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APPROXIMATION OF SOLUTION OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH CONDITIONS INSIDE THE INTERVAL $(0,1)$ USING CUBIC B-SPLINE FUNCTIONS

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Abstract. In this paper, we present a numerical algorithm, based on cubic B-splines collocation method for solving the second order differential equations with condition inside the (0,1) interval. The scheme is shown to be accurate and only a few terms are required to obtain approximate solution.

1. **Introduction**

The purpose of this paper is to approximate the solution of the following problem:

$$\begin{cases} Ly(x) = r(x),\ 0 \leq x \leq 1, \\ y(a) = A,\ y(b) = B \\ 0 < a < b < 1,\ a, b, A, B \in \mathbb{R}, \end{cases} \tag{1.1}$$

where:

$$Ly(x) := -\frac{d}{dx}(\frac{dy}{dx}) + q(x) \cdot y(x),\ 0 \leq x \leq 1 \tag{1.2}$$

and $q(x), r(x) \in C(0,1),\ q(x) < 0,\ y(x) \in C^2(0,1)$, using cubic B-splines collocation method.

In fall-back we assume that the functions $q(x)$, $r(x)$ are such that the problem (1.1) has a unique solution.

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2. **Preliminaries**

To describe the basic method in this and later sections we choose a uniform mesh of the given interval $(0,1)$

$$\Delta := 0 = x_0 < x_1 < \ldots < x_{n+1} = 1,\ x_j = j\cdot h,\ j = 0,1,\ldots,n+1,\ h = \tfrac{1}{n+1}. \tag{2.1}$$

We introduce the domain of definition of the operator L, defined by (1.2), as

$$D_B(L) := \{u \in C^2(0,1)|\ Lu \in C^2(0,1) \text{ and } u(a) = A,\ u(b) = B\},$$

and suppose that $D_B(L)$ is dense in $C^2(0,1)$ such that:

$$L : D_B(L) \subseteq C^2(0,1) \to C^2(0,1).$$

We also define

$$U := \{u \in D_B(L) : u|_{[x_i,x_{i+1}]} \in \Pi_3, u' \in C[x_j, x_{j+1}]\},$$

where Π_3 is the set of polynomials of degree at most three. We use the following notations: $q_j := q(x_j)$, $r_j := r(x_j)$.

Obviously, dim $U = n+2$. Because $U \subset C^2(0,1)$ (see [3]) there exists in U a basis consisting of cubic B-splines functions:

$$\{s_0, s_1, ...s_{n+1}\}$$

and moreover (see [4, page 555], [5, page 69]):

$$s_i(x) = S(\frac{x}{h} - i),\ i = 0,1,2,...,n+1 \tag{2.2}$$

$$s_i''(x) = \frac{1}{h^2} \cdot S''(\frac{x}{h} - i), i = 0,1,...,n+1 \tag{2.3}$$

where:

$$S(x) = \frac{1}{4!}\begin{cases} 0 & \text{if } x \in \mathbb{R}\backslash[-2,2] \\ (2-x)^3 - 4(1-x)^3 - 6x^3 + 4(1+x)^3 & \text{if } x \in [-2,-1] \\ (2-x)^3 - 4(1-x)^3 - 6x^3 & \text{if } x \in [-1,0] \\ (2-x)^3 - 4(1-x)^3 & \text{if } x \in [0,1] \\ (2-x)^3 & \text{if } x \in [1,2] \end{cases}$$

The graph of S is shown in Figure 1.

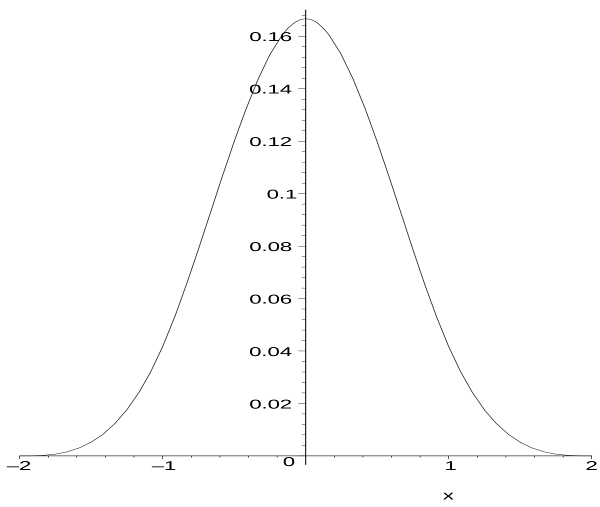


FIGURE 1. The graph of basic function S.

We observe that:

$$S(0) = \frac{1}{6};\ S(1) = S(-1) = \frac{1}{24};\ S(1) = S(-1) = \frac{1}{4}S(0) \tag{2.4}$$

$$S^{''}(x) = \frac{1}{4}\begin{cases} 0 & \text{if } x \in \mathbb{R} - (-2,2) \\ x+2 & \text{if } -2 \le x \le -1 \\ -3x-2 & \text{if } -1 < x \le 0 \\ 3x-2 & \text{if } 0 < x \le 1 \\ 2-x & \text{if } 1 < x \le 2 \end{cases}$$

The graph of S'' is shown in Figure 2.

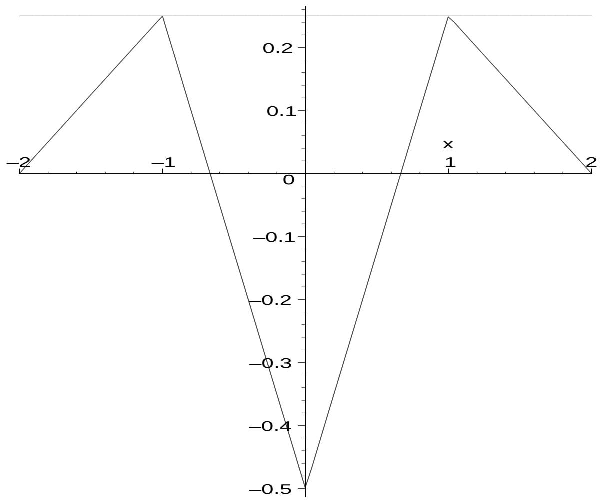


FIGURE 2. The graph of S''.

We also see:

$$S''(0) = -\frac{1}{2};\ S''(-1) = S''(1) = \frac{1}{4};\ S''(1) = S''(-1) = -\frac{1}{2} \cdot S''(0) \tag{2.5}$$

Using Orthogonal Spline Collocation Methods (see [2, pp. 2-5]) an approximate solution $s_y(x) \in U$ has the form:

$$s_y(x) = \sum_{i=0}^{n+1} c_i \cdot s_i(x),\ c_i \in \mathbb{R}, i = 0, 1, ..., n+1.$$

Moreover, from (2.2):

$$s_y(x) = \sum_{i=0}^{n+1} c_i \cdot S(\frac{x}{h} - i),\ c_i \in \mathbb{R}, i = 0, 1, ..., n+1.$$

3. **Main Result**

Lemma 3.1. *For any $i = 0, 1, 2, ..., n, n+1$, $j = 1, 2, ...n$ the following relations hold:*

$$s_i(x_j) = \begin{cases} \frac{1}{24},\ \text{if } i = j-1, i = j+1 \\ \frac{1}{6},\ \text{if } i = j \\ 0,\ \text{otherwise} \end{cases} \tag{3.1}$$

$$s_i''(x_j) = -\frac{1}{2 \cdot h^2} \begin{cases} -\frac{1}{2}, \ \textit{if } i = j-1, i = j+1 \\ 1, \ \textit{if } i = j \\ 0, \ \textit{otherwise.} \end{cases} \tag{3.2}$$

Proof. Because $s_i(x_j) \neq 0$, only for $i = j-1, j, j+1$ (see Figure 3), using (2.2) we obtain:

if $i = j-1$ => $s_i(x_j) = S(j-i) = S(1) = \frac{1}{24}$
if $i = j$ => $s_i(x_j) = S(j-i) = S(0) = \frac{1}{6}$
if $i = j+1$ => $s_i(x_j) = S(j-i) = S(-1) = \frac{1}{24}$.
Also $s_i''(x_j) \neq 0$, only for $i = j-1, j, j+1$ using (2.3) we obtain:
if $i = j-1$ => $s_i''(x_j) = \frac{1}{h^2} \cdot S''(j-i) = \frac{1}{h^2} \cdot S''(1) = \frac{1}{4 \cdot h^2}$
if $i = j$ => $s_i''(x_j) = \frac{1}{h^2} \cdot S''(j-i) = \frac{1}{h^2} \cdot S''(0) = -\frac{1}{2 \cdot h^2}$
if $i = j+1$ => $s_i''(x_j) = \frac{1}{h^2} \cdot S''(j-i) = \frac{1}{h^2} \cdot S''(-1) = \frac{1}{4 \cdot h^2}$ □

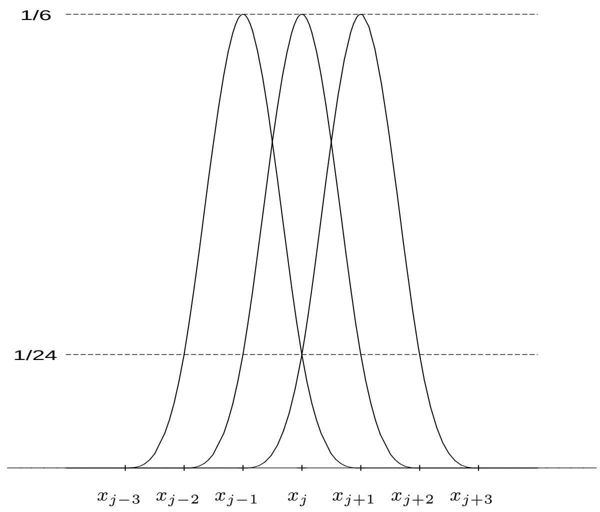


FIGURE 3. Non-zero B-splines on interval $[x_{j-3}, x_{j+3}]$.

Lemma 3.2. *For any $x_0 \in [x_j, x_{j+1}]$, $j = 1, 2, ..., n-1$ the following inequalities hold:*

$$0 < \lim_{h->0} s_{j+2}(x_0) = \lim_{h->0} s_{j-1}(x_0) < \frac{1}{24}, \tag{3.3}$$

$$\frac{1}{24} < \lim_{h->0} s_{j+1}(x_0) = \lim_{h->0} s_j(x_0) < \frac{1}{8}. \tag{3.4}$$

Proof. Let $\lambda(h) := s_{j+2}(x_0)$; $\eta(h) := s_{j+1}(x_0)$; $\rho(h) := s_j(x_0)$; $\varphi(h) := s_{j-1}(x_0)$. In ([5, page 72]) one shows that:

$$s_{j+2}(x) = S(\frac{x}{h} - j - 2) =$$

$$\frac{1}{24} \cdot \begin{cases} [(j{+}4)h - x]^3 - 4[(j{+}3)h - x]^3 + 6[(j{+}2)h - x]^3 - 4[(j{+}1)h - x]^3 \\ \quad \text{if } \ h \le x \le (j{+}1)h \\ [(j{+}4)h - x]^3 - 4[(j{+}3)h - x]^3 + 6[(j{+}2)h - x]^3 \\ \quad \text{if } (j+1)h \le x \le (j+2)h \\ [(j{+}4)h - x]^3 - 4[(j{+}3)h - x]^3; \\ \quad \text{if } (j \ +2)h \le x \le (j+3)h \\ [(j+4)h - x]^3 \\ \quad \text{if } (j+3)h \le x \le (j+4)h \\ 0, \text{ otherwise.} \end{cases}$$

Because $x_0 \in [x_j, x_{j+1}]$ and $x_j = jh$ it follows $x_0 \in [jh, (j+1)h]$. Then

$$\begin{aligned} &0 < \lim_{h->0} \lambda(h) = \\ &= \frac{1}{24} \lim_{h->0} \{[(j+4)h - x_0]^3 - 4[(j+3)h - x_0]^3 + \\ &\quad + 6[(j+2)h - x_0]^3 - 4[(j+1)h - x_0]^3\} = \frac{x_0^3}{24}. \end{aligned}$$

Since $0 < x_0 < 1$, we obtain the following relations on λ, φ, η, ρ:

$$0 < \lim_{h->0} \lambda(h) < \frac{1}{24},$$

$$0 < \lim_{h->0} \varphi(h) = \frac{1}{24} \lim_{h->0} \{[(j+1)h - x_0]^3 - 4[jh - x_0]^3 +$$

$$6[(j-1)h - x_0]^3\} = \frac{x_0^3}{24},$$

$$0 < \lim_{h->0} \varphi(h) < \frac{1}{24},$$

$$0 < \lim_{h->0} \eta(h) = \frac{1}{24} \lim_{h->0} \{[(j+3)h - x_0]^3 - 4 \cdot [j \cdot h - x_0]^3\} = \frac{x_0^3}{8},$$

$$\frac{1}{24} < \lim_{h->0} \eta(h) < \frac{1}{8},$$

and

$$0 < \lim_{h->0} \rho(h) = \frac{1}{24} \lim_{h->0} \{[(j+2)h - x_0]^3 - 4 \cdot [(j+1) \cdot h - x_0]^3\} = \frac{x_0^3}{8}.$$

Finally, $0 < x_0 < 1$, implies

$$\frac{1}{24} < \lim_{h->0} \rho(h) < \frac{1}{8}.$$

□

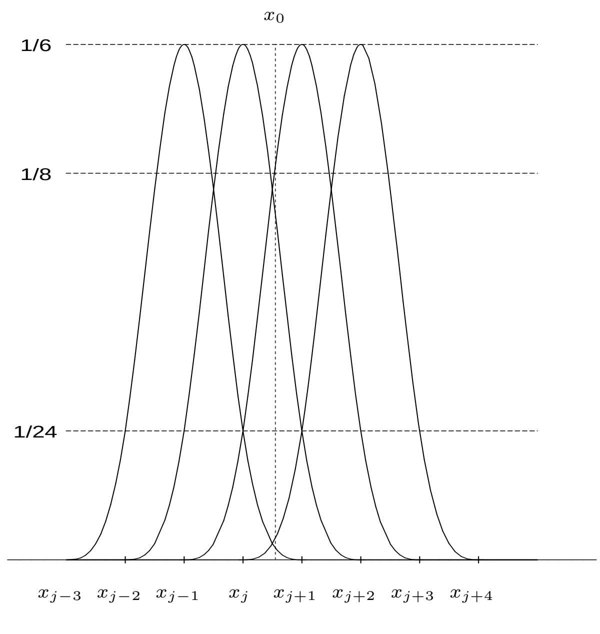


FIGURE 4. Behaviour of B-splines within the interval $[x_j, x_{j+1}]$.

Lemma 3.3. *For $h \to 0$, it holds*

$$\begin{aligned} 0 < \varphi(h) < \eta(h) < \frac{1}{8}, \\ 0 < \lambda(h) < \eta(h) < \frac{1}{8}, \\ 0 < \varphi(h) < \rho(h) < \frac{1}{8}, \\ 0 < \lambda(h) < \rho(h) < \frac{1}{8}, \end{aligned} \tag{3.5}$$

$$\frac{1}{12} < \varphi(h) + \eta(h) + \lambda(h) + \rho(h) < \frac{1}{3}. \tag{3.6}$$

Proof. The relations (3.5) and (3.6) are immediate consequences of the previous lemma. □

We show that

$$s_y(x) = \sum_{i=0}^{n+1} c_i \cdot S(\frac{x}{h} - i) \tag{3.7}$$

can be determinate to be an approximate solution of problem (1.1). We impose the conditions:

$$Ls_y(x) = r(x),\ 0 < x < 1 \tag{3.8}$$

$$s_y(a) = A, s_y(b) = B,\ 0 < a < b < 1. \tag{3.9}$$

Theorem 3.4. *If the problem (1.1) has a unique solution, then there exists and it is unique a function $s_y(x)$ which verifies (3.9) and (3.8) on mesh points*

$$x_j = jh;\ \ j = 1, 2, 3, ..., n.$$

Proof. We suppose that $a \in [x_j, x_{j+1}]$. Because $s_i(x) \neq 0$, only for $x_{j-2} < x < x_{j+2}$ then

$$S(\frac{a}{h} - i) \neq 0,\ \text{only for } i = j-1, j, j+1, j+2,$$

and

$$A = c_{j-1} \cdot S(\frac{a}{h} - j + 1) + c_j \cdot S(\frac{a}{h} - j) + c_{j+1} \cdot S(\frac{a}{h} - j - 1) + c_{j+2} \cdot S(\frac{a}{h} - j - 2).$$

Let $\alpha := S(\frac{a}{h} - j + 1); \beta := S(\frac{a}{h} - j); \gamma := S(\frac{a}{h} - j - 1); \delta := S(\frac{a}{h} - j - 2)$ then:

$$c_{j-1} \cdot \alpha + c_j \cdot \beta + c_{j+1} \cdot \gamma + c_{j+2} \cdot \delta = A \tag{3.10}$$

Since $a < b$, then $b \in [x_{j+m}, x_{j+m+1}]$, $j = 1, 2, ..., n$, $m = 1, 2, ..., n - j$ and

$$S(\frac{b}{h} - i) \neq 0, \text{ only for } i = j + m - 1, j + m, j + m + 1, j + m + 2.$$

Also,

$$\begin{aligned} B &= s_y(b) = \\ &= c_{j+m-1} \cdot S(\frac{b}{h} - j - m + 1) + c_{j+m} \cdot S(\frac{b}{h} - j - m) + \\ &\quad + c_{j+m+1} \cdot S(\frac{b}{h} - j - m - 1) + c_{j+m+2} \cdot S(\frac{b}{h} - j - m - 2). \end{aligned}$$

Let $\mu := S(\frac{b}{h} - j - m + 1); \varepsilon := S(\frac{b}{h} - j - m); \tau := S(\frac{b}{h} - j - m - 1); \xi := S(\frac{b}{h} - j - m - 2)$ then:

$$c_{j+m-1} \cdot \mu + c_{j+m} \cdot \varepsilon + c_{j+m+1} \cdot \tau + c_{j+m+2} \cdot \xi = B. \tag{3.11}$$

We impose the conditions:

$$Ls_y(x_j) = r_j; \;\; j = 1, 2, ..., n.$$

Using (2.2) and (2.3) we have

$$Ls_y(x_j) = -\sum_{i=0}^{n+1} c_i \cdot [\frac{1}{h^2} \cdot S''(j - i) - q_j \cdot S(j - i)] = r_j; \; j = 1, 2, ..., n.$$

Because $s_i(x_j) \neq 0$ and $s_i''(x_j) \neq 0$ only for $i = j - 1, j, j + 1$, using Lemma 3.1 we obtain:

$$\begin{aligned} Ls_y(x_j) = -\frac{1}{h^2} S''(0)[-\frac{1}{2}c_{j-1} + c_j - \frac{1}{2}c_{j+1}] + \\ q_j S(0)[\frac{1}{4}c_{j-1} + c_j + \frac{1}{4}c_{j+1}] = r_j; \; j = 1, 2, ..., n. \end{aligned}$$

Relations (2.5) and (2.4) yield

$$\begin{aligned} Ls_y(x_j) &= \frac{1}{4}c_{j-1}(\frac{q_j}{6} - \frac{1}{h^2}) + \frac{1}{2}c_j(\frac{q_j}{3} + \frac{1}{h^2}) + c_{j+1}(\frac{q_j}{6} - \frac{1}{h^2}) \\ &= r_j, \quad j = 1, 2, ..., n. \end{aligned} \tag{3.12}$$

Because $q(x) < 0$, for any $0 < x < 1$, then

$$\frac{1}{4} \cdot (\frac{q_j}{6} - \frac{1}{h^2}) < 0, \; j = 0, 1, ..., n + 1$$

we may divide the relation(3.12) by $\frac{1}{4} \cdot (\frac{q_j}{6} - \frac{1}{h^2})$ and let:

$$p_j := \frac{24 \cdot h^2}{h^2 \cdot q_j - 6} \cdot r_j;\ t_j := \frac{4(h^2 \cdot q_j + 3)}{h^2 \cdot q_j - 6};$$

then the relation(3.12) has the form:

$$c_{j-1} + t_j \cdot c_j + c_{j+1} = p_j. \tag{3.13}$$

We also observe that:

$$\lim_{h->0} t_j = -2. \tag{3.14}$$

The relations (3.10), (3.11), and (3.13) form a linear system of $(n+2)$ equations with $(n+2)$ unknowns $c_0, c_1, ..., c_{n+1.}$:

$$\begin{cases} c_0 \ + t_1 c_1 + c_2 & = p_0 \\ c_1 + t_2 c_2 + c_3 & = p_1 \\ \vdots & \\ c_{j-1} + t_j c_j + c_{j+1} & = p_j \\ \alpha c_{j-1} + \beta c_j + \gamma c_{j+1} + \delta c_{j+2} & = A \\ c_j + t_{j+1} c_{j+1} + c_{j+2} & = p_{j+1} \\ \vdots & \\ c_{j+m-1} + t_{j+m} c_{j+m} + c_{j+m+1} & = p_{j+m} \\ \mu c_{j+m-1} + \varepsilon c_{j+m} + \tau c_{j+m+1} + \xi c_{j+m+2} & = B \\ c_{j+m} + t_{j+m+1} c_{j+m+1} + c_{j+m+2} & = p_{j+m+1} \\ \vdots & \vdots \\ c_{n-1} + t_n c_n + c_{n+1} & = p_n \end{cases} \tag{3.15}$$

The system matrix is a band matrix with at most four nonzero diagonals. We note this matrix with C and his determinant with $\det C$. If we develop $\det C$ after the columns $0, 1, 2, ..., j-2, j+m+3, ..., n, n+1$ we obtain:

$$\det C = \begin{vmatrix} 1 & t_j & 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ \alpha & \beta & \gamma & \delta & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & t_{j+1} & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ .. & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & 1 & t_{j+m} & 1 & 0 \\ ... & ... & ... & ... & ... & ... & \mu & \varepsilon & \tau & \xi \\ ... & ... & ... & ... & ... & ... & 0 & 1 & t_{j+m+1} & 1 \end{vmatrix} \tag{3.16}$$

If in Lemma 3.2 we set $x_0 := a$; $x_0 := b$ then applying Lemma 3.3, it follows α, ξ, δ, μ are nonzero. In $\det C$ from (3.16) using the properties of determinants, we obtain the following determinant:

$$\det C = \begin{vmatrix} c & d & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & t_{j+1} & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & t_{j+2} & 1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 1 & t_{j+m-1} & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 1 & t_{j+m} & 1 \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 & e & f \end{vmatrix}, \tag{3.17}$$

where

$$\begin{aligned} c &:= \beta - \alpha t_j - \delta; \\ d &:= \gamma - \alpha - \delta t_{j+1}; \\ e &:= \varepsilon - \xi - \mu t_{j+m}; \\ f &:= \tau - \xi t_{j+m+1} - \mu. \end{aligned}$$

From (3.14) and Lemma 3.2 we have:

$$\lim_{h->0} c = \lim_{h->0} (\gamma - \alpha - \delta t_{j+1}) = \frac{a^3}{6}, \forall a \in (0,1),$$

$$\lim_{h->0} d = \lim_{h->0} (\beta - \alpha t_j - \delta) = \frac{a^3}{6}, \forall a \in (0,1),$$

$$\lim_{h->0} e = \lim_{h->0} (\varepsilon - \xi - \mu t_{j+m}) = \frac{b^3}{6}; \forall b \in (0,1),$$

$$\lim_{h->0} f = \lim_{h->0} (\tau - \xi t_{j+m+1} - \mu) = \frac{b^3}{6}; \forall b \in (0,1).$$

Let $x := \frac{\beta - \alpha t_j - \delta}{\gamma - \alpha - \delta t_{j+1}}$, $z := \frac{\tau - \xi t_{j+m+1} - \mu}{\varepsilon - \xi - \mu t_{j+m}}$, then from (3.17),

$$\det C = ce \begin{vmatrix} x & 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 1 & t_{j+1} & 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & t_{j+2} & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ .. & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & 1 & t_{j+m-1} & 1 & 0 \\ ... & ... & ... & ... & ... & ... & 0 & 1 & t_{j+m} & 1 \\ ... & ... & ... & ... & ... & ... & 0 & 0 & 1 & z \end{vmatrix}$$

But, $\lim_{h->0} x(h) = \lim_{h->0} z(h) = 1$ and from (3.14), we have

$$\lim_{h->0} |t_j(h)| = 2.$$

Let:

$$D := \begin{bmatrix} x & 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 1 & t_{j+1} & 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & t_{j+2} & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ .. & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & 1 & t_{j+m-1} & 1 & 0 \\ ... & ... & ... & ... & ... & ... & 0 & 1 & t_{j+m} & 1 \\ ... & ... & ... & ... & ... & ... & 0 & 0 & 1 & z \end{bmatrix}$$

The matrix D is symmetrical, and for $h \to 0$, is diagonal dominant, therefore is nonsingular. Then $\det D \neq 0, \det C \neq 0$ and the system (3.15) has a unique solution.

□

4. **Numerical Results**

We shall approximate the solution of following boundary value problem:

$$-Z''(t) - 243Z(t) = t; 0 \leq t \leq 1 \tag{4.1}$$

$$Z(0) = Z(1) = 0$$

with conditions:

$$Z\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{3\sqrt{3}}{2}\pi\right) - \frac{1}{6}\pi\sin\left(9\sqrt{3}\right)}{243\sin\left(9\sqrt{3}\right)}$$

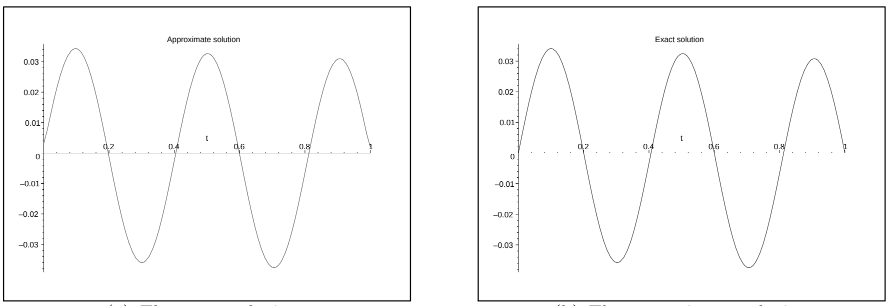
$$Z\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{9\sqrt{3}}{4}\pi\right) - \frac{1}{4}\pi\sin\left(9\sqrt{3}\right)}{243\sin\left(9\sqrt{3}\right)}.$$

The problem (4.1) has a unique solution:

$$Z(t) = \frac{\sin\left(9\sqrt{3}t\right) - t\sin\left(9\sqrt{3}\right)}{243\sin\left(9\sqrt{3}\right)}.$$

We used Maple 8 to solve the problem exactly and to approximate the solution. The mesh considered was uniform, with $h = \frac{1}{52}$.

Figure 5(a) illustrates the graph of the exact solution. Figure 5(b) shows the graph of the approximate solution. Both solution are represented on the same graph in Figure 6. The graph of error in a semilogarithmic scale is given in Figure 7.



(a) The exact solution

(b) The approximate solution

FIGURE 5. Exact (left) and approximate solution

Table 1 gives the coefficients of approximation.

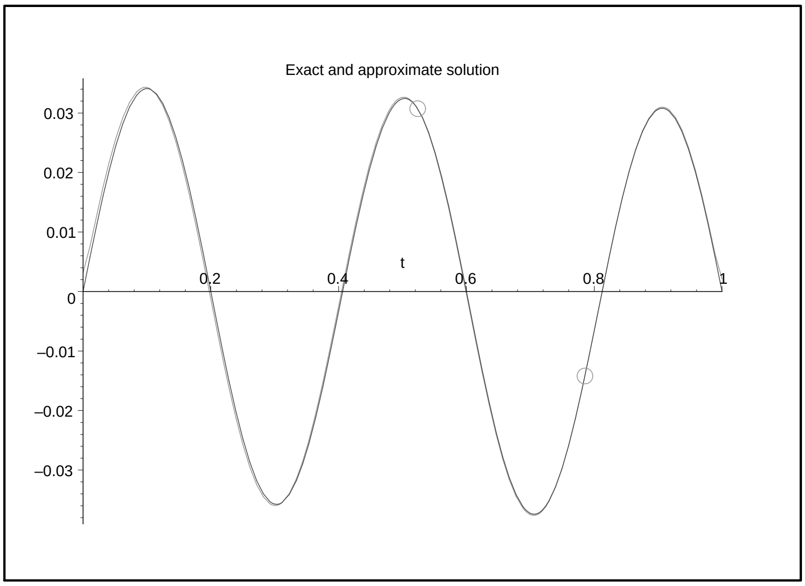


FIGURE 6. The exact and approximate solution on the same graph

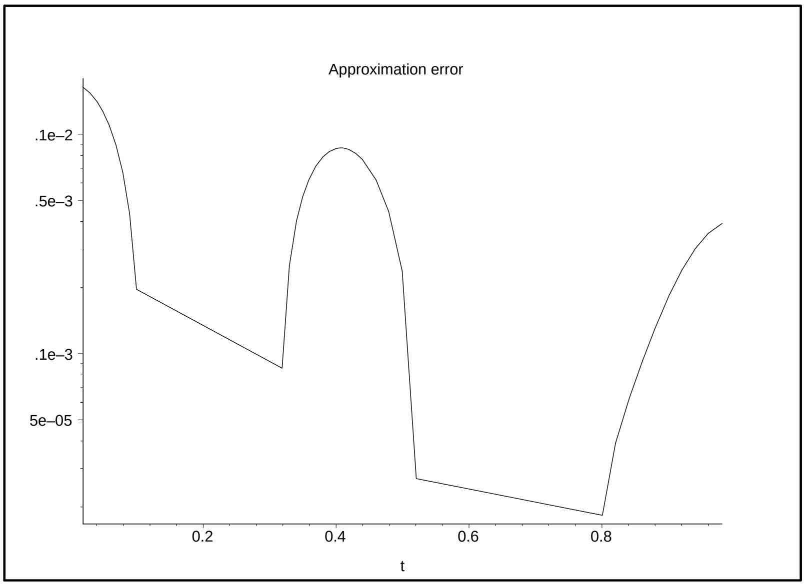


FIGURE 7. Error, plotted in semilogarithmic scale

0.00001264126797	0.01478815919	0.02952783767	0.04419585544
0.05875642949	0.07317381026	0.08741229213	0.1014362548
0.1152101532	0.1286985421	0.1418660874	0.1546775626
0.1670978974	0.1790921693	0.1906256025	0.2016636227
0.2121718270	0.2221160295	0.2314622429	0.2314622429
0.2401767329	0.2482259974	0.2555767791	0.2621961090
0.2680512859	0.2731099106	0.2773398850	0.2807094234
0.2831870848	0.2847417600	0.2853427074	0.2849595322
0.2835622375	0.2811212005	0.2776072155	0.2729914830
0.2672456113	0.2603416764	0.2522521606	0.2429500325
0.2324087120	0.2206021009	0.2075045846	0.1930910568
0.1773368827	0.1602179832	0.1417107936	0.1217922539
0.1004398914	0.07763175011	0.05334645858	0.02756317540
0.0002616852659			

TABLE 1. Coefficients of approximation.

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