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ON WIRTINGER AND OPIAL TYPE INEQUALITIES IN THREE INDEPENDENT VARIABLES

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Abstract. In this paper we establish some new integral and discrete inequalities of Wirtinger and Opial type involving functions of three independent variables. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

1. Introduction

The inequalities of Wirtinger and Opial type and their variants have played a vital role in the study of many qualitative as well as quantitative properties of solutions of differential equations. Because of their usefulness and importance these inequalities have received a wide attention and a large number of papers have appeared in the literature. During the past few years, various investigators have discovered many useful and new Wirtinger and Opial type inequalities involving functions of more than one independent variables, see [1-16] and the references given therein. The main purpose of the present paper is to establish some new integral and discrete inequalities of the Wirtinger and Opial type involving functions of three independent variables. An important feature of the inequalities established in this paper is that the analysis used in their proofs is elementary and our results provide new estimates on this type of inequalities.

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2. Integral inequalities

In what follows R denotes the set of real numbers. We use the notation $E = [a, x] \times [b, y] \times [c, z]$ for $a, b, c, x, y, z \in R$. If $f(r, s, t)$ is a differentiable function defined on E, then its partial derivatives are denoted by $D_1 f(r, s, t) = \frac{\partial}{\partial r} f(r, s, t)$, $D_2 f(r, s, t) = \frac{\partial}{\partial s} f(r, s, t)$, $D_3 f(r, s, t) = \frac{\partial}{\partial t} f(r, s, t)$, and

$$D_3 D_2 D_1 f(r, s, t) = \frac{\partial^3}{\partial t \partial s \partial r} f(r, s, t).$$

We denote by $F(E)$ the class of continuous functions $f : E \to R$ for which

$$D_1 f(r, s, t), D_2 f(r, s, t), D_3 f(r, s, t), D_3 D_2 D_1 f(r, s, t)$$

exist and continuous on E such that $f(a, s, t) = f(x, s, t) = f(r, b, t) = f(r, y, t) = f(r, s, c) = f(r, s, z) = 0$ for $a \le r \le x,\ b \le s \le y,\ c \le t \le z$.

Our main result on Wirtinger type integral inequality involving functions of three independent variables is given in the following theorem.

Theorem 1. *Let $p(r, s, t)$ be a real-valued nonnegative continuous function defined on E. Suppose that $f_i \in F(E)$ for $i = 1, 2, \ldots, n$, and let $m_i \ge 1$ for $i = 1, 2, \ldots, n$ are constants. Then*

$$\int_a^x \int_b^y \int_c^z p(r, s, t) \left[\prod_{i=1}^n |f_i(r, s, t)|^{m_i} \right]^{2/n} dt ds dr \le \tag{2.1}$$

$$\le \frac{1}{n} K(a, b, c, x, y, z, n, m_1, \ldots, m_n) \left(\int_a^x \int_b^y \int_c^z p(r, s, t) dt ds dr \right) \times$$

$$\times \int_a^x \int_b^y \int_c^z \sum_{i=1}^n |D_3 D_2 D_1 f_i(r, s, t)|^{2m_i} dt ds dr,$$

where

$$K(a, b, c, x, y, z, n, m_1, \ldots, m_n) = \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^n m_i} [(x-a)(y-b)(z-c)]^{1+\frac{2}{n}\sum_{i=1}^n (m_i - 1)}, \tag{2.2}$$

is a constant depending on $a, b, c, x, y, z, n, m_1, \ldots, m_n$.

Proof. From the hypothesis, it is easy to observe that the following identities hold for $i = 1, 2, \ldots, n$ and $(r, s, t) \in E$:

$$f_i(r,s,t) = \int_a^r \int_b^s \int_c^t D_3D_2D_1 f_i(u,v,w)\,dw dv du, \tag{2.3}$$

$$f_i(r,s,t) = -\int_a^r \int_b^s \int_t^z D_3D_2D_1 f_i(u,v,w)\,dw dv du, \tag{2.4}$$

$$f_i(r,s,t) = -\int_a^r \int_s^y \int_c^t D_3D_2D_1 f_i(u,v,w)\,dw dv du, \tag{2.5}$$

$$f_i(r,s,t) = -\int_r^x \int_b^s \int_c^t D_3D_2D_1 f_i(u,v,w)\,dw dv du, \tag{2.6}$$

$$f_i(r,s,t) = \int_a^r \int_s^y \int_t^z D_3D_2D_1 f_i(u,v,w)\,dw dv du, \tag{2.7}$$

$$f_i(r,s,t) = \int_r^x \int_s^y \int_c^t D_3D_2D_1 f_i(u,v,w)\,dw dv du, \tag{2.8}$$

$$f_i(r,s,t) = \int_r^x \int_b^s \int_t^z D_3D_2D_1 f_i(u,v,w)\,dw dv du, \tag{2.9}$$

$$f_i(r,s,t) = -\int_r^x \int_s^y \int_t^z D_3D_2D_1 f_i(u,v,w)\,dw dv du. \tag{2.10}$$

From (2.3)-(2.10) it is easy to observe that

$$|f_i(r,s,t)| \le \frac{1}{8}\int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_i(u,v,w)|\,dw dv du, \tag{2.11}$$

for $i = 1, 2, \ldots, n$ and $(r, s, t) \in E$. From (2.11) and using the Hölder's integral inequality in three dimensions with indices m_i and $m_i/(m_i - 1)$ for $i = 1, 2, \ldots, n$ we obtain

$$|f_i(r,s,t)|^{m_i} \le \left(\frac{1}{8}\right)^{m_i} [(x-a)(y-b)(z-c)]^{m_i-1} \times \tag{2.12}$$
$$\times \int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_i(u,v,w)|^{m_i}\,dw dv du.$$

From (2.12) and using the elementary inequalities (see [4])

$$\left(\prod_{i=1}^{n} b_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} b_i, \tag{2.13}$$

(for $b_1, b_2, \ldots, b_n$ nonnegative reals and $n \geq 1$) and

$$\left(\sum_{i=1}^{n} b_i\right)^{2} \leq n\sum_{i=1}^{n} b_i^2, \tag{2.14}$$

(for $b_1, b_2, \ldots, b_n$ reals) and Schwarz integral inequality in three dimensions we observe that

$$\left[\prod_{i=1}^{n} |f_i(r,s,t)|^{m_i}\right]^{2/n} \leq \tag{2.15}$$

$$\leq \left[\prod_{i=1}^{n}\left(\frac{1}{8}\right)^{m_i}[(x-a)(y-b)(z-c)]^{m_i-1}\times\right.$$

$$\left.\times\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_i(u,v,w)|^{m_i}dwdvdu\right]^{2/n} =$$

$$= \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i}[(x-a)(y-b)(z-c)]^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)}\times$$

$$\times\left[\prod_{i=1}^{n}\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_i(u,v,w)|^{m_i}dwdvdu^{1/n}\right]^2 \leq$$

$$\leq \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i}[(x-a)(y-b)(z-c)]^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)}\times$$

$$\times\left[\frac{1}{n}\sum_{i=1}^{n}\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_i(u,v,w)|^{m_i}dwdvdu\right]^2 \leq$$

$$\leq \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i}[(x-a)(y-b)(z-c)]^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)}\times$$

$$\times\frac{1}{n^2}n\sum_{i=1}^{n}\left[\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_i(u,v,w)|^{m_i}dwdvdu\right]^2 \leq$$

$$\leq \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i}[(x-a)(y-b)(z-c)]^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)}\times$$

$$\times\frac{1}{n}[(x-a)(y-b)(z-c)]\sum_{i=1}^{n}\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_i(u,v,w)|^{2m_i}dwdvdu =$$

$$= \frac{1}{n} K(a,b,c,x,y,z,n,m_1,\ldots,m_n) \times \int_a^x \int_b^y \int_c^z \sum_{i=1}^n |D_3D_2D_1 f_i(u,v,w)|^{2m_i} dw dv du.$$

Multiplying both sides of (2.15) by $p(r,s,t)$ and integrating the resulting inequality over E we get the desired inequality in (2.1). The proof is complete.

Remark 1. We note that in the special cases when (i) $m_i = 1$ for $i = 1,2,\ldots,n$, (ii) $n = 2$, (iii) $n = 1$, (iv) $n = 2$ and $m_1 = m_2 = 1$, and (v) $n = 1$ and $m_1 = 1$, the inequality established in (2.1) reduces respectively to the following inequalities

$$\int_a^x \int_b^y \int_c^z p(r,s,t) \left[\prod_{i=1}^n |f_i(r,s,t)|\right]^{2/n} dt ds dr \leq \tag{2.16}$$

$$\leq \frac{1}{n} K(a,b,c,x,y,z,n,1,\ldots,1) \left(\int_a^x \int)b^y \int_c^z p(r,s,t) dt ds dr\right) \times$$

$$\times \int_a^x \int_b^y \int_c^z \sum_{i=1}^n |D_3D_2D_1 f_i(r,s,t)|^2 dt ds dr,$$

$$\int_a^x \int_b^y \int_c^z p(r,s,t) |f_1(r,s,t)|^{m_1} |f_2(r,s,t)|^{m_2} dt ds dr \tag{2.17}$$

$$\leq \frac{1}{2} K(a,b,c,x,y,z,2,m_1,m_2) \left(\int_a^x \int_b^y \int_c^z p(r,s,t) dt ds dr\right) \times$$

$$\times \int_a^x \int_b^y \int_c^z \left[|D_3D_2D_1 f_1(r,s,t)|^{2m_i} + |D_3D_2D_1 f_2(r,s,t)|^{2m_2}\right] dt ds dr,$$

$$\int_a^x \int_b^y \int_c^z p(r,s,t) |f_1(r,s,t)|^{2m_1} dt ds dr \leq \tag{2.18}$$

$$\leq K(a,b,c,x,y,z,l,m_1) \left(\int_a^x \int)b^y \int_c^z p(r,s,t) dt ds dr\right) \times$$

$$\times \int_a^x \int_b^y \int_c^z |D_3D_2D_1 f_1(r,s,t)|^{2m_1} dt ds dr,$$

$$\int_a^x \int_b^y \int_c^z p(r,s,t) |f_1(r,s,t)||f_2(r,s,t)| dt ds dr \leq \tag{2.19}$$

$$\leq \frac{1}{2} K(a,b,c,x,y,z,2,1,1) \left(\int_a^x \int_b^y \int_c^z p(r,s,t) dt ds dr\right) \times$$

$$\times \int_a^x \int_b^y \int_c^z [|D_3D_2D_1 f_1(r,s,t)|^2 + |D_3D_2D_1 f_2(r,s,t)|^2] dt ds dr,$$

$$\int_a^x \int_b^y \int_c^z p(r,s,t) |f_1(r,s,t)|^2 dt ds dr \leq \tag{2.20}$$

$$\leq K(a,b,c,xy,z,1,1)\left(\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right)\times$$

$$\times\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_1(r,s,t)|^2dtdsdr.$$

We note that the inequalities obtained in (2.17) and (2.19) are the three independent variable analogues of the Wirtinger type inequalities established by the present author in [10] and the inequality obtained in (2.20) is a three independent variable analog of the Wirtinger type inequality established by Traple in [15, p.160].

The following theorem deals with an integral inequality of Opial type involving functions of three independent variables.

Theorem 2. *Let the functions $p(r,s,t)$, $f_i(r,s,t)$ and the constants m_i for $i = 1,2,\ldots,n$ be as in Theorem 1. Then*

$$\int_a^x\int_b^y\int_c^z \sqrt{p(r,s,t)}\left[\prod_{i=1}^n |f_i(r,s,t)|^{m_i}\right]^{1/n}\times \tag{2.21}$$

$$\times\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{m_i}dtdsdr\leq$$

$$\leq\left[K(a,b,c,x,y,z,n,m_1,\ldots,m_n)\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right]^{1/2}\times$$

$$\times\int_a^x\int_b^y\int_c^z\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{2m_i}dtdsdr,$$

where the constant $K(a,b,c,x,y,z,n,m_1,\ldots,m_n)$ is defined by (2.2).

Proof. By using the Schwarz integral inequality in three dimensions and the inequalities (2.1) and (2.14) we observe that

$$\int_a^x\int_b^y\int_c^z \sqrt{p(r,s,t)}\left[\prod_{i=1}^n |f_i(r,s,t)|^{m_i}\right]^{1/n}\times\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{m_i}dtdsdr\leq$$

$$\leq\left[\int_a^x\int_b^y\int_c^z p(r,s,t)\left[\prod_{i=1}^n |f_i(r,s,t)|\right]^{2/n}dtdsdr\right]^{1/2}\times$$

$$\times\left[\int_a^x\int_b^y\int_c^z\left(\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{m_i}\right)^2dtdsdr\right]^{1/2}\leq$$

$$\leq \left[\frac{1}{n}K(a,b,c,x,y,z,m_1,\ldots,m_n)\left(\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right)\times\right.$$

$$\left.\times\int_a^x\int_b^y\int_c^z\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{2m_i}dtdsdr\right]^{1/2}\times$$

$$\times\left[n\int_a^x\int_b^y\int_c^z\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{2m_i}dtdsdr\right]^{1/2}=$$

$$=\left[K(a,b,c,x,y,z,n,m_1,\ldots,m_n)\left(\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right)\right]^{1/2}\times$$

$$\times\int_a^x\int_b^y\int_c^z\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^{2m_i}dtdsdr.$$

This is the desired inequality in (2.21) and hence the proof is complete.

Remark 2. If we take

(i) $m_i = 1$ for $i = 1,2,\ldots,n$,

(ii) $n = 1$,

(iii) $n = 1$ and $m_1 = 1$ in (2.21),

then we get respectively the following inequalities

$$\int_a^x\int_b^y\int_c^z\sqrt{p(r,s,t)}\left[\prod_{i=1}^n |f_i(r,s,t)|\right]^{1/n}\times\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|dtdsdr\leq \tag{2.22}$$

$$\leq\left[K(a,b,c,x,y,z,n,1,\ldots,1)\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right]^{1/2}\times$$

$$\times\int_a^x\int_b^y\int_c^z\sum_{i=1}^n |D_3D_2D_1f_i(r,s,t)|^2dtdsdr,$$

$$\int_a^x\int_b^y\int_c^z\sqrt{p(r,s,t)}|f_1(r,s,t)|^{m_1}|D_3D_2D_1f_1(r,s,t)|^{m_1}dtdsdr\leq \tag{2.23}$$

$$\leq\left[K(a,b,c,x,y,z,1,m_1)\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right]^{1/2}\times$$

$$\times\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_1(r,s,t)|^{2m_1}dtdsdr,$$

$$\int_a^x\int_b^y\int_c^z\sqrt{p(r,s,t)}|f_1(r,s,t)||D_3D_2D_1f_1(r,s,t)|dtdsdr\leq \tag{2.24}$$

$$\leq \left[K(a,b,c,x,y,z,1,1)\int_a^x\int_b^y\int_c^z p(r,s,t)dtdsdr\right]^{1/2}\times$$
$$\times\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_1(r,s,t)|^2 dtdsdr.$$

We note that the inequality obtained in (2.24) is a three independent variable analog of the Opial inequality established by Traple in [15, p.160]. In the special case when $p(r,s,t)$ is constant, then from (2.24) we have the following Opial type inequality

$$\int_a^x\int_b^y\int_c^z |f_1(r,s,t)||D_3D_2D_1f(r,s,t)|dtdsdr \leq \tag{2.25}$$
$$\leq [K(a,b,c,x,y,z,1,1)(x-a)(y-b)(z-c)]^{1/2}\times$$
$$\times\int_a^x\int_b^y\int_c^z |D_3D_2D_1f_1(r,s,t)|^2 dtdsdr.$$

For similar inequalities involving functions of two independent variables, see [6,8,10,14].

3. Discrete inequalities

Let $N=\{1,2,\ldots\}$ and for x,y,z in N, we define

$$A=\{1,2,\ldots,x+1\},\quad B=\{1,2,\ldots,y+1\},\quad C=\{1,2,\ldots,z+1\}$$

and $Q=A\times B\times C$. For a function $f:N^3\to R$, we define the difference operators

$$\Delta_1 f(r,s,t)=f(r+1,s,t)-f(r,s,t),$$
$$\Delta_2 f(r,s,t)=f(r,s+1,t)-f(r,s,t),$$
$$\Delta_3 f(r,s,t)=f(r,s,t+1)-f(r,s,t),$$
$$\Delta_2\Delta_1 f(r,s,t)=\Delta_2[\Delta_1 f(r,s,t)]$$

and

$$\Delta_3\Delta_2\Delta_1 f(r,s,t)=\Delta_3[\Delta_2\Delta_1 f(r,s,t)].$$

We denote by $G(Q)$ the class of functions $f:Q\to R$ such that

$$f(1,s,r)=f(x+1,s,t)=f(r,1,t)=f(r,y+1,t)=f(r,s,1)=f(r,s,z+1)=0.$$

The discrete analogue of the inequality given in Theorem 1 is embodied in the following theorem.

Theorem 3. *Let $p(r,s,t)$ be a real-valued nonnegative function defined on Q. Suppose that $f_i \in G(Q)$ for $i = 1,2,\ldots,n$ and let $m_i \geq 1$ for $i = 1,2,\ldots,n$ are constants. Then*

$$\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z} p(r,s,t)\left[\prod_{i=1}^{n}|f_i(r,s,t)|^{m_i}\right]^{2/n} \leq \tag{3.1}$$

$$\leq \frac{1}{n}M(x,y,z,n,m_1,\ldots,m_n)\left(\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z} p(r,s,t)\right)\times$$

$$\times\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\left[\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{2m_i}\right],$$

where

$$M(x,y,z,n,m_1,\ldots,m_n) = \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i}(xyz)^{1+\frac{2}{n}\sum_{i=1}^{n}(m_i-1)}, \tag{3.2}$$

is a constant depending on $x,y,z,n,m_1,\ldots,m_n$.

Proof. From the hypotheses, it is easy to observe that the following identities hold for $i = 1,2,\ldots,n$ and $(r,s,t) \in Q$:

$$f_i(r,s,t) = \sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.3}$$

$$f_i(r,s,t) = -\sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=t}^{z} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.4}$$

$$f_i(r,s,t) = -\sum_{u=1}^{r-1}\sum_{v=s}^{y}\sum_{w=1}^{t-1} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.5}$$

$$f_i(r,s,t) = -\sum_{u=r}^{x}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.6}$$

$$f_i(r,s,t) = \sum_{u=1}^{r-1}\sum_{v=s}^{y}\sum_{w=t}^{z} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.7}$$

$$f_i(r,s,t) = \sum_{u=r}^{x}\sum_{v=s}^{y}\sum_{w=1}^{t-1} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.8}$$

$$f_i(r,s,t) = \sum_{u=r}^{x}\sum_{v=1}^{s-1}\sum_{w=t}^{z} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.9}$$

$$f_i(r,s,t) = -\sum_{u=r}^{x}\sum_{v=s}^{y}\sum_{w=t}^{z} \Delta_e\Delta_2\Delta_1 f_i(u,v,w), \tag{3.10}$$

From (3.3)-(3.10) it is easy to observe that

$$|f_i(r,s,t)| \leq \frac{1}{8}\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|, \tag{3.11}$$

for $(r,s,t) \in Q$ and $i = 1,2,\dots,n$. From (3.9) and using the Hölder's inequality for summations in three dimensions with indices m_i, $m_i/(m_i-1)$ for $i = 1,2,\dots,n$ we obtain

$$|f_i(r,s,t)|^{m_i} \leq \left(\frac{1}{8}\right)^{m_i} (xyz)^{m_i-1} \times \sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}. \tag{3.12}$$

From (3.12) and using the elementary inequalities (2.13) and (2.14) and Schwarz inequality for summation in three dimensions we observe that

$$\left[\prod_{i=1}^{n} |f_i(r,s,t)|^{m_i}\right]^{2/n} \leq \tag{3.13}$$

$$\leq \left[\prod_{i=1}^{n}\left(\frac{1}{8}\right)^{m_i} (xyz)^{m_i-1} \times \sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}\right]^{2/n} =$$

$$= \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} (xyz)^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times \left[\left[\prod_{i=1}^{n}\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}\right]^{1/n}\right]^2 \leq$$

$$\leq \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} (xyz)^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times \left[\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}\right]\right]^2 \leq$$

$$\leq \left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} (xyz)^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times \frac{1}{n^2} n \sum_{i=1}^{n}\left[\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{m_i}\right]^2 \leq$$

$$\leq \frac{1}{n}\left(\frac{1}{8}\right)^{\frac{2}{n}\sum_{i=1}^{n} m_i} (xyz)^{\frac{2}{n}\sum_{i=1}^{n}(m_i-1)} \times \sum_{i=1}^{n}(xyz)\sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{2m_i} =$$

$$= \frac{1}{n} M(x,y,z,n,m_1,\dots,m_n) \times \sum_{u=1}^{x}\sum_{v=1}^{y}\sum_{w=1}^{z}\left[\sum_{i=1}^{n} |\Delta_3\Delta_2\Delta_1 f_i(u,v,w)|^{2m_i}\right].$$

Multiplying both sides of (3.13) by $p(r,s,t)$ and taking the sum over t,s,r from 1 to z, 1 to y, 1 to x respectively we get the required inequality in (3.1). The proof is complete.

Remark 3. If we take (i) $m_i = 1$ for $i = 1, 2, \ldots, n$, (ii) $n = 2$, (iii) $n = 1$, (iv) $n = 2$ and $m_1 = m_2 = 1$ and (v) $n = 1$ and $m_1 = 1$, then the inequality established in (3.1) reduces to the various new inequalities which can be used in certain applications.

The following theorem deals with the discrete analogue of the inequality given in Theorem 2.

Theorem 4. *Let the functions $p(r,s,t), f_i(r,s,t)$ and the constants m_i for $i = 1, 2, \ldots, n$ be as in Theorem 3. Then*

$$\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\sqrt{p(r,s,t)}\left[\prod_{i=1}^{n}|f_i(r,s,t)|^{m_i}\right]^{1/n}\times\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{m_i}\le \tag{3.14}$$
$$\le\left[M(x,y,z,n,m_1,\ldots,m_n)\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}p(r,s,t)\right]^{1/2}\times$$
$$\times\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\left[\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{2m_i}\right],$$

where the constant $M(x,y,z,n,m_1,\ldots,m_n)$ is defined by (3.2).

Proof. By using Schwarz inequality for summation in three dimensions and the inequalities (3.1) and (2.14) we observe that

$$\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\sqrt{p(r,s,t)}\left[\prod_{i=1}^{n}|f_i(r,s,t)|^{m_i}\right]^{1/n}\times\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{m_i}\le$$
$$\le\left[\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}p(r,s,t)\left[\prod_{i=1}^{n}|f_i(r,s,t)|^{m_i}\right]^{2/n}\right]^{1/2}\times$$
$$\times\left[\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\left[\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{m_i}\right]^{2}\right]^{1/2}\le$$
$$\le\left[\frac{1}{n}M(x,y,z,n,m_1,\ldots,m_n)\left(\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}p(r,s,t)\right)\times\right.$$
$$\left.\times\sum_{r=1}^{x}\sum_{s=1}^{y}\sum_{t=1}^{z}\left[\sum_{i=1}^{n}|\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{2m_i}\right]\right]^{1/2}\times$$

$$\times \left[n \sum_{r=1}^{x} \sum_{s=1}^{y} \sum_{t=1}^{z} \left[\sum_{i=1}^{n} |\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{2m_i}\right]\right]^{1/2} \times$$

$$\times \left[n \sum_{r=1}^{x} \sum_{s=1}^{y} \sum_{t=1}^{z} \left[\sum_{i=1}^{n} |\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{2m_i}\right]\right]^{1/2} =$$

$$= \left[M(x,y,z,n,m_1,\ldots,m_n) \sum_{r=1}^{x} \sum_{s=1}^{y} \sum_{t=1}^{z} p(r,s,t)\right]^{1/2} \times$$

$$\times \sum_{r=1}^{x} \sum_{s=1}^{y} \sum_{t=1}^{z} \left[\sum_{i=1}^{n} |\Delta_3\Delta_2\Delta_1 f_i(r,s,t)|^{2m_i}\right].$$

This is the desired inequality in (3.14) and hence the proof is complete.

Remark 4. In the special cases, if we take (i) $m_i = 1$ for $i = 1, 2, \ldots, n$, (ii) $n = 1$, (iii) $n = 1$ and $m_1 = 1$ in (3.14), then we get the new inequalities which may be useful in certain situations. For similar inequalities, see [7,9,11,12,13] and the references given therein.

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