Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 1, 213–234
DOI: 10.24193/subbmath.2023.1.16

Global existence and blow-up of a Petrovsky equation with general nonlinear dissipative and source terms

Mosbah Kaddour and Farid Messelmi

Abstract. This work studies the initial boundary value problem for the Petrovsky equation with nonlinear damping

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u - \Delta u' + |u|^{p-2} u + \alpha g\left(u'\right) = \beta f\left(u\right) \text{ in } \Omega \times [0, +\infty[,$$

where Ω is open and bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$, α, and $\beta > 0$. For the nonlinear continuous term $f(u)$ and for g continuous, increasing, satisfying $g(0) = 0$, under suitable conditions, the global existence of the solution is proved by using the Faedo-Galerkin argument combined with the stable set method in $H_0^2(\Omega)$. Furthermore, we show that this solution blows up in a finite time when the initial energy is negative.

Mathematics Subject Classification (2010): 93C20, 93D15.

Keywords: Global existence, blow-up, nonlinear source, nonlinear dissipative, Petrovsky equation.

1. Introduction

This paper devoted to the global existence, uniqueness, and the blow-up of solutions for the nonlinear general Petrovsky equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta^2 u(t) - \Delta u'(t) + |u|^{p-2} u(t) + \alpha g(u'(t)) = \beta f(u(t)), \text{ in } \Omega \times \mathbb{R}^+, \\ u = \partial_\eta u = 0, \text{ on } \Gamma \times [0, +\infty[, \\ u(x,0) = u_0(x), \ u'(x,0) = u_1(x) \text{ in } \Omega. \end{cases} \tag{1.1}$$

Recently, in the absence of the strong damping term $-\Delta u'(t)$ and in the case where

$$\beta f(u(t)) = -q(x) u(x,t) + |u|^{p-2} u(t)$$

Received 24 June 2020; Revised 221 July 2020.

for g continuous, increasing, satisfying $g(0) = 0$, and $q : \Omega \to \mathbb{R}^+$, a bounded function, the problem (1.1) becomes the following

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u(t) + q(x)\, u(x,t) + g(u'(t)) = 0, \text{ in } \Omega \times \mathbb{R}^+.$$

This equation together with initial and boundary conditions of Dirichlet type was considered by Guesmia in [5], he proved a global existence and a regularity result of the solution, the author under suitable growth conditions on g showed that the solution decays exponentially if g behaves like a linear function, whereas the decay is of a polynomial order otherwise. Without the strong damping term $-\Delta u'(t)$ with

$$\alpha g(u'(t)) = |u'(t)|^{\sigma-2} u'(t)$$

and

$$\beta f(u(t)) = (b+1)\, |u(t)|^{p-2} u(t), \ b > 0,$$

the problem (1.1) reduced to the following problem

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u(t) + |u'(t)|^{\sigma-2} u'(t) = b\, |u(t)|^{p-2} u(t), \text{ in } \Omega \times \mathbb{R}^+,$$

this problem has been considered by Messaoudi in [9], where he investigated the global existence and blow-up of solution. More precisely, he showed that solutions with any initial data continue to exist globally in time if $\sigma \geq p$ and blow-up in finite time if $\sigma < p$ and the initial energy is negative. He used a new method introduced by Georgiev and Todorova [4] based on the fixed point theorem for the proof. In [12], Wu and Tsai showed that the solution of the problem considered in [9] is global under some conditions. Also, Chen and Zhou [11] studied the blow-up of the solution of the same problem as in [9]. In the presence of the strong damping, in the case where

$$\beta f(u(t)) = (b+1)\, |u(t)|^{p-2} u(t),$$

$$g(u'(t)) = |u'(t)|^{\sigma-1} u'(t), \ b > 0,$$

general Petrovsky problem as in (1.1) becomes

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u(t) - \Delta u'(t) + |u'(t)|^{\sigma-1} u'(t) = b\, |u(t)|^{p-1} u(t), \tag{1.2}$$

this problem was considered by Li et al. [6], in [10] and in [2], the authors obtained global existence, uniform decay of solutions without any interaction between p and σ, the blow-up of the solution result was established when $\sigma < p$. Very recently, Pişkin and Polat [10] studied the decay of the solution of the problem (1.2). In this paper, our aim is to extend the results of [9], [12] and others' established in a bounded domain to a general problem as in (1.1). The nonlinear term f in (1.1) likes

$$f(u(x,t)) = a(x)\, |u(t)|^{r-2} u(t) - b(x)\, |u(t)|^{q-2} u(t)$$

with $r > q \geq 1$ and $a(x), b(x) > 0$, and g in (1.1) likes

$$g(u'(x,t)) = \alpha(x)\, |u'(t)|^{\sigma-2} u'(t)$$

with $\sigma \geq 2$ for $\alpha : \Omega \to \mathbb{R}^+$ a function, satisfying $\alpha_1 \geq \alpha(x) \geq \alpha_0 > 0$. For these purposes, we must establish the global existence of solution for (1.1), we use the variational approach of Faedo–Galerkin approximation combined with the monotonous, compactness, and the stable set method as in [9], [11] and in [10] with some modification in some passages to derive the blow-up result in the infinite time of the solution.

2. Hypotheses

Let us state the precise hypotheses on p, g, and f. Let p be a positive number with

$$2 < p \leq \frac{2n-6}{n-4}\,(n \geq 5)\,(2 \leq p < \infty \text{ if } n = 1,2,3,4)\,, \tag{H1}$$

g is an odd increasing C^1 function and

$$\begin{cases} xg(x) \geq d_0 |x|^\sigma\,, & \forall x \in \mathbb{R},\ p > \sigma \geq 2, \\ |g(x)| \leq d_1 |x| + d_2 |x|^{\sigma-1}\,, & \forall x \in \mathbb{R},\ p > \sigma \geq 2,\ d_i \geq 0. \end{cases} \tag{H2}$$

Let $f(x,s) \in C^1(\Omega \times \mathbb{R})$, satisfies:

$$sf(x,s) + k_1(x)|s| \geq pF(x,s)\,,\ p > 2, \tag{H3}$$

and the growth conditions

$$\begin{cases} |f(x,s)| \leq l_1\left(|s|^\theta + k_2(x)\right), \\ |f_s(x,s)| \leq l_1\left(|s|^{\theta-1} + k_3(x)\right) \text{ in } \Omega \times \mathbb{R}, \end{cases} \tag{H4}$$

where $F(x,s) = \int_0^s f(x,\zeta)\,d\zeta$, with some l_0, $l_1 > 0$ and the non-negative functions $k_1(x), k_2(x), k_3(x) \in L^\infty(\Omega)$, a.e. $x \in \Omega$, and $1 < \theta \leq \frac{\sigma}{2} < \frac{p}{2}$.

3. Local existence

In this section, we establish a local existence result for (1.1) under the assumptions on f, g, and p.

Theorem 3.1. *Let $(u_0, u_1) \in W \cap L^p(\Omega) \times H_0^2(\Omega) \cap L^{2\sigma-2}(\Omega)$. Assume that (H1)-(H4) hold. Then problem (1.1) has a unique weak solution $u(t)$ satisfying:*

$$u \in L^\infty(0,T; W \cap L^p(\Omega)), \tag{3.1}$$

$$u' \in L^\infty(0,T; H_0^2(\Omega)), \tag{3.2}$$

$$g(u'(t)).u'(t) \in L^1\left(0,T; L^1(\Omega)\right), \tag{3.3}$$

$$u'' \in L^\infty(0,T; L^2(\Omega)), \tag{3.4}$$

where

$$H_0^2(\Omega) = \left\{\varphi \in H^2(\Omega) : \varphi = \partial_\eta \varphi = 0 \text{ on } \partial\Omega\right\},$$

and

$$W = \left\{\varphi \in H^4(\Omega) \cap H_0^2(\Omega) : \Delta\varphi = \partial_\eta \Delta\varphi = 0 \text{ on } \partial\Omega\right\}.$$

Note that throughout this paper, C denotes a generic positive constant depending on Ω and as all given constants, which may be different from line to line, and is capable of being examined and modified.

Proof. We adopt the Galerkin method to construct a global solution. Let $T > 0$ be a fixed, and denote by V_m the space generated by $\{\varphi_1,\ \varphi_2, ...,\ \varphi_m\}$, where the set $\{\varphi_m;\ m \in \mathbb{N}\}$ is a basis of $L^2(\Omega)$, $H_0^2(\Omega)$, and $H^4(\Omega) \cap H_0^2(\Omega)$. We construct approximate solutions u_m $(m = 1, 2, 3, ...)$ in the form

$$u_m(t) = \sum_{j=1}^{m} K_{jm}(t) w_j,$$

where K_{jm} are determined by the following ordinary differential equations:

$$\begin{aligned}(u_m'', w_j) + (\Delta u_m, \Delta w_j) + (\nabla u_m', \nabla w_j) \\ + \left(|u_m|^{p-2} u_m, w_j\right) + \alpha\left(g\left(u_m'\right), w_j\right) &= \beta\left(f\left(u_m\right), w_j\right),\end{aligned} \tag{3.5}$$

$$\begin{aligned}u_m(0) &= u_{0m} = \sum_{i=1}^{m} (u_0, w_j)\, w_j \overset{\text{as } m \longrightarrow \infty}{\longrightarrow} u_0 \\ &\text{in } H^4(\Omega) \cap H_0^2(\Omega) \cap L^p(\Omega),\end{aligned} \tag{3.6}$$

$$\begin{aligned}u_m'(0) &= u_{1m} = \sum_{i=1}^{m} (u_1, w_j)\, w_j \overset{\text{as } m \longrightarrow \infty}{\longrightarrow} u_1 \\ &\text{in } H_0^2(\Omega) \cap L^{2\sigma-2}(\Omega),\end{aligned} \tag{3.7}$$

with u_0, u_1 are given functions on Ω, by virtue of the theory of ordinary differential equations, the system (3.5)-(3.7) has a unique local solution on some interval $[0, t_m)$. We claim that for any $T > 0$, such a solution can be extended to the whole interval $[0, T]$, as a consequence of the a priori estimates that shall be proven in the next step. We denote by C, C_k or c_k the constants which are independent of m, the initial data u_0 and u_1.

Multiplying the equation (3.5) by $K_{jm}'(t)$ and performing the summation over $j = 1, ..., m$, the integration par parts gives

$$E_m'(t) + |\nabla u_m'(t)|^2 + \alpha\left(g\left(u_m'(t)\right), u_m'(t)\right) = 0,\ \forall t \geq 0, \tag{3.8}$$

where

$$E_m(t) = \frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}|\Delta u_m(t)|^2 + \frac{1}{p}\|u_m(t)\|_p^p - \beta\int_\Omega F(x, u_m(t))\, dx, \tag{3.9}$$

by (H3), and Young inequality, we have

$$\begin{aligned}-\int_\Omega F(x, u_m)\, dx &\geq -\frac{1}{p}\int_\Omega k_1(x)\, |u_m|\, dx - \frac{1}{p}\int_\Omega u_m f(x, u_m)\, dx \\ &\geq -\varepsilon C_*^2 |\Delta u_m(t)|^2 - C_\varepsilon |k_1(x)|^2 - \frac{1}{p}\int_\Omega u_m f(x, u_m)\, dx,\end{aligned} \tag{3.10}$$

by using hypotheses (H4), Young's inequality yields

$$
\begin{aligned}
\frac{1}{p}\int_{\Omega} u_m f\left(x,u_m\right)dx &\leq \frac{1}{p}\left|f\left(x,u_m\right)\right|\left|u_m\right| \\
\leq \frac{l_1^2}{p}\varepsilon\int_{\Omega}\left(\left|u_m\right|^{2\theta}+\left|k_2\left(x\right)\right|^2\right)dx &+ \frac{c\left(\varepsilon,p\right)}{p^2}\int_{\Omega}\left|u_m\right|^2 dx \\
&= \frac{l_1^2}{p}\varepsilon\left\|u_m\right\|_{2\theta}^{2\theta}+\frac{l_1^2}{p}\varepsilon\left|k_2\left(x\right)\right|^2+\frac{c\left(\varepsilon,p\right)}{p^2}\left\|u_m\right\|_p^2 \\
&\leq \frac{l_1^2}{p}\varepsilon\left(\frac{p-2\theta}{p}+\frac{2\theta}{p}\left\|u_m\right\|_p^p\right)+\frac{l_1^2}{p}\varepsilon\left|k_2\left(x\right)\right|^2 \\
&\quad +C'\left(\varepsilon,p\right)+\frac{1}{p^2}\left\|u_m\right\|_p^p,
\end{aligned} \tag{3.11}
$$

substituting (3.11) in (3.10), and chosen $\varepsilon \leq C_0 = \min\left(\frac{1}{2C_*^2};\frac{p}{2\theta l_1^2+1}\right)$, (3.9) becomes

$$
E_m\left(t\right) \geq \frac{1}{2}\left|u_m'(t)\right|^2+C_1\left|\Delta u_m(t)\right|^2+C_2\left\|u_m\right\|_p^p-C_3\left(1+K_1+K_2\right), \tag{3.12}
$$

or

$$
\left|u_m'(t)\right|^2+\left|\Delta u_m(t)\right|^2+\left\|u_m\right\|_p^p \leq C_4\left(E_m\left(t\right)+K_1+K_2+1\right), \tag{3.13}
$$

where

$$
\begin{gathered}
0<C_1\leq\left(1-C_0C_*^2\right),\ 0<C_2\leq\left(\frac{1}{p}-\frac{2\theta l_1^2+1}{p^2}C_0\right), \\
C_3=\max\left(C_\varepsilon;\frac{l_1^2}{p}\varepsilon;C'\left(\varepsilon,p\right)+\frac{l_1^2}{p}\varepsilon\frac{p-2\theta}{p}\right), \\
C_4=\max\left(\frac{1}{\min\left(\frac{1}{2},C_1,C_2\right)},C_3\right).
\end{gathered}
$$

Thus, it follows from (3.8), and (3.12) that, for any $m=1,2,...$, and $t\geq 0$,

$$
\begin{aligned}
&\left|u_m'(t)\right|^2+\left|\Delta u_m(t)\right|^2+\left\|u_m\left(t\right)\right\|_p^p+\int_0^t\left|\nabla u_m'\left(s\right)\right|^2 ds \\
&+\alpha\int_0^t\left(g\left(u_m'\left(s\right)\right),u_m'\left(s\right)\right)ds \leq C_4\left(E_m\left(0\right)+K_1+K_2+1\right).
\end{aligned} \tag{3.14}
$$

By assumption (H2)-(H4), according to the Hölder's inequality, we have

$$
\begin{aligned}
\left|\int_{\Omega}F\left(x,u_{0m}\right)dx\right| &\leq \frac{1}{p}\int_{\Omega}k_1\left(x\right)\left|u_{0m}\right|dx+\frac{1}{p}\int_{\Omega}u_{0m}f\left(x,u_{0m}\right)dx \\
&\leq C\left(\left|u_m(0)\right|^2+\left|k_1\left(x\right)\right|^2+\left\|u_m\left(0\right)\right\|_p^p+\left|k_2\left(x\right)\right|^2+\left|u_m\left(0\right)\right|^2\right).
\end{aligned} \tag{3.15}
$$

Then using (3.6), (3.7), (3.8), and (3.9) we obtain that

$$
\begin{aligned}
E_m(t) \leq E_m(0) &= \frac{1}{2}|u_{1m}|^2 + \frac{1}{p}\|u_{0m}\|_p^p \\
&+ \frac{1}{2}|\Delta u_{0m}|^2 - \beta \int_\Omega F(x, u_{0m})\, dx \qquad (3.16)\\
&\leq C_4\left(|u_{1m}|^2 + \|u_{0m}\|_p^p + |\Delta u_{0m}|^2 + |u_{0m}|^2 + K_1 + K_2\right) \leq C,
\end{aligned}
$$

for some $C > 0$, where $K_1 = \|k_1\|_\infty^2$, $K_2 = \|k_2\|_\infty^2$.
Hence, for any $t \geq 0$, and $m = 1, 2, ...$, from (3.14), and (3.16) we get

$$
\begin{aligned}
|u_m'(t)|^2 + |\Delta u_m(t)|^2 + \int_0^t |\nabla u_m'(s)|^2\, ds + \|u_m(t)\|_p^p \\
+ \alpha \int_0^t \int_\Omega g(u_m'(s))\, u_m'(s)\, dxds \\
\leq C. \qquad (3.17)
\end{aligned}
$$

By the growth conditions, the estimate (3.17), and as $2\theta \leq p$, we have

$$
|f(u_m)|^2 \leq Cl_1\left(|u_m|^{2\theta} + |k_2(x)|^2\right) \leq C\left(||u_m||_p^{2\theta} + \|k_2\|_\infty^2\right) \leq C.
$$

With this estimate we can extend the approximate solution $u_m(t)$ to the interval $[0, T]$ and the following a priori estimates

$$
\begin{cases}
u_m \text{ is bounded in } L^\infty(0, T; L^p(\Omega)), \\
u_m' \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right), \\
\nabla u_m' \text{ is bounded in } L^2\left(0, T; L^2(\Omega)\right), \\
g(u_m').u_m' \text{ is bounded in } L^1(\Omega \times (0, T)), \\
\Delta u_m(t) \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right), \\
f(u_m) \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right),
\end{cases} \qquad (3.18)
$$

hold. □

Lemma 3.2. *There exists a constant $K > 0$ such that*

$$
||g(u_m'(t))||_{L^{\frac{\sigma}{\sigma-1}}(\Omega\times[0,T])} \leq K,
$$

for all $m \in \mathbb{N}$.

Proof. From (H2), Holder's, and Young's inequalities gives

$$
\begin{aligned}
\int_0^T \int_\Omega |g(u_m')|^{\frac{\sigma}{\sigma-1}}\, dxdt &= \int_0^T \int_\Omega |g(u_m')|\, |g(u_m')|^{\frac{1}{\sigma-1}}\, dxdt \\
\leq \int_0^T \int_\Omega |g(u_m'(t))| &\left(d_1 |u_m'(t)| + d_2 |u_m'(t)|^{\sigma-1}\right)^{\frac{1}{\sigma-1}} dxdt \\
\leq C \int_0^T \int_\Omega &|g(u_m'(t))| \left(|u_m'(t)|^{\frac{1}{\sigma-1}} + |u_m'(t)|\right) dxdt \\
= C \int_0^T &\int_\Omega |g(u_m'(t))|\, |u_m'(t)|^{\frac{1}{\sigma-1}}\, dxdt
\end{aligned}
$$

$$+C\int_0^T\int_\Omega |g(u_m'(t))|\,|u_m'(t)|\,dxdt$$

$$\leq \frac{\sigma-1}{\sigma}\int_0^T\int_\Omega |g(u_m')|^{\frac{\sigma}{\sigma-1}}\,dxdt + C(\sigma)\int_0^T\int_\Omega |u_m'(t)|^{\frac{\sigma}{\sigma-1}}\,dxdt$$

$$+C\int_0^T\int_\Omega |g(u_m'(t))|\,|u_m'(t)|\,dxdt,$$

therefore

$$\frac{1}{\sigma}\int_0^T\int_\Omega |g(u_m'(t))|^{\frac{\sigma}{\sigma-1}}\,dxdt \leq C(\sigma)\int_0^T\int_\Omega |u_m'(t)|^{\frac{\sigma}{\sigma-1}}\,dxdt$$

$$+C\int_0^T\int_\Omega |g(u_m'(t))|\,|u_m'(t)|\,dxdt$$

$$\leq C\int_0^T ||u_m'(t)||_2^{\frac{\sigma}{\sigma-1}}\,dt + C\int_0^T\int_\Omega |g(u_m'(t))|\,|u_m'(t)|\,dxdt,$$

hence, by (3.18), we deduce

$$\int_0^T\int_\Omega |g(u_m'(t))|^{\frac{\sigma}{\sigma-1}}\,dxdt \leq K. \qquad \square$$

Lemma 3.3. *There exists a constant $M > 0$ such that*

$$|u_m''(t)| + |\Delta u_m'(t)| + \int_0^T |\nabla u_m''(t)|\,dt \leq M,$$

for all $m \in \mathbb{N}$.

Proof. From (3.5) we obtain

$$|u_m''(0)| \leq |u_{0m}|^{p-1} + \left|\Delta^2 u_{0m}\right| + |\Delta u_{1m}| + \alpha\,|g(u_{1m})| + \beta\,|f(u_{0m})|,$$

by (H4) we have

$$|f(u_{0m})|^2 \leq l_1\left(|u_{0m}|^{2\theta} + |k_2(x)|^2\right) \leq C\left(||\Delta u_{0m}||_2^{2\theta} + \|k_2\|_\infty^2\right),$$

Since $g(u_{1m})$ is bounded in $L^2(\Omega)$ by (H2), from (3.6) and (3.7) we obtain

$$|u_m''(0)| \leq C.$$

Differentiating (3.5) with respect to t, we get

$$(u_m''', w_j) + \left(\Delta^2 u_m', w_j\right) - (\Delta u_m'', w_j) + (p-1)\left(|u_m|^{p-2} u_m', w_j\right)$$
$$+\alpha\,(g'(u_m')\,u_m'', w_j) = \beta\,(f'(u_m)\,u_m', w_j). \tag{3.19}$$

Multiplying it by $K_{jm}''(t)$ and summing over j from 1 to m , according to the Hölder's inequality, to find

$$\frac{1}{2}\frac{d}{dt}\left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2\right) + |\nabla u_m''(t)|^2 + \alpha\,(g'(u_m')\,u_m'', u_m'') \tag{3.20}$$
$$\leq (p-1)\int_\Omega |u_m|^{p-2}\,|u_m'|\,|u_m''|\,dx + \beta\int_\Omega |f'(u_m)|\,|u_m'|\,|u_m''|\,dx.$$

By choosing λ satisfies the inequalities

$$\begin{cases} \lambda+1 \leq \min\left(\frac{p}{2(\theta-1)}, \frac{n}{n-4}\right) & \text{if } n \geq 5, \\ \lambda+1 \leq \frac{p}{2(\theta-1)} & \text{if } n = 1,2,3,4, \end{cases}$$

then by using (H4), estimates (3.18) and generalized Hölder's inequality, we deduce that

$$\begin{aligned} &\int_\Omega |f'(u_m)|\,|u_m'|\,|u_m''|\,dx \\ &\leq \left\| l_1\left(|u_m|^{\theta-1} + k_3(x)\right)\right\|_{2(\lambda+1)}^{\lambda} ||u_m'||_{2(\lambda+1)} \|u_m''\|_2 \\ &\leq C\left(\left\| |u_m|^{\theta-1}\right\|_{2(\lambda+1)}^{\lambda} + \|k_3(x)\|_{2(\lambda+1)}^{\lambda}\right) ||u_m'||_{2(\lambda+1)} \|u_m''\|_2 \\ &\leq C\left(\|u_m\|_p^{\lambda(\theta-1)} + \|k_3(x)\|_p^{\lambda}\right) ||\Delta u_m'||_2 \|u_m''\|_2 \\ &\leq C_5\left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2\right), \end{aligned} \tag{3.21}$$

where C_1 and C_2 are positive constants independent of m and $t \in [0,T]$.
By same manner, using condition (H1), Young's inequality, Sobolev embedding, and estimate (3.18) we reach to

$$\begin{aligned} \int_\Omega |u_m|^{p-2} |u_m'|\,|u_m''|\,dx &\leq \left\| |u_m|^{p-2}\right\|_n \|u_m'\|_{\frac{2n}{n-2}} \|u_m''\|_2 \\ &\leq C\,||\Delta u_m'||_2 \|u_m''\|_2 \leq C_5\left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2\right). \end{aligned} \tag{3.22}$$

Combining (3.20), (3.21) and (3.22) we deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2\right) + |\nabla u_m''(t)|^2 + \alpha\left(g'(u_m')\,u_m'', u_m''\right) \\ \leq C_6\left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2\right). \end{aligned}$$

Integrating the last inequality over $(0,t)$ and applying Gronwall's lemma, we obtain

$$|u_m''(t)| + |\Delta u_m'(t)| + \int_0^t |\nabla u_m''(t)|^2\,ds \leq C \text{ for all } t \geq 0.$$

Therefore

$$\begin{aligned} u_m'' &\text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \Delta u_m' &\text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \nabla u_m'' &\text{ is bounded in } L^2\left(0,T;L^2(\Omega)\right), \end{aligned} \tag{3.23}$$

it follows from (3.23), (u_m') is bounded in $L^\infty\left(0,T;H_0^2(\Omega)\right)$.
Furthermore, by applying the Lions-Aubin compactness Lemma in [7], we claim that

$$u_m' \text{ is compact in } L^2\left(0,T;L^2(\Omega)\right), \tag{3.24}$$

From (3.18) and (3.23), there exists a subsequence of (u_m), still denote by (u_m), such that

$$\begin{cases} u_m \longrightarrow u \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\right), \\ u_m \longrightarrow u \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right), \\ u'_m \longrightarrow u' \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\right), \\ u'_m \longrightarrow u' \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right), \\ u''_m \longrightarrow u'' \text{ weak star in } L^\infty\left(0,T;L^2(\Omega)\right), \\ g\left(u'_m\right) \longrightarrow \chi \text{ weak star in } L^{\frac{\sigma}{\sigma-1}}\left(\Omega\times(0,T)\right), \\ f\left(u_m\right) \longrightarrow \zeta \text{ weak star in } L^\infty\left(0,T;L^2(\Omega)\right). \end{cases} \tag{3.25}$$

Using the compactness of $H_0^2(\Omega)$ to $L^2(\Omega)$, it is easy to see that

$$\int_0^T\int_\Omega |u_m|^{p-2}\, u_m v dxdt \to \int_0^T\int_\Omega |u|^{p-2}\, uvdxdt, \text{ for all } v\in L^\sigma\left(0,T;H_0^2(\Omega)\right),$$

as $m\to\infty$.

By (H2), and estimates (3.25) we have

$$g\left(u'_m\right) \longrightarrow g\left(u'\right) \text{ a.e.in } \Omega\times(0,T).$$

Therefore, from [7, Chapter1,Lemma1.3], we infer that

$$g\left(u'_m\right) \longrightarrow g\left(u'\right) \text{ weak star in } L^{\frac{\sigma}{\sigma-1}}\left(0,T;L^{\frac{\sigma}{\sigma-1}}\right),$$

as $m\to\infty$, and this implies that

$$\int_0^T\int_\Omega g\left(u'_m\right) vdxdt \to \int_0^T\int_\Omega g\left(u'\right) vdxdt \text{ for all } v\in L^\sigma\left(0,T;H_0^2(\Omega)\right).$$

By the same manner using the growth conditions in (H4) and estimate (3.25), we see that

$$\int_0^T\int_\Omega \left|f\left(u_m\right)\right|^{\frac{\theta+1}{\theta}} dxdt$$

is bounded and

$$f\left(u_m\right) \longrightarrow f(u) \text{ a.e.in } \Omega\times(0,T),$$

then

$$f\left(u_m\right) \longrightarrow f(u) \text{ weak star in } L^{\frac{\theta+1}{\theta}}\left(0,T;L^{\frac{\theta+1}{\theta}}\right),$$

as $m\to\infty$, and this implies that

$$\int_0^T\int_\Omega f\left(u_m\right) vdxdt \to \int_0^T\int_\Omega f(u)\, vdxdt \text{ for all } v\in L^\theta\left(0,T;H_0^2(\Omega)\right).$$

It follows at once from all estimates that for each fixed $v\in L^\theta\left(0,T;H_0^2(\Omega)\right)\cap L^\sigma\left(0,T;H_0^2(\Omega)\right)$,

$$\int_0^T\int_\Omega \left(u''_m + \Delta^2 u_m - \Delta u'_m + |u_m|^\rho\, u_m + \alpha g\left(u'_m\right) - \beta f\left(u_m\right)\right) vdxdt$$
$$\to \int_0^T\int_\Omega \left(u'' + \Delta^2 u - \Delta u' + |u|^{p-2}\, u + \alpha g\left(u'\right) - \beta f(u)\right) vdxdt,$$

as $m\to\infty$.

Consequently

$$\int_0^T \int_\Omega \left(u'' + \Delta^2 u - \Delta u' + |u|^{p-2} u + \alpha g(u') - \beta f(u)\right) v dx dt = 0,$$
$$\forall v \in L^\theta\left(0,T; H_0^2(\Omega)\right) \cap L^\sigma\left(0,T; H_0^2(\Omega)\right).$$

This means that the problem admit a weak solution u satisfying (1.1), and (3.1)-(3.4). ☐

Theorem 3.4. *Under the hypotheses of the Theorem* 3.1*, we have the solution u given by Theorem* 3.1*, is unique.*

Proof. Let u and v are two solutions, in the sense of the Theorem 3.1. Then $w = u - v$ satisfies

$$w'' + \left(\Delta^2 u - \Delta^2 v\right) - \Delta w' + \alpha\left(g(u') - g(v')\right)$$
$$+(|u|^{p-2} u - |v|^{p-2} v) = \beta\left(f(u) - f(v)\right), \tag{3.26}$$

$$w(0) = w'(0) = 0 \text{ in } \Omega, \tag{3.27}$$

$$w = \partial_\eta w = 0 \text{ on } \Sigma, \tag{3.28}$$

$$w \in L^p(0,T; W \cap L^p(\Omega)), \tag{3.29}$$

$$w' \in L^2(0,T; H_0^2(\Omega)). \tag{3.30}$$

Let's multiply the two members of (3.26) by w' and integrate on Ω. According to the Green's formula and conditions (3.28), integrating by part the result on $[0,t]$, using conditions (3.27) to find that

$$\frac{1}{2}\left(|w'(t)|^2 + |\Delta w|^2\right) \le \int_0^t \int_\Omega \left||u|^{p-2} u - |v|^{p-2} v\right| |w'| dx ds \tag{3.31}$$
$$+\beta \int_0^t \int_\Omega |f(u) - f(v)| |w'| dx ds.$$

According to the Hölder's, Young's inequalities, condition (H1), the estimates (3.25) the first term on the right-hand side of (3.31) can be estimated as follows:

$$\int_0^t \int_\Omega \left||u|^{p-2} u - |v|^{p-2} v\right| |w'| dx ds$$
$$\le (p-1) \int_0^t \left(\left\||u|^{p-2}\right\|_{L^n(\Omega)} + \left\||v|^{p-2}\right\|_{L^n(\Omega)}\right) \|w\|_{L^{\frac{2n}{n-2}}(\Omega)} \|w'\|_{L^2(\Omega)} ds$$
$$\le C \int_0^t \left(\|u\|_{L^{n(p-2)}(\Omega)}^{p-2} + \|v\|_{L^{n(p-2)}(\Omega)}^{p-2}\right) \|\Delta w\|_{L^2(\Omega)} \|w'\|_{L^2(\Omega)} ds \tag{3.32}$$
$$\le C \int_0^t \left(\|\Delta u\|_{L^2(\Omega)}^{p-2} + \|\Delta v\|_{L^2(\Omega)}^{p-2}\right) \|\Delta w\|_{L^2(\Omega)} \|w'\|_{L^2(\Omega)} ds$$
$$\le C \int_0^t \left(|w'(s)|^2 + |\Delta w(s)|^2\right) ds.$$

Now let $U_\varepsilon = \varepsilon u + (1-\varepsilon) v$, $0 \le \varepsilon \le 1$, by the growth conditions, for the second term of the right side to (3.31), we have

$$\begin{aligned}
\left|\int_0^t \int_\Omega |f(u) - f(v)| \, |w'| \, dxdt\right| &= \left|\int_0^t \int_\Omega \int_0^1 \frac{d}{d\varepsilon} f(U_\varepsilon) \, d\varepsilon w' dxds\right| \\
&\le \int_0^t \int_\Omega \left|\int_0^1 \frac{d}{d\varepsilon} f(U_\varepsilon) \, d\varepsilon\right| |w'| \, dxds \\
&\le \int_0^t \int_\Omega \int_0^1 \left|\frac{d}{d\varepsilon} f(U_\varepsilon) \, d\varepsilon\right| |w'| \, dxds \\
&\le l_1 \int_0^t \int_\Omega \int_0^1 \left(|U_\varepsilon|^{\theta-1} + |k_3(x)|\right) |u - v| \, |w'| \, d\varepsilon dxds \\
&\le C \int_0^t \int_\Omega \left(|u|^{\theta-1} + |v|^{\theta-1} + |k_3(x)|\right) |w(s)| \, |w'(s)| \, dxds = I.
\end{aligned}$$

Using the generalized Hölder's, Young's inequalities, and the estimates (3.25), and choosing λ such that

$$\begin{cases} \lambda + 1 \le \frac{n}{(\theta-1)(n-4)} \text{ if } n \ge 5, \\ 2 \le \lambda + 1 < \infty \text{ if } n = 1, 2, 3, 4, \end{cases}$$

we infer

$$\begin{aligned}
I &\le C \int_0^t \left\| |u|^{\theta-1} + |v|^{\theta-1} + |k_3(x)| \right\|_{2(\lambda+1)}^{\lambda} ||w||_{2(\lambda+1)} \|w'\|_2 \\
&\le C \int_0^t \left(\left\| |u|^{\theta-1} \right\|_{2(\lambda+1)}^{\lambda} + \left\| |v|^{\theta-1} \right\|_{2(\lambda+1)}^{\lambda} + \|k_3(x)\|_{2(\lambda+1)}^{\lambda} \right) ||w||_{2(\lambda+1)} \|w'\|_2 \, ds \\
&\le C \int_0^t \left(\|\Delta u\|_2^{\lambda(\theta-1)} + \|\Delta v\|_2^{\lambda(\theta-1)} + \|k_3(x)\|_\infty^{\lambda} \right) ||\Delta w||_2 \|w'\|_2 \, ds \\
&\le C \int_0^t ||\Delta w||_2 \|w'\|_2 \, ds \le C \int_0^t \left(|w'(s)|^2 + |\Delta w(s)|^2 \right) ds. \qquad (3.33)
\end{aligned}$$

Combining (3.31), (3.32) and (3.33) to obtain

$$|w'(t)|^2 + |\Delta w(t)|^2 \le C \int_0^t \left(|w'(s)|^2 + |\Delta w(s)|^2 \right) ds.$$

The integral inequality and Gronwall's lemma show that $w = 0$. □

4. Global existence

In this section, we discuss the global existence of the solution for problem (1.1). In order to state and prove our main results, we first introduce the following functions

$$I(t) = I(u(t)) = |\Delta u(t)|^2 - \beta \int_\Omega f(u(t)) \, u(x,t) \, dx - \beta \int_\Omega k_1(x) \, |u(x,t)| \, dx, \quad (4.1)$$

$$J(t) = J(u(t)) = \frac{1}{2} |\Delta u|^2 - \beta \int_\Omega F(x,u) \, dx, \quad (4.2)$$

$$E(t) = E(u(t), u'(t)) = J\left(u\left(t\right)\right) + \frac{1}{2}\left|u_t(t)\right|_2^2 + \frac{1}{p}\left\|u(t)\right\|_p^p. \tag{4.3}$$

And the stable set as

$$W = \left\{u : u \in H_0^2\left(\Omega\right), I\left(t\right) > 0\right\} \cup \left\{0\right\}. \tag{4.4}$$

The next lemma shows that our energy functional (4.3) is a nonincreasing function along with the solution of (1.1).

Lemma 4.1. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$E'\left(t\right) = -\left|\nabla u'\left(t\right)\right|^2 - \alpha \int_\Omega u'\left(t\right) g\left(u'\left(t\right)\right) dx \leq 0. \tag{4.5}$$

Proof. By multiplying equation (1.1) by u' and integrate over Ω, using integrate by parts and summing up the product results,

$$E\left(t\right) - E\left(0\right) = -\int_0^t \left|\nabla u'\left(s\right)\right|^2 ds - \alpha \int_0^t \int_\Omega u'\left(s\right) g\left(u'\left(s\right)\right) dxds \text{ for } t \geq 0. \qquad \square$$

Lemma 4.2. *Suppose that (H1)-(H4) hold, let $u_0 \in W$ and $u_1 \in H_0^2(\Omega)$ such that*

$$\gamma = \beta C_*^{\theta+1}\left(\frac{2p}{p-2}E\left(0\right)\right)^{\frac{\theta-1}{2}}\left(l_1 + l_1\left|\left|k_2\left(x\right)\right|\right|_\infty + \left|\left|k_1\left(x\right)\right|\right|_\infty\right) < 1. \tag{4.6}$$

Then $u \in W$ for each $t \geq 0$, where C_* is the Sobolev–Poincaré embedding such that for all $2 < p \leq \frac{2n}{n-4}$ $(n \geq 5)$, $(2 \leq p < \infty$ if $n = 1, 2, 3, 4)$ we have

$$\left\|u\left(t\right)\right\|_p \leq C_* \left\|\Delta u\left(t\right)\right\|_2, \ \forall u \in H_0^2(\Omega).$$

Proof. Since $I\left(0\right) > 0$, by the continuity, there exists $0 < T_m < T$ such

$$I\left(t\right) \geq 0, \ \forall t \in \left[0, T_m\right],$$

this gives from (4.2), and (H3),

$$E\left(t\right) \geq J\left(t\right) = \frac{1}{p}I\left(t\right) + \frac{p-2}{2p}\left|\Delta u\right|^2$$
$$+\frac{\beta}{p}\left(\int_\Omega f\left(u\right) u dx + \int_\Omega k_1\left(x\right)\left|u\right| dx - p\int_\Omega F\left(x, u\right) dx\right) \geq \frac{p-2}{2p}\left|\Delta u\right|^2. \tag{4.7}$$

By using (4.7), (4.3), and (4.5),

$$\left|\Delta u\right|^2 \leq \frac{2p}{p-2}J\left(t\right) \leq \frac{2p}{p-2}E\left(t\right) \leq \frac{2p}{p-2}E\left(0\right). \tag{4.8}$$

By recalling (H1), (H2), (4.8), (4.6), Cauchy-Schwartz inequality, and Sobolev embedding we have

$$
\begin{aligned}
\beta \int_\Omega f(u)\, u dx + \beta \int_\Omega k_1(x)\, |u|\, dx &\le \beta \int_\Omega |f(u)|\, |u|\, dx + \beta \int_\Omega |k_1(x)|\, |u|\, dx \\
&\le \beta l_1 \int_\Omega |u|^{\theta+1} dx + \beta l_1 \int_\Omega |k_2(x)|\, |u|\, dx + \beta \int_\Omega |k_1(x)|\, |u|\, dx \\
&\le \beta l_1 \|u(t)\|_{\theta+1}^{\theta+1} + \beta \left(l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty\right) \|u(t)\|_{\theta+1}^{\theta+1} \\
&\le \beta l_1 C_*^{\theta+1} |\Delta u(t)|^{\theta+1} + \beta C_*^{\theta+1} \left(l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty\right) |\Delta u(t)|^{\theta+1} \qquad (4.9) \\
&= \beta l_1 C_*^{\theta+1} |\Delta u(t)|^{\theta-1} |\Delta u(t)|^2 \\
&\quad + \beta C_*^{\theta+1} \left(l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty\right) |\Delta u(t)|^{\theta-1} |\Delta u(t)|^2 \\
&\le \beta C_*^{\theta+1} \left(\frac{2p}{p-2} E(0)\right)^{\frac{\theta-1}{2}} \left(l_1 + l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty\right) |\Delta u|^2 \\
&< |\Delta u|^2 \text{ on } [0, T_m].
\end{aligned}
$$

Therefore, by using (4.1), we conclude that $I(t) > 0$ for all $t \in [0, T_m]$. By repeating this procedure, and using the fact that

$$
\lim_{t \to T_m} \beta C_*^{\theta+1} \left(\frac{2p}{p-2} E(t)\right)^{\frac{\theta-1}{2}} \left(l_1 + l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty\right) \le D < 1,
$$

T_m is extended to T. □

Lemma 4.3. *Let the assumptions (4.6) holds. Then there exists $\eta = 1 - \gamma$ such that*

$$
\beta \int_\Omega f(u)\, u dx + \beta \int_\Omega k_1(x)\, |u|\, dx \le (1 - \eta) |\Delta u|^2, \tag{4.10}
$$

and therefore

$$
|\Delta u|^2 \le \frac{1}{\eta} I(t). \tag{4.11}
$$

Proof. From (4.9) we have

$$
\beta \int_\Omega f(u)\, u dx + \beta \int_\Omega k_1(x)\, |u|\, dx \le \gamma |\Delta u|^2.
$$

We get (4.10) by taking $\eta = 1 - \gamma > 0$, and by using (4.10), from (4.1) we get the result (4.11). □

Theorem 4.4. *Suppose that (H1)-(H4) hold. Let $u_0 \in W$ satisfying (4.6). Then the solution of problem (1.1) is global.*

Proof. It sufficient to show that $\|u_t\|_2^2 + |\Delta u|^2$ is bounded independently to t. To see this we use (4.1), (4.3), and (H3) to obtain

$$E(0) \geq E(t) = \frac{1}{2} |\Delta u|^2 - \beta \int_\Omega F(x,u)\, dx + \frac{1}{2} \|u'(t)\|_2^2 + \frac{1}{p} \|u(t)\|_p^p$$
$$\geq \frac{1}{2} |\Delta u|^2 - \frac{\beta}{p} \int_\Omega f(u)\, u dx - \frac{\beta}{p} \int_\Omega k_1(x) |u|\, dx$$
$$+ \frac{1}{2} \|u'(t)\|_2^2 + \frac{1}{p} \|u(t)\|_p^p = \frac{1}{2} |\Delta u|^2 + \frac{1}{p} \left(I(t) - |\Delta u|^2 \right)$$
$$+ \frac{1}{2} \|u'(t)\|_2^2 + \frac{1}{p} \|u(t)\|_p^p$$
$$= \frac{p-2}{2p} |\Delta u|^2 + \frac{1}{p} I(t) + \frac{1}{2} \|u'(t)\|_2^2 + \frac{1}{p} \|u(t)\|_p^p$$
$$\geq \frac{1}{2} \|u'(t)\|_2^2 + \frac{p-2}{2p} |\Delta u(t)|^2,$$

since $I(t) \geq 0$, and $p > 2$. Therefore

$$\|u'(t)\|_2^2 + |\Delta u|^2 \leq \max\left(2, \frac{2p}{p-2}\right) E(0).$$

These estimates imply that the solution $u(t)$ exist globally in $[0, +\infty[$. □

5. Blow-up of solution

In this section, after some estimates, we show that the solution of problem (1.1) blows up in finite time under the assumption $E(0) < 0$, where

$$E(t) = E(u(t), u'(t)) = \frac{1}{2} |u'(t)|^2 + \frac{1}{2} |\Delta u(t)|^2 + \frac{1}{p} \|u(t)\|_p^p - \beta \int_\Omega F(x, u(t))\, dx. \quad (5.1)$$

Remark 5.1. We set

$$H(t) = -E(t), \quad (5.2)$$

we multiply Eq.(1.1) by $-u'$ and integrate over Ω, using (H2) to get

$$H'(t) = |\nabla u'(t)|^2 + \alpha \int_\Omega u'(t)\, g(u'(t))\, dx \geq \alpha d_0 \|u'(t)\|_\sigma^\sigma \text{ a.e. } t \in [0,T], \quad (5.3)$$

$H(t)$ is absolutely continuous, hence

$$0 < H(0) \leq H(t) \leq \beta \int_\Omega F(x,u)\, dx, \quad (5.4)$$

when

$$E(0) < 0.$$

We need the following lemma, easy to prove by using the definition of the energy corresponding to the solution

Lemma 5.2. *Let $2 < p \leq \frac{2n}{n-4}$ if $n \geq 5$ and $2 < p < \infty$ if $n \leq 4$. Then there exists a positive constant $C > 1$, depending only on Ω, such that*

$$\|u(t)\|_p^s \leq C\left(\|u(t)\|_p^p + |\Delta u(t)|^2\right), \text{ with } 2 \leq s \leq p, \tag{5.5}$$

for any $u \in H_0^2(\Omega)$. If u is the solution constructed in Theorem 3.1, then

$$\|u(t)\|_p^s \leq C\left(H(t) + \|u(t)\|_p^p + |u'(t)|^2 + \beta\int_\Omega F(x,u(t))\,dx\right), \tag{5.6}$$

with $2 \leq s \leq p$ on $[0,T)$.

Theorem 5.3. *Let the conditions of the Theorem 3.1 be satisfied. Assume further that*

$$E(0) < 0. \tag{5.7}$$

Then the solution (3.1) blows up in a finite time T.

Proof. We pose

$$\begin{cases} L(t) = |u(t)|^2 = \int\limits_\Omega |u(x,t)|^2\,dx, \\ L'(t) = 2(u(t), u'(t)), \\ L''(t) = 2|u'(t)|^2 + 2(u(t), u''(t)), \end{cases}$$

we define the function

$$G(t) = H^{1-a}(t) + \varepsilon L'(t) - 3\varepsilon p e^{T-t}\beta\int_\Omega F(x,u(t))\,dx$$
$$+\gamma_1\varepsilon t\|k_1(x)\|_\infty + \gamma_2\varepsilon t\|k_2(x)\|_\infty^\sigma, \; t \geq 0, \tag{5.8}$$

where γ_1, γ_2, $\varepsilon > 0$ are positives constants to be specified later, and

$$0 < a \leq \min\left(\frac{p-2}{2p}, \frac{p-\sigma}{(\theta+1)(\sigma-1)}\right) < 1, \tag{5.9}$$

derivative the Eq. (5.8), using Eq. (1.1), and hypotheses (H3) we obtain

$$\frac{d}{dt}G(t) = (1-a)H^{-a}(t)H'(t) + \varepsilon L''(t) + \gamma_1\varepsilon\|k_1(x)\|_\infty$$
$$+\gamma_2\varepsilon\|k_2(x)\|_\infty^\sigma + \frac{d}{dt}\left(-3p\varepsilon e^{T-t}\beta\int_\Omega F(x,u(t))\,dx\right)$$
$$= (1-a)H^{-a}(t)H'(t) + 2\varepsilon|u'(t)|^2 + 2\varepsilon(u(t), u''(t))$$
$$+\gamma_1\varepsilon\|k_1(x)\|_\infty + \gamma_2\varepsilon\|k_2(x)\|_\infty^\sigma$$
$$+3p\varepsilon e^{T-t}\beta\int_\Omega F(x,u(t))\,dx - 3p\varepsilon e^{T-t}\beta\int_\Omega f(u(t))u'(t)\,dx \tag{5.10}$$
$$= (1-a)H^{-a}(t)H'(t) + 2\varepsilon|u'(t)|^2 + 2\beta\varepsilon\int_\Omega u(t)f(u(t))\,dx - 2\varepsilon|\Delta u(t)|^2$$
$$-2\varepsilon\int_\Omega u(t)\Delta u'(t)\,dx - 2\varepsilon\|u(t)\|_p^p + \gamma_1\varepsilon\|k_1(x)\|_\infty + \gamma_2\varepsilon\|k_2(x)\|_\infty^\sigma$$
$$+3p\varepsilon e^{T-t}\beta\int_\Omega F(x,u(t))\,dx - 3p\varepsilon e^{T-t}\beta\int_\Omega f(u(t))u'(t)\,dx - 2\alpha\varepsilon\int_\Omega u(t)g(u'(t))\,dx.$$

We then exploit Holder's, Young's inequalities, and the hypotheses on g, to estimate the last term in (5.10) as

$$\begin{aligned} 2\alpha\varepsilon\left|\int_\Omega u(t)\,g(u'(t))\,dx\right| &\le 2\alpha\varepsilon d_1\int_\Omega |u'(t)|\,|u(t)|\,dx + 2\alpha\varepsilon d_2\int_\Omega |u'(t)|^{\sigma-1}|u(t)|\,dx \\ &\le 2\alpha\varepsilon d_1\frac{\delta^\sigma}{\sigma}\|u(t)\|_\sigma^\sigma + 2\alpha\varepsilon d_1\frac{\sigma-1}{\sigma}\delta^{\frac{\sigma}{1-\sigma}}\|u'(t)\|_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} \qquad (5.11)\\ &+2\alpha\varepsilon d_2\frac{\delta^\sigma}{\sigma}\|u(t)\|_\sigma^\sigma + 2\alpha\varepsilon d_2\frac{\sigma-1}{\sigma}\delta^{\frac{\sigma}{1-\sigma}}\|u'(t)\|_\sigma^\sigma \\ &= 2(d_1+d_2)\frac{\delta^\sigma}{\sigma}\alpha\varepsilon\|u(t)\|_\sigma^\sigma \\ &+2\alpha\varepsilon\frac{\sigma-1}{\sigma}\delta^{\frac{\sigma}{1-\sigma}}\left(d_1\|u'(t)\|_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + d_2\|u'(t)\|_\sigma^\sigma\right),\ \delta>0, \end{aligned}$$

because $\frac{\sigma}{\sigma-1}\le\sigma$, then by (5.3) we have

$$\begin{aligned} d_1\|u'(t)\|_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + d_2\|u'(t)\|_\sigma^\sigma &\le C(\Omega)^{\frac{\sigma-2}{\sigma}}d_1\|u'(t)\|_\sigma^{\frac{\sigma}{\sigma-1}} + \frac{d_2}{\alpha d_0}H'(t) \\ &\le C^*d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}\|u'(t)\|_\sigma^\sigma + \frac{d_2}{\alpha d_0}H'(t) \\ &\le \frac{1}{\alpha d_0}\left(C^*d_1C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right)H'(t). \qquad (5.12) \end{aligned}$$

By the boundary conditions we derive the following estimates

$$\int_\Omega u(t)\,\Delta u'(t)\,dx = \int_\Omega \Delta u(t)\,u'(t)\,dx \le \frac{1}{4}|\Delta u(t)|^2 + |u'(t)|^2. \qquad (5.13)$$

Using hypotheses (H4), Holder's, Young's inequalities, conditions (5.9), and (5.3) we have

$$\begin{aligned} \int_\Omega |f(u(t))|\,|u'(t)|\,dx &\le l_1\int_\Omega\left(|u|^\theta|u'(t)| + |k_2(x)|\,|u'(t)|\right)dx \\ &\le l_1\|u(t)\|_{2\theta}^\theta\|u'(t)\|_2 + l_1\frac{\sigma-1}{\sigma}\delta^{\frac{\sigma}{1-\sigma}}\|u'(t)\|_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + l_1\frac{\delta^\sigma}{\sigma}\|k_2(x)\|_\infty^\sigma \\ &\le \frac{l_1}{\sigma}C(\delta,\sigma)\delta^\sigma\|u(t)\|_{2\theta}^{2\theta} + \frac{1}{\sigma}l_1\delta^{\frac{\sigma}{1-\sigma}}\|u'(t)\|_2^2 \\ &+l_1\frac{\sigma-1}{\sigma}\delta^{\frac{\sigma}{1-\sigma}}\|u'(t)\|_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + l_1\frac{\delta^\sigma}{\sigma}\|k_2(x)\|_\infty^\sigma \\ &\le \frac{l_1}{\sigma}C^*C(\delta,\sigma)C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\|u\|_\sigma^\sigma \\ &+\frac{1}{\sigma}l_1C^*C(\Omega)^{\frac{\sigma-2}{2\sigma}}\delta^{\frac{\sigma}{1-\sigma}}\|u'(t)\|_\sigma^\sigma \\ &+l_1\frac{\sigma-1}{\sigma}\delta^{\frac{\sigma}{1-\sigma}}C^*C(\Omega)^{\frac{\sigma-2}{2\sigma}}\|u'(t)\|_\sigma^\sigma + l_1\frac{\delta^\sigma}{\sigma}\|k_2(x)\|_\infty^\sigma \\ &\le \frac{l_1}{\alpha d_0}C^*C(\Omega)^{\frac{\sigma-2}{2\sigma}}\delta^{\frac{\sigma}{1-\sigma}}H'(t) \\ &+\frac{l_1}{\sigma}C(\delta,\sigma)\delta^\sigma C^*C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\|u\|_\sigma^\sigma + l_1\frac{\delta^\sigma}{\sigma}\|k_2(x)\|_\infty^\sigma. \end{aligned}$$

By the hypotheses (H3), and the estimate (5.4) we have

$$\begin{aligned} 2\beta\int_\Omega u(t)\, f(u(t))\,dx &\geq 2\beta p\int_\Omega F(x)\,dx - 2\beta\int_\Omega k_1(x)\,|u(x)|\,dx \\ &\geq 2pH(t) - 2\beta\int_\Omega k_1(x)\,|u(x)|\,dx, \end{aligned} \tag{5.14}$$

and by Holder's, Young's inequalities,

$$\int_\Omega k_1(x)\,|u(x)|\,dx \leq C(\sigma,\alpha)\,\|k_1(x)\|_\infty + 2\alpha\frac{\delta^\sigma}{\sigma}\,\|u(t)\|_\sigma^\sigma. \tag{5.15}$$

By substituting in (5.10), and using (5.11)-(5.15), yields,

$$\begin{aligned} &\frac{d}{dt}G(t) \\ \geq&\left(\begin{array}{c}(1-a)\,H^{-a}(t)\\ -\frac{1}{\alpha d_0}\left(3p\varepsilon e^{T-t}\beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} + 2\alpha\varepsilon\frac{\sigma-1}{\sigma}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right)\right)\delta^{\frac{\sigma}{1-\sigma}}\end{array}\right)H'(t) \\ &+2p\varepsilon H(t) - 2\varepsilon\,\|u(t)\|_p^p - \frac{5}{2}\varepsilon\,|\Delta u(t)|^2 + (\gamma_1 - 2\beta C(\sigma,\alpha))\,\varepsilon\,\|k_1(x)\|_\infty \\ &+\left(\gamma_2 - 3p\varepsilon e^{T-t}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\varepsilon\,\|k_2(x)\|_\infty^\sigma + 3p\beta\varepsilon\int_\Omega F(x,u(s))\,dx \\ &-\varepsilon\left(3\theta p e^{T-t}\beta l_1 C(\delta,\sigma)\,C^* C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}} + 2\beta\alpha(d_1+d_2)\right)\frac{\delta^\sigma}{\sigma}\,\|u(t)\|_\sigma^\sigma, \\ &\forall\delta,\varepsilon>0. \end{aligned} \tag{5.16}$$

At this point, for a large positive constant λ to be chosen later, picking δ such that $\delta^{\frac{\sigma}{1-\sigma}} = \lambda H^{-a}(t) > 0$ in (5.16) we arrive for all $t > 0$ at

$$\begin{aligned} &\frac{d}{dt}G(t) \\ \geq&\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_0}\left(3p\varepsilon e^{T}\beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} + 2\alpha\varepsilon\frac{\sigma-1}{\sigma}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right)\right)\end{array}\right)H^{-a}(t)\,H'(t) \\ &+3\beta p\varepsilon\int_\Omega F(x,u)\,dx - 2\varepsilon\,\|u(t)\|_p^p - \frac{5}{2}\varepsilon\,|\Delta u(t)|^2 + 2p\varepsilon H(t) \\ &+(\gamma_1 - 2\beta C(\sigma,\alpha))\,\varepsilon\,\|k_1(x)\|_\infty \\ &+\left(\gamma_2 - 3p\varepsilon e^{T}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\varepsilon\,\|k_2(x)\|_\infty^\sigma \\ &-\varepsilon\left(3\theta p e^{T}\beta l_1 C(\delta,\sigma)\,C^* C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}} + 2\beta\alpha(d_1+d_2)\right)\frac{\lambda^{1-\sigma}}{\sigma}H^{a(\sigma-1)}(t)\,\|u(t)\|_\sigma^\sigma, \\ &\forall\delta,\ \varepsilon>0. \end{aligned} \tag{5.17}$$

By exploiting (5.4), we have

$$H^{a(\sigma-1)}(t)\,||u(t)||_{\sigma}^{\sigma} \leq \beta^{a(\sigma-1)} \left(\int_{\Omega} F(x,u)\,dx\right)^{a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma}, \tag{5.18}$$

from (H3) we have

$$\begin{aligned}\int_{\Omega} F(x,u)\,dx &\leq \frac{l_1}{p}\left(\int_{\Omega} |u(t)|^{\theta+1}\,dx + (|k_2(x)| + |k_1(x)|)\,|u|\right)\\
&\leq \frac{l_1}{p}\,||u(t)||_{\theta+1}^{\theta+1} + C\frac{l_1}{p}\left(\|k_1(x)\|_{\infty} + \|k_2(x)\|_{\infty}\right)||u(t)||_{\theta+1}^{\theta+1},\\
&\leq C\frac{l_1}{p}\,||u(t)||_{\theta+1}^{\theta+1}\end{aligned} \tag{5.19}$$

by condition (5.9), and the estimates (5.6) we confirm that

$$\begin{aligned}&\beta^{a(\sigma-1)}\left|\int_{\Omega} F(x,u)\,dx\right|^{a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma}\\
&\leq C\frac{l_1}{p}\beta^{a(\sigma-1)}\left(||u(t)||_{\theta+1}^{\theta+1}\right)^{a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma}\\
&= C\frac{l_1}{p}\beta^{a(\sigma-1)}\,||u(t)||_{\theta+1}^{(\theta+1)a(\sigma-1)}\,||u(t)||_{\sigma}^{\sigma}\\
&\leq C\frac{l_1}{p}\beta^{a(\sigma-1)}\,||u(t)||_{\theta+1}^{(\theta+1)a(\sigma-1)}\,||u(t)||_{\theta+1}^{\sigma}\\
&= C\frac{l_1}{p}\beta^{a(\sigma-1)}\,||u(t)||_{\theta+1}^{(\theta+1)a(\sigma-1)+\sigma}\\
&\leq \frac{l_1}{p}\beta^{a(\sigma-1)}C\left(H(t) + \|u(t)\|_p^p + |u'(t)|^2 + \beta\int_{\Omega} F(x,u)\,dx\right)\\
&\leq C\frac{l_1}{p}\beta^{a(\sigma-1)}\left(\begin{array}{c} H(t) + \|u(t)\|_p^p + |u'(t)|^2 + \beta\int_{\Omega} F(x,u)\,dx\\ + \|k_1(x)\|_{\infty} + \|k_2(x)\|_{\infty}^{\sigma}\end{array}\right)\end{aligned} \tag{5.20}$$

substituting (5.20) in (5.17) we obtain

$$\begin{aligned}\frac{d}{dt}G(t) \geq{}& \left((1-a) - \frac{\lambda}{\alpha d_0}\left(\begin{array}{c} 3p\varepsilon e^{T}\beta C^{*}C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\alpha\varepsilon^{\frac{\sigma-1}{\sigma}}\left(C^{*}d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right)\end{array}\right)\right) H^{-a}(t)\,H'(t)\\
&+3p\beta\varepsilon\int_{\Omega} F(x,u)\,dx - \frac{5}{2}\varepsilon\,|\Delta u(t)|^2 - 2\varepsilon\,\|u(t)\|_p^p\\
&+\varepsilon\left(\gamma_1 - 2\beta C(\sigma,\alpha)\right)\|k_1(x)\|_{\infty}\\
&+\varepsilon\left(\gamma_2 - 3p\varepsilon e^{T}\beta l_1\frac{\delta^{\sigma}}{\sigma}\right)\|k_2(x)\|_{\infty}^{\sigma}\end{aligned} \tag{5.21}$$

$$+\varepsilon\left(\begin{array}{c} 2pH(t)-\left(\begin{array}{c}3\theta pe^{T}\beta l_1 C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}+2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)} \\ \times C\left(\begin{array}{c} H(t)+\|u(t)\|_p^p+|u'(t)|^2+\beta\int\limits_\Omega F(x,u)\,dx \\ +\|k_1(x)\|_\infty+\|k_2(x)\|_\infty^\sigma\end{array}\right)\end{array}\right)$$

or

$$\begin{aligned}\frac{d}{dt}G(t)\geq&\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_0}\left(\begin{array}{c}3p\varepsilon e^{T}\beta C^{*}C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\alpha\varepsilon\frac{\sigma-1}{\sigma}\left(C^{*}d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)\end{array}\right)H^{-a}(t)H'(t)\\ &+3p\beta\varepsilon\int_\Omega F(x,u)\,dx-\frac{5}{2}\varepsilon|\Delta u(t)|^2-2\varepsilon\|u(t)\|_p^p\\ &+\varepsilon(\gamma_1-2\beta C(\sigma,\alpha))\|k_1(x)\|_\infty \qquad (5.22)\\ &+\varepsilon\left(\gamma_2-3p\varepsilon e^{T}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\|k_2(x)\|_\infty^\sigma\end{aligned}$$

$$+\varepsilon\left(\begin{array}{c}(5p-1)H(t)\\ -\left(\begin{array}{c}3\theta pe^{T-t}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\\ \times C\left(\begin{array}{c}H(t)+\|u(t)\|_p^p+|u'(t)|^2+\beta\int_\Omega F(x,u)\,dx\\ +\|k_1(x)\|_\infty+\|k_2(x)\|_\infty^\sigma\end{array}\right)\end{array}\right)-\varepsilon(3p-1)H(t).$$

By using the definition (5.2), the estimate (5.22) gives

$$\frac{d}{dt}G(t)\geq\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_0}\left(\begin{array}{c}3p\varepsilon e^{T}\beta C^{*}C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\alpha\varepsilon\frac{\sigma-1}{\sigma}\left(C^{*}d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)\end{array}\right)\times H^{-a}(t)H'(t)$$

$$+\varepsilon\left[\begin{array}{c}\left(\frac{3p-1}{2}\right)\\ -\left(C\left(\begin{array}{c}3\theta pe^{T}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\times|u'(t)|^2$$

$$+\left(\frac{3p-1}{2}-\frac{5}{2}\right)\varepsilon|\Delta u(t)|^2$$

$$+\varepsilon\left[\begin{array}{c}(\gamma_1-2\beta C(\sigma,\alpha))\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\|k_1(x)\|_\infty$$

$$+\varepsilon\left[\begin{array}{c}\left(\gamma_2-3p\varepsilon e^{T}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\|k_2(x)\|_\infty^\sigma$$

$$+\varepsilon\left[\begin{array}{c}\left(\frac{3p-1}{p}-2\right)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_{1}C\left(\delta,\sigma\right)C^{*}C\left(\Omega\right)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha\left(d_{1}+d_{2}\right)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_{1}}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]$$
$$\times\left\|u\left(t\right)\right\|_{p}^{p}$$

$$+\varepsilon\left[\begin{array}{c}3p-(3p-1)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_{1}C\left(\delta,\sigma\right)C^{*}C\left(\Omega\right)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha\left(d_{1}+d_{2}\right)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_{1}}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\beta\int_{\Omega}F\left(x,u\right)dx$$

$$+\varepsilon\left[\begin{array}{c}(5p-1)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_{1}C\left(\delta,\sigma\right)C^{*}C\left(\Omega\right)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha\left(d_{1}+d_{2}\right)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_{1}}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]H\left(t\right).$$

pose

$$C_{1}=C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_{1}C\left(\delta,\sigma\right)C^{*}C\left(\Omega\right)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha\left(d_{1}+d_{2}\right)\end{array}\right)\frac{1}{\sigma}\frac{l_{1}}{p}\beta^{a(\sigma-1)}\right),$$

we arrive at

$$\begin{aligned}\frac{d}{dt}G\left(t\right)\geq&\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_{0}}\varepsilon\left(\begin{array}{c}3pe^{T}\beta C^{*}C\left(\Omega\right)^{\frac{\sigma-2}{2\sigma}}\\ +2^{\frac{\sigma-1}{\sigma}}\left(C^{*}d_{1}C\left(\Omega\right)^{\frac{\sigma-2}{\sigma}}+d_{2}\right)\end{array}\right)\end{array}\right)H^{-a}\left(t\right)H'\left(t\right)\\ &+\varepsilon\left[\frac{3p-1}{2}-C_{1}\lambda^{1-\sigma}\right]\left|u'\left(t\right)\right|^{2}+\left(\frac{3p-1}{2}-\frac{5}{2}\right)\varepsilon\left|\Delta u\left(t\right)\right|^{2}\\ &+\varepsilon\left(\left(\gamma_{1}-2\beta C\left(\sigma,\alpha\right)\right)-C_{1}\lambda^{1-\sigma}\right)\left\|k_{1}\left(x\right)\right\|_{\infty}\\ &+\varepsilon\left(\left(\gamma_{2}-3p\varepsilon e^{T}\beta l_{1}\frac{\delta^{\sigma}}{\sigma}\right)-C_{1}\lambda^{1-\sigma}\right)\left\|k_{2}\left(x\right)\right\|_{\infty}^{\sigma}\\ &+\varepsilon\left[\frac{p-1}{p}-C_{1}\lambda^{1-\sigma}\right]\left\|u\left(t\right)\right\|_{p}^{p}+\varepsilon\left[1-C_{1}\lambda^{1-\sigma}\right]\beta\int_{\Omega}F\left(x,u\right)dx\\ &+\varepsilon\left(\left(5p-1\right)-C_{1}\lambda^{1-\sigma}\right)H\left(t\right).\end{aligned}\tag{5.23}$$

chosen $\gamma_{1}=1+2\beta C\left(\sigma,\alpha\right),\gamma_{2}=1+3p\varepsilon e^{T}\beta l_{1}\frac{\delta^{\sigma}}{\sigma}$ and λ satisfying the following inequality

$$\lambda\geq\lambda_{0}=\min\left(\sqrt[\sigma-1]{\frac{2C_{1}}{3p-1}},\ \sqrt[\sigma-1]{\frac{pC_{1}}{p-1}},\ \sqrt[\sigma-1]{C_{1}},\ \sqrt[\sigma-1]{\frac{C_{1}}{5p-1}}\right)$$

so that the coefficients of $H(t)$, $|u'(t)|^2$, $|\Delta u(t)|^2$, $\|u(t)\|_p^p$, $\|k_1(x)\|_\infty$, $\|k_2(x)\|_\infty$ and $\int_\Omega F(x,u)\,dx$ in (5.23) are strictly positive, hence we get

$$\frac{d}{dt}G(t) \geq \left(\begin{array}{c} (1-a) \\ -\frac{\lambda}{\alpha d_0}\varepsilon \left(\begin{array}{c} 3pe^T\beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \\ +2\alpha\frac{\sigma-1}{\sigma}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right) \end{array} \right) \end{array} \right) H^{-a}(t) H'(t)$$

$$+\omega\varepsilon \left(\begin{array}{c} H(t) + |u'(t)|^2 + \|u(t)\|_p^p + \int_\Omega F(x,u)\,dx \\ + \|k_1(x)\|_\infty + \|k_2(x)\|_\infty^\sigma \end{array} \right), \tag{5.24}$$

where ω is the minimum of these coefficients. We pick ε small enough, so that

$$0 < \varepsilon \leq \varepsilon_0 = \min \left(\begin{array}{c} \dfrac{1-a}{\frac{\lambda}{\alpha d_0}\left(\begin{array}{c} 3pe^T\beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \\ +2\alpha\frac{\sigma-1}{\sigma}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right) \end{array} \right)}; \\ \frac{H^{1-a}(0)}{-L'(0)+3pe^T\beta\int_\Omega F(x,u_0)dx} \end{array} \right)$$

therefore (5.24) take the form

$$\frac{d}{dt}G(t) \geq \omega\varepsilon \left(\begin{array}{c} H(t) + |u'(t)|^2 + \|u(t)\|_p^p \\ + \int_\Omega F(x,u)\,dx + \|k_1(x)\|_\infty + \|k_2(x)\|_\infty^\sigma \end{array} \right), \tag{5.25}$$

hence

$$G(t) \geq G(0) > 0 \text{ for all } t \geq 0.$$

The second term in (5.8), by applying Young's inequality we can estimate as follows

$$\frac{1}{2}L'(t) = (u(t), u'(t)) \leq c|u'(t)|\,\|u(t)\|_p \leq c\left(|u'(t)|^{2(1-a)} + \|u(t)\|_p^{\frac{2(1-a)}{1-2a}}\right),$$

so

$$|(u(t), u'(t))|^{\frac{1}{1-a}} \leq C\left(|u'(t)|^2 + \|u(t)\|_p^{\frac{2}{1-2a}}\right)$$

using Lemma (5.2) and the condition (5.9) we obtain

$$|(u(t), u'(t))|^{\frac{1}{1-a}}$$

$$\leq C\left(H(t) + |u'(t)|^2 + \|u(t)\|_p^p + \int_\Omega F(x,u)\,dx\right), \ \forall t \geq 0. \tag{5.26}$$

Consequently we have

$$G(t)^{\frac{1}{1-a}} = \left(H^{1-a}(t) + 2\varepsilon\int_\Omega u(x,t)u'(t)\,dx + \gamma_1\varepsilon t\|k_1(x)\|_\infty + \gamma_2\varepsilon t\|k_2(x)\|_\infty^\sigma\right)^{\frac{1}{1-a}}$$

$$\leq C\left(H(t) + \left|2\varepsilon\int_\Omega u(x,t)u'(t)\,dx\right|^{\frac{1}{1-a}} + |\gamma_1\varepsilon t\|k_1(x)\|_\infty|^{\frac{1}{1-a}} + |\gamma_2\varepsilon t\|k_2(x)\|_\infty^\sigma|^{\frac{1}{1-a}}\right)$$

$$\leq C\left(H(t) + |u'(t)|^2 + \|u(t)\|_p^p + \int_\Omega F(x,u)\,dx + \|k_1(x)\|_\infty + \|k_2(x)\|_\infty^\sigma\right). \tag{5.27}$$

We then combine (5.25), (5.26), and (5.27), to arrive at

$$\frac{d}{dt}G(t) \geq \rho G(t)^{\frac{1}{1-a}}, \tag{5.28}$$

where ρ is a constant depending on C, ω, and ε only, and not depend of u.
Integrate (5.28) over $(0,t)$ to get

$$G\left(t\right)^{\frac{a}{1-a}} \geq \frac{1}{G^{\frac{a-1}{a}}\left(0\right) - t\frac{a}{(1-a)}\rho}.$$

Therefore $G\ (t)$ blows up in a finite time T^{*} where

$$T^{*} \leq \frac{1-a}{a\rho G^{\frac{a}{1-a}}\left(0\right)}.$$

□

References

[1] Adams, R.A, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] Amroun, N.E., Benaissa, A., *Global existence and energy decay of solution to a Petrovesky equation with general dissipation and source term*, Georgian Math. J., **13**(2006), no. 3, 397–410.

[3] Dautray, R., Lions, J.L., *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*, vol. 3, Masson, Paris, 1985.

[4] Georgiev, V., Todorova, G., *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations., **109**(1994), no. 2, 295–308.

[5] Guesmia, A., *Existence globale et stabilisation interne non linéaire d'un systèm de Petrovsky*, Bull. Belg. Math. Soc. Simon Stevin, **5**(1998), 583–594.

[6] Li, G., Sun, Y., Liu, W., *Global existence and blow-up of solutions for a strongly damped Petrovsky system with nonlinear damping*, Appl. Anal., **91**(2012), no. 3, 575–586.

[7] Lions, J.L., *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod Gautier-Villars, Paris, 1969.

[8] Lions, J.L., Magenes, E., *Problèmes aux Limites Non Homogènes et Applications*, vol. 1, 2. Dunod, Paris, 1968.

[9] Messaoudi, S.A., *Global existence and nonexistence in a system of Petrovsky*, J. Math. Anal. Appl., **265**(2002), no. 2, 296–308.

[10] Piskin, E., Polat, N., *On the decay of solutions for a nonlinear Petrovsky equation*, Math. Sci. Lett., **3**(2014), no. 1, 43–47.

[11] Wenying, C., Yong, Z., *Global nonexistence for a semilinear Petrovsky equation*, Nonlinear Anal., **70**(2009), 3203–3208.

[12] Wu, S.T., Tsai, L.Y., *On global solutions and blow-up of solutions for a nonlinearly damped Petrovsky system*, Taiwanese J. Math., **19**(2009), no. 2A, 545–558.

Mosbah Kaddour
Faculty of Mathematics and Computer Science,
Mohamed Boudiaf University-M'Sila 28000, Algeria
e-mail: `mosbah_kaddour@yahoo.fr`

Farid Messelmi
Department of Mathematics and LDMM Laboratory Ziane Achour,
University of Djelfa 17000, Algeria
e-mail: `foudimath@yahoo.fr`