{\|s_k\|^2} + \frac{\mu \alpha_k}{\|s_k\|}
\]
Hence
\[
\|d_{k+1}\| \leq \|g_{k+1}\| + |\beta_k^{HLB}| \|d_k\|
\leq \mu + \frac{\mu \|s_k\|}{\eta_1 \alpha_k \|s_k\|} + \frac{\mu \|s_k\| \alpha_k^2}{\alpha_k \|s_k\|} + \frac{\mu \alpha_k \|s_k\|}{\alpha_k \|s_k\|}
\leq \mu + \frac{\mu \alpha_k}{\eta_1 \alpha_k} + \frac{\mu \alpha_k}{\eta_1 \alpha_k}
\]
which implies that (3.1) is true. Therefore, by Lemma 1 we have (3.2), which for uniformly convex functions is equivalent to (3.7). □

4. Numerical results and discussion

In the present numerical experiments, we analyze the efficiency of $\beta^{HLB}$, as compared to the classic methods: $\beta^{PRP}$ and $\beta^{RMIL}$. These comparisons are based on the number of iterations and CPU time per second to reach the optimum. All the comparisons are done with two or three different initial points and different number of variables ranging from 2 to 20000. All variables have been experimented to each function test [3]. For the numerical tests, the strong Wolfe line searches parameters have been experimentally fixed to $\rho = 10^{-3}$ and $\delta = 10^{-4}$. All tests were terminated when the stopping criteria $\|g_k\| \leq \varepsilon$ is fulfilled, where $\varepsilon = 10^{-6}$. When the iteration number exceeds 2000 or the CPU execution time exceeded 500 seconds, the test is considered as failed.
Figures 1 and 2 show that the method of $\beta^{HLB}$ is superior when compared to $\beta^{PRP}$ and $\beta^{RMIL+}$ with the least duration of CPU time. The highest percentage of successful comparison is with $\beta^{HLB}$ at 98.34%, followed by $\beta^{RMIL+}$ with 93.72%. However, the successful rate comparison for $\beta^{PRP}$ is low at 90.05%. Hence, our method ($\beta^{HLB}$) successfully solves the test problems, and it is competitive with the well-known conjugate gradient methods for unconstrained optimization.
5. Conclusion

Numerous studies have been devoted to develop and improve hybrid conjugate gradient methods. In this paper we have presented a new convex hybridization of the PRP and the RMIL+ conjugate gradient algorithms; HLB. The global convergence of our method is demonstrated for 0 < $\theta$ < 1. Numerical experiments reveal that our method is reaching the optimum in less iteration number and CPU time comparing to RMIL+ and PRP.

References

CGHLB, hybrid convex combination of PRP and RMIL+


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