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CRITICAL POINT METHODS IN DEGENERATE ANISOTROPIC PROBLEMS WITH VARIABLE EXPONENT

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Abstract. We work on the anisotropic variable exponent Sobolev spaces and we consider the problem: $-\sum_{i=1}^{N}\partial_{x_i}a_i\left(x,\partial_{x_i}u\right)+b(x)|u|^{P_+^+-2}u=f(x,u)$ in Ω, $u\geq 0$ in Ω and $u=0$ on $\partial\Omega$, where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary and $\sum_{i=1}^{N}\partial_{x_i}a_i\left(x,\partial_{x_i}u\right)$ is a $\overrightarrow{p}(\cdot)$ - Laplace type operator. Relying on the critical point theory and using the mountain-pass theorem, we prove the existence of a unique nontrivial weak solution for our problem.

1. **Introduction**

We are interested in discussing the problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N}\partial_{x_i}a_i\left(x,\partial_{x_i}u\right)+b(x)|u|^{P_+^+-2}u=f(x,u) & \text{in } \Omega,\\ u\geq 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary, the operator $\sum_{i=1}^{N}\partial_{x_i}a_i\left(x,\partial_{x_i}u\right)$ is a $\overrightarrow{p}(\cdot)$ - Laplace type operator, $\overrightarrow{p}(x)=(p_1(x),p_2(x),...p_N(x))$, $b\in L^\infty\left(\overline{\Omega}\right)$, $P_+^+=\max_{i\in\{1,...,N\}}\left\{\sup_{x\in\Omega}p_i(x)\right\}$ and a_i, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions fulfilling some adequate hypotheses. In order to detail the conditions imposed on the functions involved in our problem we make the following notation. We denote by $A_i:\Omega\times\mathbb{R}\to\mathbb{R}$

$$A_i(x,s)=\int_0^s a_i(x,t)dt \quad \text{for all } i\in\{1,...,N\},$$

and by $F:\Omega\times\mathbb{R}\to\mathbb{R}$

$$F(x,s)=\int_0^s f(x,t)dt.$$

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We set $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \,:\, \min_{x\in\overline{\Omega}} p(x) > 1\}$ and we denote, for any $p \in C_+(\overline{\Omega})$,

$$p^+ = \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\Omega} p(x).$$

For $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$, $\overrightarrow{p}(x) = (p_1(x), p_2(x), ...p_N(x))$ with $p_i \in C_+(\overline{\Omega})$, $i \in \{1, ..., N\}$ we denote by $\overrightarrow{P}_+$, $\overrightarrow{P}_- \in \mathbb{R}^N$ the vectors

$$\overrightarrow{P}_+ = (p_1^+, ..., p_N^+), \quad \overrightarrow{P}_- = (p_1^-, ..., p_N^-),$$

and by P_+^+, P_-^+, $P_-^- \in \mathbb{R}^+$ the following:

$$P_+^+ = \max\{p_1^+, ..., p_N^+\}, \quad P_-^+ = \max\{p_1^-, ..., p_N^-\}, \quad P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

We define $P_-^\star \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^\star = \frac{N}{\sum_{i=1}^N 1/p_i^- - 1}\,, \qquad P_{-,\infty} = \max\{P_-^+, P_-^\star\}\,.$$

Now we can state the conditions satisfied by the functions b, p_i, A_i, a_i, f, for all $i \in \{1, ..., N\}$:

(b) there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$;

(p) $p_i \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous (that is, there exists $M > 0$ such that $|p_i(x) - p_i(y)| \leq -M/\log(|x-y|)$ for all x, $y \in \Omega$ with $|x - y| \leq 1/2$), $p_i(x) < N$ for all $x \in \Omega$ and $\sum_{i=1}^N 1/p_i^- > 1$;

(A1) there exists a positive constant $c_{1,i}$ such that a_i satisfies the growth condition

$$|a_i(x, s)| \leq c_{1,i}(1 + |s|^{p_i(x)-1}),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

(A2) the following inequalities hold:

$$|s|^{p_i(x)} \leq a_i(x, s)s \leq p_i(x)\; A_i(x, s),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

(A3) a_i is fulfilling

$$(a_i(x, s) - a_i(x, t))(s - t) > 0,$$

for all $x \in \Omega$ and s, $t \in \mathbb{R}$ with $s \neq t$;

(f1) there exist a positive constant c_3 and $q \in C(\overline{\Omega})$ with $1 < P_-^- < P_+^+ < q^- < q^+ < P_-^\star$, such that f satisfies the growth condition

$$|f(x, s)| \leq c_3|s|^{q(x)-1},$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

(f2) f verifies the Ambrosetti-Rabinowitz type condition: there exists a constant $\mu > P_+^+$ such that for every $x \in \Omega$

$$0 < \mu F(x,s) \leq s f(x,s), \quad \forall s > 0;$$

(f3) f is fulfilling

$$(f(x,s) - f(x,t))(s-t) < 0,$$

for all $x \in \overline{\Omega}$ and $s,\, t \in \mathbb{R}$ with $s \neq t$.

Our main result is stated by the following theorem.

Theorem 1.1. *Suppose that conditions* (b)*,* (p)*,* (A1)-(A3)*,* (f1)-(f3) *are fulfilled, where* $b \in L^{\infty}\left(\overline{\Omega}\right)$ *and* $a_i,\, f : \Omega \times \mathbb{R} \to \mathbb{R}$ *are Carathéodory functions. Then there is a unique nontrivial weak solution to problem (1.1).*

Notice that $\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right)$ is a $\overrightarrow{p}(\cdot)$ - Laplace type operator, since by choosing $a_i(x,s) = |s|^{p_i(x)-2} s$ for all $i \in \{1, ..., N\}$, we have $A_i(x,s) = \frac{1}{p_i(x)} |s|^{p_i(x)}$ for all $i \in \{1, ..., N\}$, and we arrive at the anisotropic variable exponent Laplace operator

$$\Delta_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i} \left(\left|\partial_{x_i} u\right|^{p_i(x)-2} \partial_{x_i} u \right). \tag{1.2}$$

We bring to your attention that when choosing $p_1, ..., p_N$ to be constant functions and $\overrightarrow{p} = (p_1, ..., p_N)$, we obtain the anisotropic $\overrightarrow{p}$ - Laplace operator

$$\Delta_{\overrightarrow{p}}(u) = \sum_{i=1}^{N} \partial_{x_i} \left(\left|\partial_{x_i} u\right|^{p_i-2} \partial_{x_i} u \right),$$

while when choosing $p_1 = ... = p_N = p$ in (1.2) we obtain an operator similar to the variable exponent $p(\cdot)$ - Laplace operator

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u),$$

where p is a continuous function. Therefore it is not only a study of boundary value problems, it is also a "boundary" study, if we take into consideration the fact that the theory of anisotropic variable exponent Lebesgue-Sobolev spaces is situated at the boundary between the the anisotropic Sobolev spaces theory developed by [25, 27, 28, 30, 31] and the variable exponent Sobolev spaces theory developed by [5, 6, 7, 8, 9, 11, 12, 15, 22, 23, 24, 29]. In this newly formed direction of PDEs, various articles appeared and continue to appear. For the proof of our main result we need to explore techniques similar to those used by [2, 3, 4, 14, 20, 21]. In order to not repeat the same arguments we import some of the results presented in these papers, indicating the place where all the calculus details may be found. Generally speaking, we are relying on the critical point theory, that is, we associate to (1.1) a functional energy whose critical points represent the weak solutions of the problem. Among the previously enumerated papers, our work is more closely related to [2, 14, 19], where are also used general $\overrightarrow{p}(\cdot)$ - Laplace type operators under conditions resembling to (A1)-(A3). In [14] the authors prove that the functional energy is proper, weakly lower semi-continuous and coercive, thus it has a minimizer which is a weak solution to their problem. In [2, 19] the authors establish the multiplicity of the solution in addition to its existence. The first paper utilize the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz, while the second one is combining the mountain-pass theorem of Ambrosetti and Rabinowitz with the Ekeland's variational principle. For more variational methods that could prove useful we send the reader to [10, 17, 18].

An interesting remark is that operators fulfilling conditions like (A1)-(A3) were not just considered when working on anisotropic variable exponent Lebesgue-Sobolev spaces. To give some examples, we refer to [16, 22], where it was discussed the following type of problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = f(x,u) & \text{for} \quad x \in \Omega, \\ u = 0 & \text{for} \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary. The preference for conditions (A1)-(A3) comes from the fact that, as said before, the anisotropic variable exponent Laplace operator satisfies them. But the association with the $\overrightarrow{p}(\cdot)$ - Laplace is not the only reason, since there are other well known operators that satisfy these conditions. Indeed, when choosing $a_i(x,s) = (1+|s|^2)^{(p_i(x)-2)/2}s$ for all $i \in \{1,...,N\}$, we have $A_i(x,s) = \frac{1}{p_i(x)}[(1+|s|^2)^{p_i(x)/2} - 1]$ for all $i \in \{1,...,N\}$, and we obtain the anisotropic variable mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i} \left[\left(1 + |\partial_{x_i} u|^2\right)^{(p_i(x)-2)/2} \partial_{x_i} u \right].$$

Now that we have examples of appropriate operators we can pass to shortly describing the structure of the rest of the paper. In the second section we recall the definition and some important properties of the variable exponent spaces, anisotropic spaces and anisotropic variable exponent spaces. In the third section we define the notion of weak solution to problem (1.1) and we introduce the functional energy associated to this problem. Then we present several auxiliary results and we use them to prove the main theorem.

2. **Abstract framework**

In what follows we consider $p,\, p_i \in C_+(\overline{\Omega})$, $i \in \{1, \ldots, N\}$ to be logarithmic Hölder continuous. We define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega)=\{u : u \text{ is a measurable real–valued function such that } \int_\Omega |u(x)|^{p(x)}dx<\infty\}$$

endowed with the so-called Luxemburg norm

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0 \ : \quad \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

We mention that for p constant this norm becomes the norm of the classical Lebesgue space L^p, that is,

$$|u|_p = \left(\int_\Omega |u|^p\right)^{1/p}.$$

The space $\big(L^{p(\cdot)}(\Omega),\, |\cdot|_{p(\cdot)}\big)$ has some important properties. It is a separable and reflexive Banach space ([15, Theorem 2.5, Corollary 2.7]). The inclusion between spaces generalizes naturally: if $0 < |\Omega| < \infty$ and $p_1,\ p_2$ are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous ([15, Theorem 2.8]). The following Hölder-type inequality

$$\left|\int_\Omega uv\ dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(\cdot)}|v|_{p'(\cdot)} \le 2|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{2.1}$$

holds true for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ ([15, Theorem 2.1]), where we denoted by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise i.e. $1/p(x) + 1/p'(x) = 1$ ([15, Corollary 2.7]).
The $p(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space, that is, the function $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\ dx,$$

plays a key role in handling this space. We remind some of its properties (see again [15]): if $u \in L^{p(\cdot)}(\Omega)$, $(u_n) \subset L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$, then,

$$\begin{aligned}
|u|_{p(\cdot)} < 1\ (= 1; > 1) \quad &\Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1\ (= 1; > 1)\\
|u|_{p(\cdot)} > 1 \quad &\Rightarrow \quad |u|^{p^-}_{p(\cdot)} \le \rho_{p(\cdot)}(u) \le |u|^{p^+}_{p(\cdot)}\\
|u|_{p(\cdot)} < 1 \quad &\Rightarrow \quad |u|^{p^+}_{p(\cdot)} \le \rho_{p(\cdot)}(u) \le |u|^{p^-}_{p(\cdot)}\\
|u|_{p(\cdot)} \to 0\ (\to \infty) \quad &\Leftrightarrow \quad \rho_{p(\cdot)}(u) \to 0\ (\to \infty)\\
\lim_{n\to\infty} |u_n - u|_{p(\cdot)} = 0 \quad &\Leftrightarrow \quad \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0.
\end{aligned} \tag{2.2}$$

We define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$,

$$W^{1,p(\cdot)}(\Omega) = \Big\{ u \in L^{p(\cdot)}(\Omega) : \partial_{x_i} u \in L^{p(\cdot)}(\Omega),\ i \in \{1, 2, ...N\} \Big\}$$

endowed with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}. \tag{2.3}$$

The space $\big(W^{1,p(\cdot)}(\Omega), \|\cdot\|\big)$ is a separable and reflexive Banach space. In what concerns the Sobolev space with zero boundary values, we denote it by $W_0^{1,p(\cdot)}(\Omega)$ and we define it as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|$. We consider the norms

$$\|u\|_{1,p(\cdot)} = |\nabla u|_{p(\cdot)},$$

and

$$\|u\|_{p(\cdot)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p(\cdot)}$$

which are equivalent to (2.3) on $W_0^{1,p(\cdot)}(\Omega)$. The space $W_0^{1,p(\cdot)}(\Omega)$ is also a separable and reflexive Banach space. Furthermore, if $s \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, where $p^\star(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^\star(x) = \infty$ if $p(x) \geq N$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous.

We define now the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}\,.$$

The space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ can be considered a natural generalization of both the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ and the classical anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1, ..., p_N)$. The space $W_0^{1,\overrightarrow{p}}(\Omega)$ endowed with the norm

$$\|u\|_{1,\overrightarrow{p}} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i}$$

is a reflexive Banach space for all $\overrightarrow{p} \in \mathbb{R}^N$ with $p_i > 1$, $i \in \{1, ..., N\}$. This result can be easily extended to $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, see [21]. Another extension can be made in what concerns the embedding between $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega)$ [21, Theorem 1]: if $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, the components of $\overrightarrow{p}$ verify (p) and $q \in C(\overline{\Omega})$ verifies $1 < q(x) < P_{-,\infty}$ for all $x \in \overline{\Omega}$, then the embedding

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is continuous and compact.

3. **Proof of the main result**

Working under the hypotheses of Theorem 1.1, we denote $W_0^{1,\vec{p}(\cdot)}(\Omega)$ by E and we start by giving the definition of the weak solution for problem (1.1).

Definition 3.1. By a weak solution to problem (1.1) we understand a function $u \in E$ such that

$$\int_\Omega \left[\sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i}\varphi + b(x)|u|^{P_+^+-2}u\varphi - f(x,u)\varphi \right] dx = 0, \tag{3.1}$$

for all $\varphi \in E$.

As said in the introductory section, we base our proof on the critical point theory, thus we associate to problem (1.1) the energy functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right)\, dx + \frac{1}{P_+^+}\int_\Omega b(x)|u|^{P_+^+}\, dx - \int_\Omega F(x,u_+)\, dx,$$

where $u_+(x) = \max\{u(x), 0\}$.

For all $i \in \{1, 2, ...N\}$, we denote by J_i, $K : E \to \mathbb{R}$ the functionals

$$J_i(u) = \int_\Omega A_i\left(x, \partial_{x_i} u\right) dx \quad \text{and} \quad K(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right)\, dx - \int_\Omega F(x,u_+)\, dx.$$

In what follows we present several results concerning the functionals J_i, K or other terms of I.

Lemma 3.2. *(*[14, Lemma 3.4]*) For $i \in \{1, 2, ...N\}$,*

(i) *the functional J_i is well-defined on E;*
(ii) *the functional J_i is of class $C^1(E, \mathbb{R})$ and*

$$\langle J_i'(u), \varphi\rangle = \int_\Omega a_i(x, \partial_{x_i} u)\partial_{x_i}\varphi\, dx,$$

for all $u, \varphi \in E$.

Lemma 3.3. *(*[2, Section 4, Claim 2]*) There exist ρ, $r > 0$ such that $K(u) \geq r > 0$, for any $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$.*

Remark 3.4. In the proof of [2, Section 4, Claim 2] appeared the fact that f was considered odd in its second variable, thus F was even in its second variable. In our case, this hypothesis is not necessary, since we are interested in $F(x, u_+)$, which has its second variable nonnegative.

Lemma 3.5. *(see* [2, Section 4, Claim 1]*)* *(i) For all $u \in E$,*

$$\sum_{i=1}^{N} \int_{\Omega} \left[A_i(x, \partial_{x_i} u) - \frac{1}{\mu} a_i(x, \partial_{x_i} u) \partial_{x_i} u \right] dx \geq \left(\frac{1}{P_+^+} - \frac{1}{\mu} \right) \left(\frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- - 1}} - N \right),$$

where μ is the constant from (f2).

(ii) For any sequence $(u_n)_n \subset E$ weakly convergent to $u \in E$,

$$\lim_{n \to \infty} \int_{\Omega} f(x, u_n)(u_n - u) dx = 0.$$

Lemma 3.6. *(*[2, Section 3, Lemma 2]*) Let $(u_n)_n \subset E$ be a sequence which is weakly convergent to $u \in E$ and*

$$\limsup_{n \to \infty} \int_{\Omega} \sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u) dx \leq 0.$$

Then $(u_n)_n$ converges strongly to u in E.

By Lemma 3.2 and by a standard calculus, I is well-defined on E and $I \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle I'(u), \varphi \rangle = \int_{\Omega} \sum_{i=1}^{N} a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi \, dx + \int_{\Omega} b(x) |u|^{P_+^+ - 2} u \varphi \, dx - \int_{\Omega} f(x, u) \varphi \, dx,$$

for all $u, \varphi \in E$, therefore the critical points of I are weak solutions to (1.1). Being concerned with the existence of critical points, we search for help in the mini-max principles theory (see for example [1, 13, 26]) and we find it in the mountain-pass theorem of Ambrosetti and Rabinowitz without the Palais-Smale condition. Following the steps described by the statement of this theorem, we first prove two auxiliary results.

Lemma 3.7. *There exist ρ, $r > 0$ such that $I(u) \geq r > 0$, for any $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$.*

Proof. By hypothesis (b),

$$\frac{1}{P_+^+} \int_{\Omega} b(x) |u|^{P_+^+} \, dx \geq \frac{b_0}{P_+^+} |u|_{P_+^+}^{P_+^+} \geq 0,$$

for all $u \in E$. Hence

$$I(u) \geq \int_{\Omega} \sum_{i=1}^{N} A_i \left(x, \partial_{x_i} u \right) \, dx - \int_{\Omega} F(x, u_+) \, dx,$$

for all $u \in E$. Using Lemma 3.3, we deduce that we can find ρ, $r > 0$ such that $I(u) \geq r > 0$, for all $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$. ☐

Lemma 3.8. *There exists $e \in E$ with $\|e\|_{\overrightarrow{p}(\cdot)} > \rho$ (ρ given by Lemma 3.7) such that $I(e) < 0$.*

Proof. Since for all $i \in \{1, ..., N\}$, $A_i(x,s) = \int_0^s a_i(x,t)\,dt$, by a simple change of variables and by condition (A1) we get

$$A_i(x,s) \le c_{1,i} \int_0^1 |a_i(x,ts)s|\;dt \le c_{1,i}\left(|s| + \frac{|s|^{p_i(x)}}{p_i(x)}\right) \quad \text{for all } x \in \Omega,\ s \in \mathbb{R}.$$

Setting $C_1 = \max\{c_{1,i} : \ i \in \{1, 2, ...N\}\}$, by the above relation we obtain that

$$\int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i}(tu)\right)\,dx \le C_1 \int_\Omega \sum_{i=1}^N |\partial_{x_i}(tu)|\,dx + \frac{C_1}{P_-^-}\int_\Omega \sum_{i=1}^N |\partial_{x_i}(tu)|^{p_i(x)}\,dx$$

for all $u \in E$.

On the other hand, by rewriting condition (f2), we deduce that there exists a positive constant c_4 such that

$$F(x,s) \ge c_4\,|s|^\mu, \quad \forall x \in \Omega,\ \forall s \ge 0,$$

therefore

$$I(tu) \le C_1\,t \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|\,dx + \frac{C_1\,t^{P_+^+}}{P_-^-}\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,dx$$

$$+\frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|u|^{P_+^+}\,dx - c_4\,t^\mu \int_\Omega |u|^\mu\,dx,$$

for all $u \in E$ and $t > 1$. Then, due to the fact that $\mu > P_+^+ > 1$, for $u \not\equiv 0$ we have $I(tu) \to -\infty$ when $t \to \infty$ and we can choose t large enough and $e = tu \in E$ with $\|e\|_{\overrightarrow{p}(\cdot)} > \rho$ such that

$$I(e) < 0.$$

□

Proof of Theorem 1.1. Proof of existence. By Lemma 3.7, Lemma 3.8 and the mountain-pass theorem of Ambrosetti and Rabinowitz, there exist a sequence $(u_n)_n \subset E$ such that

$$I(u_n) \to \alpha > 0 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Notice that from the definition of the functional I we can consider $(u_n)_n$ to be a sequence of nonnegative functions. We will prove that $(u_n)_n$ is bounded in E by arguing by contradiction, more exactly by assuming that, passing eventually to a subsequence still denoted by $(u_n)_n$,

$$\|u_n\|_{\overrightarrow{p}(\cdot)} \to \infty \quad \text{when } n \to \infty. \tag{3.3}$$

Combining relations (3.2) and(3.3) we infer

$$\begin{aligned}
1+\alpha+\|u_n\|_{\vec{p}(\cdot)} &\geq I(u_n)-\frac{1}{\mu}\langle I'(u_n),u_n\rangle \\
&\geq \sum_{i=1}^{N}\int_\Omega\left[A_i(x,\partial_{x_i}u_n)-\frac{1}{\mu}a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n\right]dx+ \\
&\quad +\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\int_\Omega b(x)|u|^{P_+^+}\,dx- \\
&\quad -\int_\Omega\left[F(x,u_n)-\frac{1}{\mu}u_nf(x,u_n)\right]dx \\
&\geq \sum_{i=1}^{N}\int_\Omega\left[A_i(x,\partial_{x_i}u_n)-\frac{1}{\mu}a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n\right]dx,
\end{aligned}$$

for sufficiently large n, since μ is the constant from (f2). By Lemma 3.5(i) we come to

$$1+\alpha+\|u_n\|_{\vec{p}(\cdot)}\geq\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\left(\frac{\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- -1}}-N\right),$$

hence by dividing by $\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}$ and passing to the limit as $n\to\infty$ we obtain the desired contradiction and we conclude that $(u_n)_n$ is bounded in E. We know that the space E is reflexive, thus there is $u_0\in E$ such that, up to a subsequence, $(u_n)_n$ converges weakly to u_0 in E. We need to show that $(u_n)_n$ converges strongly to u_0 in E.

The fact that $P_+^+<P_{-,\infty}$ implies that the embedding $E\hookrightarrow L^{P_+^+}(\Omega)$ is compact. Thus $(u_n)_n$ converges strongly to u_0 in $L^{P_+^+}(\Omega)$. By the Hölder-type inequality (2.1),

$$\left|\int_\Omega b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0)dx\right|\leq 2\|b\|_{L^\infty(\Omega)}\left||u_n|^{P_+^+-2}u_n\right|_{\frac{P_+^+}{P_+^+-1}}|u_n-u_0|_{P_+^+}.$$

Using the strong convergence of $(u_n)_n$ to u_0 in $L^{P_+^+}(\Omega)$, the above relation and (2.2) we come to

$$\lim_{n\to\infty}\int_\Omega b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0)dx=0. \tag{3.4}$$

Let us consider the relations

$$\begin{aligned}
<I'(u_n),u_n-u_0>=\int_\Omega\Big[\sum_{i=1}^{N}a_i(x,\partial_{x_i}u_n)(\partial_{x_i}u_n-\partial_{x_i}u_0)+b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0) \\
-f(x,u_n)(u_n-u_0)\Big]dx
\end{aligned}$$

and, from (3.2),

$$\lim_{n\to\infty} < I'(u_n), u_n - u_0 >= 0.$$

Combining these relations with (3.4) and Lemma 3.5(ii) we obtain

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0) dx = 0.$$

Using Lemma 3.6 we deduce that $(u_n)_n$ converges strongly to u_0 in E. By (3.2) u_0 is a critical point to I and $I(u_0) = \alpha > 0$. Since $I(0) = 0$ it follows that u_0 is a nontrivial weak solution to (1.1). □

Proof of uniqueness. Let us assume that there exist two nontrivial solutions to problem (1.1), that is, u_1 and u_2. We replace the solution u by u_1 in (3.1) and we choose $\varphi = u_1 - u_2$. We obtain

$$\begin{aligned} \int_\Omega \Big[\sum_{i=1}^N a_i \left(x, \partial_{x_i} u_1\right) \partial_{x_i} (u_1 - u_2) + b(x) |u_1|^{P_+^+ - 2} u_1 (u_1 - u_2) \\ - f(x, u_1)(u_1 - u_2) \Big] dx = 0. \end{aligned} \tag{3.5}$$

Next, we replace the solution u by u_2 in (3.1) and we choose $\varphi = u_2 - u_1$. We infer

$$\begin{aligned} \int_\Omega \Big[\sum_{i=1}^N a_i \left(x, \partial_{x_i} u_2\right) \partial_{x_i} (u_2 - u_1) + b(x) |u_2|^{P_+^+ - 2} u_2 (u_2 - u_1) \\ - f(x, u_2)(u_2 - u_1) \Big] dx = 0. \end{aligned} \tag{3.6}$$

Putting together (3.5) and (3.6) we arrive at

$$\begin{aligned} \int_\Omega \left\{ \sum_{i=1}^N \left[a_i \left(x, \partial_{x_i} u_1\right) - a_i \left(x, \partial_{x_i} u_2\right)\right] \left(\partial_{x_i} u_1 - \partial_{x_i} u_2\right) \right\} dx + \\ + \int_\Omega b(x) \left[|u_1|^{P_+^+ - 2} u_1 - |u_2|^{P_+^+ - 2} u_2 \right] (u_1 - u_2)\, dx - \\ - \int_\Omega \left[f(x, u_1) - f(x, u_2)\right] (u_1 - u_2)\, dx = 0. \end{aligned}$$

By hypotheses (A3) and (f3), all the terms in the above equality are positive unless $u_1 = u_2$, and this yields the uniqueness of the solution. □

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