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The isotomic transformation in the hyperbolic plane

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Abstract. In this note, we introduce a hyperbolic analogue of the isotomic transformation, originally defined for Euclidean triangle and we investigate some of its properties.

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1. Introduction

The aim of this paper is to introduce the *isotomic transformation* in the hyperbolic geometry and investigate some of its basic properties. There are several approaches to the geometry of the hyperbolic plane. We found that the approach most suitable for our purposes is the so-called *Cayley-Klein approach* (see [4], [9] or [5]). In this approach, the hyperbolic plane is thought of as being a region of the projective plane, bounded by a real, nondegenerate projective conic. This conic is defined by a polarity of the real projective plane,

$$\begin{cases} x_\mu = c_{\mu\nu}\xi_\nu, \\ \xi_\mu = C_{\mu\nu}x_\nu, \end{cases} \tag{1.1}$$

where we sum after all the possible values of the indices ($\mu, \nu = 0, 1, 2$). Here (x_μ) are the point coordinates, while (ξ_μ) are the line coordinates. As we are given a hyperbolic polarity, the matrices $[c]$ and $[C]$ are both symmetric and inverse to each other. Moreover, the conic given by the point equation

$$c_{\mu\nu}x_\mu x_\nu = 0 \tag{1.2}$$

is a nondegenerate real conic, called the *Absolute*. The equation of the Absolute, written in line coordinates, is

$$C_{\mu\nu}\xi_\mu\xi_\nu = 0. \tag{1.3}$$

We notice that, usually, we prescribe the polarity, therefore the equations (1.1) give, implicitly, the definition of the coordinates (homogeneous coordinates, of course).

The system of coordinates we are going to use is the system of *barycentric* (or *areal*) coordinates, introduced by Sommerville, in 1932 ([10]) by another method. For these coordinates, the two matrices that define the polarity are

$$[c_{\mu\nu}] = \begin{pmatrix} 1 & \cosh c & \cosh b \\ \cosh c & 1 & \cosh a \\ \cosh b & \cosh a & 1 \end{pmatrix} \tag{1.4}$$

and

$$[C_{\mu\nu}] = \frac{1}{\gamma} \begin{pmatrix} -\sinh^2 a & \sinh a \sinh b \cos C & \sinh a \sinh c \cos B \\ \sinh a \sinh b \cos C & -\sinh^2 b & \sinh b \sinh b \cos A \\ \sinh a \sinh c \cos B & \sinh b \sinh b \cos A & -\sinh^2 c \end{pmatrix}, \tag{1.5}$$

where

$$\gamma = 1 + 2\cosh a \cosh b \cosh c - \cosh^2 a - \cosh^2 b - \cosh^2 c > 0$$

is the determinant of the matrix $[c_{\mu\nu}]$.

It should be clear that the coordinates defined by a polarity are defined for any point of the projective plane, not just for the ordinary points. In contrast, the barycentric coordinates, defined by Sommerville by using a triangle and a unit point, as it is standard in projective geometry, are valid only for ordinary points. Thus, for Sommerville, the barycentric (point) coordinates are defined by

$$\begin{cases} X_0 = \sinh a \sinh u, \\ X_1 = \sinh b \sinh v, \\ X_0 = \sinh c \sinh w, \end{cases} \tag{1.6}$$

where a, b, c are the lengths of the sides BC, CA and AB, respectively, of the reference triangle, while u, v, w are the distances from the current point to these sides. These definitions are equivalent to those given by polarisation for ordinary points in the hyperbolic plane, but they don't make sense for ideal or ultra-infinite points for the very simple reason that the lengths u, v, w are not defined.

The homogeneous coordinates in the hyperbolic plane have not been very popular, lately. Most of the works on analytic hyperbolic geometry prefer the use of Cartesian or polar coordinates. Nevertheless, if somebody wants to investigate problems related to a hyperbolic triangle, it is more convenient to use some coordinates related closely to the triangle itself. Recently, (see [11]) Ungar introduced a set of barycentric coordinates, in the framework of the so-called *Einstein velocity space* model of hyperbolic geometry. We feel, however, that the Cayley-Klein (projective) model is closer to the intuition and, therefore, we use the barycentric coordinates introduced by Sommerville ([10]) and, afterwards, reformulated by Coxeter ([4]).

We introduce the following notations, that we will use again and again. For more details, see [2]. First of all, we denote by H^2 the hyperbolic plane, as a subset of the real projective plane. If (x) and (y) are two points, while $[\xi]$ and $[\eta]$ are two lines (from the real projective plane!), then

1. $(x, y) = c_{\mu\nu} x_\mu y_\nu$;
2. $[\xi, \eta] = C_{\mu\nu} \xi_\mu \eta_\nu$;
3. $\{x, \eta\} = x_\mu \xi_\eta$;

4. $\{\xi, y\} = \xi_\mu y_\nu$,

where, as usually, we sum after all the possible values of the indices. We mention that, if the lines $[\xi]$ and $[\eta]$ are the polars of the points (x) and (y), the all the all four brackets defined above are equal.

The brackets we introduce are very convenient for describing different entities related to hyperbolic geometry. We mention only some of them, that will be used in the paper.

1. The equation of the Absolute is $(x, x) = 0$ (in point coordinates) or $[\xi, \xi] = 0$ (in line coordinates);
2. a point (x) is an ordinary point iff $(x, x) > 0$;
3. a point (x) is an ultra-infinite point iff $(x, x) < 0$;
4. a line $[\xi]$ is ultra-infinite (i.e. lies outside the hyperbolic plane) iff $[\xi, \xi] > 0$;
5. the polar of any ordinary point of the hyperbolic plane is ultra-infinite and the polar of ultra-infinite point is an ordinary line;
6. the lines $[\xi]$ and $[\eta]$ are perpendicular iff $[\xi, \eta] = 0$;
7. if α is the angle between two lines, $[\xi]$ and $[\eta]$, then
$$\cos^2 \alpha = \frac{[\xi, \eta]^2}{[\xi, \xi] \cdot [\eta, \eta]}.$$
Notice that this relations makes sense iff the two lines are either both ordinary, either both ultra-infinite. We cannot compute, for instance, the angle between an ordinar line and an ultra-ideal one.
8. If (x) and (y) are two points and d is the distance between them, then
$$\cosh d = \frac{|(x, y)|}{\sqrt{(x, x) \cot (y, y)}}.$$
Again, we can only compute distances between two ordinary points or two ultra-infinite points, but not between an ordinary point and an ultra-infinite point.
9. We can, also, compute the distance d between a point (x) and a line $[\xi]$, by using the formula
$$\sinh d = \frac{|\{x, \xi\}|}{\sqrt{(x, x)} \cdot \sqrt{-[\xi, \xi]}}.$$
This distance can be computed iff both (x) and (ξ) are ordinary.

From now on, we shall use exclusively barycentric coordinates and we shall denote them with capital letters, (X_0, X_1, X_2). In this coordinates, as we saw, we have
$$\begin{aligned}(X, X) = (X_0)^2 + (X_1)^2 + (X_2)^2 + 2 \cosh c \cdot X_0 X_1 + \\ + 2 \cosh b \cdot X_0 X_2 + 2 \cosh a \cdot X_1 X_2.\end{aligned} \tag{1.7}$$

2. The transformation

The isotomic transformation for Euclidean triangles has been introduced by G. de Longchamps in 1866 (see [6] and [7]). We shall give here a similar definition, using the hyperbolic barycentric coordinates.

Definition 2.1. *We define, by analogy to the Euclidean case, the* isotomic transformation *as being a map*

$$\mathrm{Isot} : \mathbb{P}^2(\mathbb{R}) \setminus \mathcal{T} \to \mathbb{P}^2(\mathbb{R}),$$

defined by

$$\mathrm{Isot}\,(X_0, X_1, X_2) = \left(\frac{1}{X_0}, \frac{1}{X_1}, \frac{1}{X_2}\right). \tag{2.1}$$

Here $\mathcal{T}$ is the union of the three sides of the triangle ABC (thought of as projective *lines). We shall say that the points M and M' form an* isotomic pair*. We shall also say that M' is the* isotomic conjugate *or the* isotomic inverse *of M. We may, as well, say, again inspired from the classical case, that the two points are* reciprocal *(with respect to the triangle ABC).*

As Isot is defined on $\mathbb{P}^2(\mathbb{R}) \setminus \mathcal{T}$, none of the coordinates X_i vanishes, hence Isot is well defined.

Remark 2.2.
1. We might have tried, as well, to define the isotomic transformation just on points of the hyperbolic plane. Nevertheless, as we shall see later, the image of an ordinary point through the isotomic transformation is not always an ordinary point, it might be ideal or ultra-infinite.
2. By looking at the formula (2.1), the reader may think that the definition of the isotomic transformation is identical to the definition from the Euclidean/projective case. This is not the case, however, because the barycentric coordinates from the hyperbolic case are not the same with the classical barycenter coordinates.

Definition 2.3. *We shall say that two points on the side BC of hyperbolic triangle ABC (ordinary, ideal or ultra-infinite) are* isotomically symmetric *with respect to the midpoint A' of the side BC if they coordinates are $A_1(0, \alpha_1, \alpha_2)$ and $A'(0, 1/\alpha_1, 1/\alpha_2)$ or $A_1'(0, \alpha_2, \alpha_1)$.*

The following theorem justifies the name of "isotomic transformation".

Theorem 2.4. *If A_1 is an* ordinary *point on BC, then its isotomic symmetric A_1' is, also, ordinary, and $A'A_1 = A'A_1'$ (as hyperbolic lengths). Moreover, is A_1 is either ideal or ultra-infinite, the same holds true for A_1'.*

Proof. We notice, first of all, that $(A_1, A_1) = (A_1', A_1')$, therefore the two numbers are simultaneously zero, positive or negative. As such, the points A_1 and A_1' have the same character(ordinary, ideal or ultra-infinite).

Thus, we will consider the particular case when A_1 is an ordinary point, and, of course, the same is true for A_1'. We know, already, that the barycentric coordinates of A' are $(0, 1, 1)$. We compute, first, the length of the segment $A'A_1$. We have

$$\cosh A'A_1 = \frac{|(A', A_1)|}{\sqrt{(A', A') \cdot (A_1, A_1)}}.$$

On the other hand,

$$(A', A') = 2(1 + \cosh a) = 4\cosh^2 \frac{a}{2},$$
$$(A_1, A_1) = \alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \cosh a,$$
$$(A', A_1) = 2(\alpha_1 + \alpha_2)\cosh^2 \frac{a}{2}.$$

We have, therefore

$$\cosh A'A_1 = \frac{\left|2(\alpha_1 + \alpha_2)\cosh^2 \frac{a}{2}\right|}{2\cosh \frac{a}{2}\sqrt{\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \cosh a}} = \frac{|\alpha_1 + \alpha_2| \cosh \frac{a}{2}}{\sqrt{\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \cosh a}}.$$

Now, it is easy to check that $(A_1', A_1') = (A_1, A_1)$ and $(A', A_1') = (A', A_1)$, therefore $\cosh A'A_1' = \cosh A'A_1$, hence $A'A_1' = A'A_1$. □

The previous theorem justifies the following definition:

Definition 2.5. *Two cevians of a hyperbolic triangle ABC, starting from the same vertex, are called* isotomic *if they cut the opposite side at isotomically symmetric points. We shall, also, say that the cevians are* isotomically conjugated.

Theorem 2.6. *If three cevians (starting from different vertices) are concurrent at a point, then their isotomic conjugates are also concurrent and the intersection points are, as well, isotomically conjugated.*

Proof. Let $M\left(X_0^0, X_1^0, X_2^0\right)$ be the intersection point of the three given cevians. It is easy to see that the equations of these cevians are

$$AM : X_2^0 X_1 - X_1^0 X_2 = 0,$$
$$BM : X_2^0 X_0 - X_0^0 X_2 = 0,$$
$$CM : X_1^0 X_0 - X_0^0 X_1 = 0.$$

As such, their intersection points with the sides BC, CA and AB, respectively, will be $A_1\left(0, X_1^0, X_2^0\right)$, $B_1\left(X_0^0, 0, X_2^0\right)$ and $C_1\left(X_0^0, X_1^0, 0\right)$, respectively. Then, according to the previous lemma, their symmetrics with respect to the midpoints of the respective sides will be $A_1'\left(0, X_2^0, X_1^0\right)$, $B_1'\left(X_2^0, 0, X_0^0\right)$ and $C_1'\left(X_1^0, X_0^0, 0\right)$, respectively.

We are, thus, led to the equations of the isotomically conjugated of the cevians AM, BM and CM:

$$AA_1' : X_1^0 X_1 - X_2^0 X_2 = 0,$$
$$BB_1' : X_0^0 X_0 - X_2^0 X_2 = 0,$$
$$CC_1' : X_0^0 X_0 - X_1^0 X_1 = 0.$$

It turns out that the three cevians *do* intersect, at the point

$$M'\left(1/X_0^0, 1/X_1^0, 1/X_2^0\right),$$

as we expected. □

3. The Steiner quadratix

As we mentioned before, one of the advantages of the Cayley-Klein approach to the hyperbolic plane is that, in this model, the hyperbolic geometry is, in a certain sense, a "sub-geometry" of the real projective plane. As such, we have access to all the points of the projective plane, although they are not treated on the same footing. We can treat any pair of lines as being intersecting lines, but some of them intersect at ideal or ultra-infinite points. We can write the equation of the line passing through an arbitrary pair of points, but some of the lines are either ultra-infinite (they don't intersect the hyperbolic plane) or lines at infinity (they are tangents to the Absolute). The downside is that we cannot compute distances and lengths when ideal or ultra-infinite points and lengths are involved.

We turn, for a while, to the "classical" language of hyperbolic geometry. Then, for instance, the theorem 2.6 can be reformulated as

Theorem 3.1. *Consider three cevians of a given triangle, starting from different vertices. If the three cevians belong to the same pencil of lines (concurrent, ultra-parallel or parallel), then the their isotomic conjugates also belong to the same pencil.*

The point is that we don't know *what kind* of pencil.

We ask the following question: *When three concurrent cevians of a given hyperbolic triangle, starting from different vertices turn, through the isotomic transformation, into three parallel lines?*

We know the answer in the classical Euclidean (or, rather, projective case): when they intersect on the line at infinity. But the things are similar, here, only that the line at infinity gets replaced by the Absolute. Indeed, three lines belong to the same pencil of parallel lines iff they intersect (according to the Cayley-Klein view of the hyperbolic geometry) on the Absolute. Therefore, we have the following theorem:

Theorem 3.2. *Let us assume that the cevians AA_1, BB_1 and CC_1 of the triangle ABC intersect at a point $M(X_0^0, X_1^0, X_2^0)$ (ordinary, ideal or ultra-infinite). Then the isotomic conjugates of the cevians are parallel (i.e. they intersect at an ideal point) iff M belongs to the curve*

$$\begin{aligned} &X_0^2X_1^2 + X_0^2X_2^2 + X_1^2X_2^2 + 2\cosh c X_0X_1X_2^2 + \\ &+ 2\cosh b X_0X_1^2X_2 + 2\cosh a X_0^2X_1X_2 = 0. \end{aligned} \tag{3.1}$$

*We shall call the curve (*3.1*) the* Steiner quadratix*. It is the hyperbolic analogue of the first Steiner ellipse.*

Proof. The isotomic conjugates of the cevians are parallel to each other iff they intersect at a point of the Absolute. But, as we saw earlier, the conjugates intersect at the point $M'\left(1/X_0^0, 1/X_1^0, 1/X_2^0\right)$. M' belongs to the Absolute iff

$$\frac{1}{\left(X_0^0\right)^2} + \frac{1}{\left(X_1^0\right)^2} + \frac{1}{\left(X_2^0\right)^2} + \frac{2}{X_0^0\cdot X_1^0}\cosh c + \frac{2}{X_0^0\cdot X_2^0}\cosh b + \frac{2}{X_1^0\cdot X_2^0}\cosh a = 0$$

or

$$\begin{aligned}&\left(X_0^0\right)^2 \cdot \left(X_1^0\right)^2 + \left(X_0^0\right)^2 \cdot \left(X_2^0\right)^2 + \left(X_1^0\right)^2 \cdot \left(X_2^0\right)^2 + \\ &+ 2\cosh c \cdot X_0^0 \cdot X_1^0 \cdot \left(X_2^0\right)^2 + 2\cosh b \cdot X_0^0 \cdot \left(X_1^0\right)^2 \cdot X_2^0 + \\ &+ 2\cosh a \cdot \left(X_0^0\right)^2 \cdot X_1^0 \cdot X_2^0 = 0,\end{aligned}$$

which shows that the point M belongs to the Steiner quadratix. □

Remark 3.3. Clearly, the vertices of the triangle ABC belong to the Steiner quadratix, which, thus, is not empty.

4. Some remarkable pairs of isotomic points

As examples, we use the hyperbolic analogs of some classical remarkable points from the geometry of the Euclidean triangles, the Gergonne group of points and the Nagel group of points. For the Euclidean points, see [1] and [8].

4.1. The Gergonne and Nagel Points

In [3], we introduced the Gergonne and Nagel points associated to a hyperbolic triangle. Exactly as it happens for a Euclidean triangle, the Gergonne point is the point obtained by intersecting the lines connecting the vertices of a hyperbolic triangle ABC to the points of contact of the incircle with the opposite sides. The incircle is, for any hyperbolic triangle, a proper circle. For the Nagel point, the definition has to be a little bit adapted to work for an arbitrary hyperbolic triangle. Thus, the Nagel point is obtained as intersection of the lines connecting the vertices of the triangle to the points of contact of the opposite sides to the corresponding excycles. Unlike the Euclidean case, an arbitrary hyperbolic triangle doesn't always have excircles. In some cases, these circles become equidistants or horocycles. We use the term "cycle" to cover all the possible situations.

In [3], we prove that, for each situation, the three cevians really intersect and, moreover, the intersection points are always ordinary. More specifically, we were able to prove that they barycentric coordinates are identical to the barycentric coordinates of their Euclidean analogues, i.e.

- for the Gergonne point we obtain

$$\Gamma = \Gamma\left(\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}\right);$$

- for the Nagel point, we obtain

$$N = N\left(\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}\right).$$

Thus, the Gergonne and Nagel points are isotomic to each other.

We can see immediately, without any computation, that the points are ordinary (as all the coordinates are strictly positive, they are in the interior of the triangle) and they are isotomic to each other.

4.2. The adjoint Gergonne and Nagel points

We introduce these points in [3], by analogy to the classical case. Thus, the *adjoint Gergonne points* $\Gamma_a, \Gamma_b, \Gamma_c$ are the analogues of the Gergonne points, for the excycles. Thus, consider, for instance the excycles that is tangent to the side BC in an interior points. We connect the tangency points with the opposite vertices. We proved in [3] that they intersect at a point Γ_a (which is not necessarily ordinary) and the same happens with the other two vertices of the triangle ABC.

We get, thus, three points

$$\Gamma_a = \Gamma_a\left(-\tan\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}\right),$$
$$\Gamma_b = \Gamma_b\left(\cot\frac{A}{2}, -\tan\frac{B}{2}, \cot\frac{C}{2}\right),$$
$$\Gamma_c = \Gamma_c\left(\cot\frac{A}{2}, \cot\frac{B}{2}, -\tan\frac{C}{2}\right).$$

The lines connecting the extremities of a side of the triangle ABC to the contact points of the excycles lying within the angles adjacent to this side, situated on the extensions of the opposite sides of the one considered and the line that connects the third vertex to the contact point of the incircle to the opposite side are concurrent at a point (ordinary, ideal or ultra-infinite). We get, thus (see [3]), three points N_a, N_b, N_c, called the *adjoint Nagel points* of the triangle ABC:

$$N_a = N_a\left(-\cot\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}\right),$$
$$N_b = N_b\left(\tan\frac{A}{2}, -\cot\frac{B}{2}, \tan\frac{C}{2}\right),$$
$$N_c = N_c\left(\tan\frac{A}{2}, \tan\frac{B}{2}, -\cot\frac{C}{2}\right).$$

It can be seen that each adjoint Nagel point is the isotomic conjugate of the corresponding adjoint Gergonne point.

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