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SOME PROPERTIES OF THE ω-LIMIT POINTS SET OF AN OPERATOR

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Abstract. In this paper we study the ω -limit points set of an operator, in the terms of the fixed points set, the periodic points set and the recurrent points set.

1. Introduction

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the following notations and notions:

$P(x) := \{Y \subset X | Y \neq \emptyset\}$,

$I(A) := \{Y \in P(X) | A(Y) \subset Y\}$,

$F_A := \{x \in X | A(x) = x\}$ - the fixed points set of A,

$P_A := \bigcup\limits_{n \in N^*} F_{A^n}$ - the periodic points set of A,

$P_A^n := \{x \in X | A^k(x) \neq x, k = \overline{1, n-1}, A^n(x) = x\}$-the n-order periodic points set of A,

$\omega_A(x) := \{y \in X |\ \exists\ n_k \to \infty$, such that $f^{n_k}(x) \to y$ as $n \to \infty\}$ - the ω - limit points set of A,

$\omega_A(X) := \bigcup\limits_{x \in X} \omega_A(x)$,

$R_A := \{x \in X | x \in \omega_A(x)\}$ - the recurrent points set of A,

$O_A(x) := \{x, A(x), \ldots, A^n(x), \ldots\}$.

The purpose of this paper is to study the ω - limit points set of an operator A in the terms of F_A, P_A, R_A.

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2. F_A, P_A, P_A^n and ω_A. Examples. Basic problems

Let (X, d) be a metric space and $A : X \to X$ an operator. It is clear that

$$F_A \subset P_A \subset R_A \subset \omega_A$$

In what follow we give some examples and counterexamples to these notions.

Example 2.1 (see[20], [21] and [23]). Let (X, d) be a metric space and $A : X \to X$ a weakly Picard operator. Then

$$F_A = P_A = R_A = \omega_A(X).$$

Example 2.2 (see [1], [2]). $X = \{z \in \mathbf{C} | |z| = 1\}$ and $A(z) := e^{i\alpha}z$. If $\alpha = 1$, then

$$F_A = \emptyset,\ P_A = R_A = \omega_A(X) = X.$$

If α/π is an irrational real number, then

$$F_A = P_A = \emptyset,\ R_A = \omega_A(X) = X.$$

Example 2.3 (see [3], [5], [7]). Let $A \in C([0,1],[0,1])$. If $P_A^3 \neq \emptyset$, then $P_A^n \neq \emptyset$, for all $n \in N^*$ (Sarkovskii's theorem).

Example 2.4 (see [19]). Let $A \in C(R,R)$ such that $A^2 = 1_R$ (an involution). Then $F_A = \{x^*\}$ and $P_A = R$.

Example 2.5 Let (X, d) be a metric space and $A : X \to X$ an operator. We suppose that

(i)$X = \bigcup_{i \in I} X_i,\ X_i \neq \emptyset,\ X_i \bigcap X_j \neq \emptyset,\ i \neq j;$

(ii)$X_i \in I(A)$,

(iii)$cl(X_i) = int(X_i), i \in I.$

Then

$$\omega_A(X_i) \bigcap \omega_A(X_j) = \emptyset,$$

for all $i, j \in I, i \neq j$.

Example 2.6 (see [4]). Let $A \in C([0,1],[0,1])$. Then,

$$\overline{P_A} = \overline{R_A}.$$

Example 2.7 (see [26]). Consider the nonlinear Cauchy problem

$$\frac{du}{dt} = -u - u^3,\ u(0) = U \in \mathbf{R},$$

where $\omega(U) = \{0\}$ for all $U \in \mathbf{R}$. Application of the forward Euler numerical method gives

$$U_{n+1} = U_n - \Delta t(U_n + U_n^3),\ U_0 = U,$$

where $U_n \approx u(n\Delta t)$, $n = 0, 1, \ldots$ and Δt is the time step. If $A(u) = u - \Delta t(u + u^3)$, it may be shown that

$$\begin{aligned} &\omega_A(U) = 0 && for \quad \Delta t(1+U^2) \in (0,2) \\ &\omega_A(U) = \{-U, U\} && for \quad \Delta t(1+U^2) = 2 \\ &|U_n| \to \infty \ as\ n \to \infty && for \quad \Delta t(1+U^2) \in (2, +\infty) \end{aligned}$$

Thus, if $\Delta t < \frac{2}{1+U^2}$ we obtain the correct asymptotic behaviour of the differential equation. If $U = \sqrt{\frac{2}{\Delta t} - 1}$ we obtain a spurious period two solution $U_n = (-1)^n\sqrt{\frac{2}{\Delta t} - 1}$, i.e. $F_A \neq \emptyset$, $P_A^2 \neq \emptyset$ for all $\Delta t \in [0,2]$.

The following problems arise:

Let (X, d) be a metric space and $A : X \to X$ an operator.

Problem 2.1 (see [4], [24]). Establish conditions on X and A which imply that

a) $P_A \neq \emptyset$;

b) $P_A^n \neq \emptyset$.

Problem 2.2 (see [11]). Which are the operators with the following property:

$$P_A \neq \emptyset \ \Rightarrow \ F_A \neq \emptyset?$$

Problem 2.3 ([4]). Which are the metric spaces, X, with the following property

$$A \in C(X,X) \ \Rightarrow \ \overline{P_A} = \overline{R_A}$$

Problem 2.4 Let $n \in N$. In which conditions on X, A and n we have:

$$F_A = P_A^k = \emptyset,\ k = \overline{2, n-1} \text{ and } P_A^k \neq \emptyset,\ k \geq n.$$

Problem 2.5 (see [3], [4], [6], [9], [12], [16], [21], [25]).

Establish conditions on X and A which imply that:

a)$\omega_A(x) \neq \emptyset,\ \forall x \in X$;

b)$R_A \neq \emptyset$;

c)$\omega_A(X) = F_A$;

d)$\omega_A(X) = P_A$;

e)$\omega_A(X) = R_A$;

f) there exists $x \in X : \overline{\omega_A(x)} = X$.

For other examples and countraexamples to the above problems and for some results see [2], [5], [7], [8], [10] and [25].

3. Periodic points

Theorem 3.1. *Let $(X, \leq)$ be a complete lattice and $A : X \to X$ a monoton operator. Then $P_A \neq \emptyset$.*

Proof. If the operator A is monoton increasing, then by the fixed point theorem of Tarski we have that $F_A \neq \emptyset$. If the operator A is monoton decreasing then A^2 is monoton increasing, so, $F_{A^2} \neq \emptyset$.

Theorem 3.2. *Let (X, S, M) be a fixed point structure (see [22]) and $A : X \to X$ an operator. We suppose that there exists $k \in N^*$ such that*

(i)$A^k \in M(X)$;

(ii) there exists $Y \in S(X)$ such that $A^k(X) \subset Y$.

Then $P_A \neq \emptyset$.

Proof. From $A^k : X \to X$ and $A^k(X) \subset Y \subset X$ we have that $Y \in I(A^K)$. On the other hand, $A^k \in M(X)$ implies that $A^k|_Y \in M(Y)$, so, $F_{A^k} \neq \emptyset$.

If in the Theorem 3.2 we consider the fixed point structure of Schauder (X - Banach space, $S(X) = P_{cp,cv}(X)$ and $M(Y) = C(Y, Y)$) we have

Theorem 3.3. *Let X be a Banach space and $A : X \to X$ a continous operator such that there exists $k \in N^*$ such that $A^k(X)$ is relatively compact. Then, $P_A \neq \emptyset$.*

Proof. If we take $Y = \overline{co}A^k(X)$, then $Y \in P_{cp,cv}(X)$ and we are in the conditions of the Theorem 3.2.

Theorem 3.4. *Let $X = [-a, a] \subset \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ such that $A(u) = u + \Delta t f(u)$ is a contraction on X, $A : X \to X$ and $A(0) = 0$, where $\Delta t > 0$. Then the numerical method $U_{n+1} = A(U_n)$, $U_0 = U \in X$ (forward Euler method for the Cauchy problem $\frac{du}{dt} = f(u), u(0) = U \in X$) has no spurious period two solutions in X.*

Proof. By the above conditions, A^2 is a contraction on X and by theorem 3.2. for the fixed point structure of Banach, A^2 has a unique fixed point in X and this point is 0.

Example 3.1. Let $f(u) = -u - u^3$ and $A(u) = u - \Delta t(u + u^3)$ for $\Delta t \in (0, 1)$. Let $X = [-\sqrt{\frac{1-\Delta t}{3\Delta t}}, \sqrt{\frac{1-\Delta t}{3\Delta t}}]$. It may be shown that $A : X \to X$ and A is a contraction on X. Thus we have no spurious period two solutions.

Note that all Runge-Kutta methods retain all the equilibria of $\frac{du}{dt} = f(u)$ (see [26], Th. 5.3.3.). Consequently, the forward Euler method gives the correct asymptotic behaviour of this differential equation on X.

4. Recurrent points

Lemma 4.1 (see [9], [18]). *Let (X, d) be a compact metric space and $A : X \to X$ a continuous operator. Then $R_A \neq \emptyset$.*

Theorem 4.1. *Let (X, d) be a metric space and $A : X \to X$ such that*

(i)A is continous;

(ii) there exists $k \in N^*$ such that $A^k(X)$ is relatively compact.

Then, $R_A \neq \emptyset$.

Proof. It is clear that $clA^k(X) \in I(f)$. So, the operator $A : \overline{A^k(X)} \to \overline{A^k(X)}$ satisfies the conditions in the Lemma 4.1.

Theorem 4.2. *Let (X, d) be a complete metric space, $\alpha : P_b(X) \to R_+$ a measure of noncompactness (see [23]) on X and $A : X \to X$ an operator. We suppose that:*

(i)A is continuous;

(ii)A is a (α, a) - contraction.

Then, $R_A \neq \emptyset$.

Proof. Let $Y_1 := \overline{A(X)}, \ldots, Y_{n+1} := \overline{A(Y_n)}, n \in \mathbf{N}^*$. We remark that

$$Y_n \in P_{b,cl}(X) \bigcap I(A), n \in \mathbf{N}^*$$

From the condition (ii) we have that

$$\alpha(Y_n) \leq a\alpha(Y_{n-1}) \leq \cdots \leq a^n \alpha(Y) \to 0 \text{ as } n \to \infty.$$

But α is a measure of noncompactness on X, i.e, α satisfies the following conditions (see [23]):

(a)$\alpha(A) = 0 \Rightarrow \overline{A} \in P_{cp}(X)$,

(b)$\alpha(A) = \alpha(\overline{A})$, for all $A \in P_b(X)$,

(c)$A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$, for all $A, B \in P_b(X)$,

(d) If $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n$, $n \in \mathbf{N}$, and $\alpha(A_n) \to 0$ *as* $n \to \infty$, *then*

$$A_\infty := \bigcap_{n \in N} A_n \neq \emptyset \text{ and } \alpha(A_\infty) = 0.$$

From the condition (d) and (a) we have that

$$Y_\infty := \bigcap_{n \in N} Y_n \in I_{cp}(A).$$

Now the theorem follows from the Lemma 4.1.

Theorem 4.3. *Let (X, d) be a bounded metric space, α_{DP} a Danes-Pasicki measure of noncompactness (see [23]) and $A : X \to X$ an operator. We suppose that*

(i) the operator A is continous,

(ii) the operator A is α_{DP} - condensing.

Then, $R_A \neq \emptyset$.

Proof. Let $x_0 \in X$. By Lemma 3.1. in [22], there exists $A_0 \subset X$ such that

$$cl(f(A_0) \bigcup \{x_0\}) = A_0$$

This implies that $\alpha_{DP}(A_0) = 0$. Thus

$$A_0 \in P_{cp}(X) \bigcap I(A).$$

Now the proof follows from Lemma 4.1.

5. The set ω_A

In what follow we consider operators on ordered metric space (for the ordered Banach spaces see [6], [10], [11], [25]). We have

Theorem 5.1. *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an increasing operator. Then*

(i) $x \leq Ax \;\Rightarrow\; x \leq \omega_A(x)$

and $y \geq A(y) \;\Rightarrow\; y \geq \omega_A(y)$;

(ii) $\omega_A(x) \leq y \;\Rightarrow\; \omega_A(x) \leq \omega_A(y)$ *and*

$x \leq \omega_A(y) \;\Rightarrow\; \omega_A(x) \leq \omega_A(y)$;

Proof. (i) Let, for example, $x \leq Ax$. Then $x \leq A^n(x)$ for all $n \in \mathbf{N}$. This implies that $x \leq \omega_A(x)$.

(ii) Let, for example, $x, y \in X$, such that $\omega_A(x) \leq y$. Since $\omega_A(x) \in I(A)$, it follows that $\omega_A(x) \leq T^n y$ for all $n \in \mathbf{N}$. Hence we have $\omega_A(x) \leq \omega_A(y)$.

Remark 5.1. The above results improve some results given by E.N. Dancer in [6].

Remark 5.2. From the Theorem 5.1. we have the following results given in [23]:

Theorem 5.2. *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator and $x, y \in X$ such that $x \leq y$, $x \leq A(x)$, $y \geq A(y)$. We suppose that*

(i) A is weakly Picard operator;

(ii) A is monoton increasing.

Then

(a) $x \leq A^\infty(x) \leq A^\infty(y) \leq y$;

(b) $A^\infty(x)$ is the minimal fixed point of A in $[x, y]$ and $A^\infty(y)$ is the maximal fixed point of A in $[x, y]$.

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