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ON SOME CLASSES OF ANALYTIC FUNCTIONS DEFINED BY A MULTIPLIER TRANSFORMATION

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Abstract. We introduce two new classes of analytic functions defined by applying a multiplier transformation to functions $f \in \mathcal{A}(p)$ and study some containment properties of this classes.

1. Preliminaries

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + ...\}.$$

For $p \in \mathbb{N}^*$, we consider $\mathcal{A}(p)$ to be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

which are analytic in the unit disk U.

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since we use the terms of subordination and superordination, we review here those definitions. Let $f, F \in \mathcal{H}$. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, if there exists a function w analytic in U, with

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$w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

The functions considered in this paper and conditions on them are defined uniformly in the unit disk U, so we shall omit the requirement "$z \in U$".

For $c > -p$, $\delta \in \mathbb{R}$ and for a given function $f \in \mathcal{A}(p)$, we consider the multiplier transformation of functions $f \in \mathcal{A}(p)$, introduced in [5] by

$$K_p^\delta f(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{c+p}{c+n}\right)^\delta a_n z^n.$$

For $\delta \geq 0$ we find that K_p^δ is the Komatu linear operator, defined in [2] by

$$K_p^\delta f(z) = \frac{(c+p)^\delta}{\Gamma(\delta)} \frac{1}{z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t)\, dt.$$

We introduce and study some properties of the following classes of functions.

Definition 1.1. *Let ϕ be an analytic functions in the unit disk, with $\phi(0) = 1$ and $\lambda \geq 0$. A function $f \in \mathcal{A}(p)$ is said to be in the class $\Omega_p^\delta(\phi, \lambda)$ if it satisfies the following subordination:*

$$\frac{\lambda}{p} \frac{K_p^{\delta-1} f(z)}{z^p} + \frac{p-\lambda}{p} \frac{K_p^\delta f(z)}{z^p} \prec \phi(z),$$

and is said to be in the class $\overline{\Omega}_p^\delta(\phi, \lambda)$ if it satisfies the superordination

$$\phi(z) \prec \frac{\lambda}{p} \frac{K_p^{\delta-1} f(z)}{z^p} + \frac{p-\lambda}{p} \frac{K_p^\delta f(z)}{z^p}.$$

In our investigation we shall need the following results.

Theorem 1.2 ([1]). *Let h be a convex function in U, with $h(0) = a$, $\gamma \neq 0$ and* $\mathrm{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a, 1]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)\, t^{\gamma-1} dt.$$

The function q is convex and is the best dominant.

Theorem 1.3 ([3])**.** *Let h be a convex function in U, with $h(0) = a$, $\gamma \neq 0$ and* $\mathrm{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in U and*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma},$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)\, t^{\gamma-1} dt.$$

The function q is convex and is the best subordinant.

2. Main results

Theorem 2.1. *Let ϕ be a convex function in the unit disk, with $\phi(0) = 1$ and $\lambda > 0$. If $f \in \Omega_p^{\delta}(\phi, \lambda)$, then there exists a convex function q, such that $q(z) \prec \phi(z)$ and $f \in \Omega_p^{\delta}(q, 0)$.*

Proof. We set

$$p(z) = \frac{K_p^{\delta} f(z)}{z^p} = 1 + \sum_{n=1}^{\infty} \left(\frac{c+p}{c+p+n} \right)^{\delta} a_{p+n} z^n$$

and observe that $p \in \mathcal{H}[1,1]$.

A short calculation leads us to

$$\frac{z\left\{K_p^{\delta} f(z)\right\}'}{pz^p} = p(z) + \frac{zp'(z)}{p}$$

and, by using the identity

$$z\left\{K_p^{\delta} f(z)\right\}' = (c+p)\, K_p^{\delta-1} f(z) - cK_p^{\delta} f(z),$$

we get

$$\frac{K_p^{\delta-1} f(z)}{z^p} = p(z) + \frac{zp'(z)}{c+p}.$$

Therefore, since $f \in \Omega_p^{\delta}(\phi, \lambda)$, we can conclude that

$$p(z) + \frac{\lambda}{p(c+p)} z p'(z) \prec \phi(z).$$

By Theorem 1.2 for $\gamma = \frac{p(c+p)}{\lambda}$ it follows now that

$$\frac{K_p^{\delta} f(z)}{z^p} \prec q(z) \prec \phi(z),$$

where

$$q(z) = \frac{p(c+p)}{\lambda} z^{-p(c+p)/\lambda} \int_0^z \phi(t)\, t^{p(c+p)/\lambda - 1} dt$$

is convex and the best dominant.

Thus, $f \in \Omega_p^{\delta}(q, 0)$ and $f \in \Omega_p^{\delta}(\widetilde{q}, 0)$ for all convex functions $\widetilde{q}$ that satisfy $q \prec \widetilde{q}$. □

For suitable choises of the function ϕ, we can obtain some corollaries. Let us first consider the function $\phi(z) = \frac{1+Az}{1+Bz}$, for $-1 \le B < A \le 1$. The class $\Omega_p^{\delta}(\phi, \lambda)$ becomes in this case the class $\Omega_p^{\delta}(A, B, \lambda)$ from [4].

Corollary 2.2 ([4])**.** *Let $\lambda > 0$ and $f \in \Omega_p^{\delta}(A, B, \lambda)$. Then $f \in \Omega_p^{\delta}(A, B, 0)$.*

We take now ϕ to be the function given by $\phi(z) = \frac{1+\beta z}{1-\alpha\beta z}$, with $0 < \alpha \le 1$ and $0 < \beta < 1$. In this case let us denote the class $\Omega_p^{\delta}(\phi, \lambda)$ by $\Omega_p^{\delta}(\alpha, \beta, \lambda)$.

Corollary 2.3. *Let $\lambda > 0$ and $f \in \Omega_p^{\delta}(\alpha, \beta, \lambda)$. Then $f \in \Omega_p^{\delta}(\alpha, \beta, 0)$.*

Theorem 2.4. *Let ϕ be a convex function in the unit disk, with $\phi(0) = 1$ and $\lambda > 0$. If $f \in \overline{\Omega}_p^{\delta}(\phi, \lambda)$, $\frac{K_p^{\delta} f(z)}{z^p} \in \mathcal{H}[a, 1] \cap \mathcal{Q}$ and $\frac{\lambda}{p} \frac{K_p^{\delta-1} f(z)}{z^p} + \frac{p-\lambda}{p} \frac{K_p^{\delta} f(z)}{z^p}$ is univalent in U, then there exists a convex function q such that $f \in \overline{\Omega}_p^{\delta}(q, 0)$.*

Proof. We set

$$p(z) = \frac{K_p^{\delta} f(z)}{z^p} = 1 + \sum_{n=1}^{\infty} \left(\frac{c+p}{c+p+n} \right)^{\delta} a_{p+n} z^n$$

and observe that $p \in \mathcal{H}[1, 1] \cap \mathcal{Q}$.

After a short calculation and considering that $f \in \overline{\Omega}_p^{\delta}(\phi, \lambda)$, we can conclude that

$$\phi(z) \prec p(z) + \frac{\lambda}{p(c+p)} z p'(z)$$

and $p(z)+\dfrac{\lambda}{p(c+p)}zp'(z)$ is univalent in U. We can apply now Theorem 1.3 for $\gamma=\dfrac{p(c+p)}{\lambda}$ and it follows that

$$q(z)\prec\frac{K_p^{\delta}f(z)}{z^p},$$

where

$$q(z)=\frac{p(c+p)}{\lambda}z^{-p(c+p)/\lambda}\int_0^z\phi(t)\,t^{p(c+p)/\lambda-1}dt$$

is convex and the best subordinant.

Thus, $f\in\overline{\Omega}_p^{\delta}(q,0)$ and $f\in\overline{\Omega}_p^{\delta}(\widetilde{q},0)$, for all convex functions $\widetilde{q}$ that satisfy $\widetilde{q}\prec q$. □

If we combine the results of Theorem 2.1 and Theorem 2.4, we obtain the following differential "sandwich theorem".

Corollary 2.5. *Let ϕ_1, ϕ_2 be convex functions in the unit disk, with $\phi_1(0)=\phi_2(0)=1$ and $\lambda>0$. If $f\in\Omega_p^{\delta}(\phi_1,\lambda)\cap\overline{\Omega}_p^{\delta}(\phi_2,\lambda)$, $\dfrac{K_p^{\delta}f(z)}{z^p}\in\mathcal{H}[a,1]\cap\mathcal{Q}$ and $\dfrac{\lambda}{p}\dfrac{K_p^{\delta-1}f(z)}{z^p}+\dfrac{p-\lambda}{p}\dfrac{K_p^{\delta}f(z)}{z^p}$ is univalent in U, then*

$$f\in\Omega_p^{\delta}(q_1,0)\cap\overline{\Omega}_p^{\delta}(q_2,0)$$

where

$$q_1(z)=\frac{p(c+p)}{\lambda}z^{-p(c+p)/\lambda}\int_0^z\phi_1(t)\,t^{p(c+p)/\lambda-1}dt$$

and

$$q_2(z)=\frac{p(c+p)}{\lambda}z^{-p(c+p)/\lambda}\int_0^z\phi_2(t)\,t^{p(c+p)/\lambda-1}dt.$$

The functions q_1 and q_2 are convex.

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