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APPROXIMATION PROCEDURES IN CONNECTION WITH A PROBLEM OF STURM-LIOUVILLE TYPE

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Abstract. Some results concerning the superdense unbounded divergence or the convergence of a family of interpolating operators and point-interpolatory functionals, associated to a problem of Sturm-Liouville type, are established.

1. **Introduction**

Let consider the Sturm-Liouville problem

$$\begin{cases} u''(x) + [\lambda^2 - B(x)]u(x) = 0;\ 0 \le x \le \pi \\ u'(0) = au(0);\ u'(\pi) = Au(\pi), \end{cases} \tag{1.1}$$

where $B(x)$ is a continuous function with bounded variation and $a, A \in \overline{\mathbb{R}}$, with the convention $u(0) = 0$, if $|a| = \infty$ and $u(\pi) = 0$, if $|A| = \infty$, [4].

It is known that there exists an orthonormal system of eigenfunctions $u_n \in C^2[0, \pi]$, $n \ge 1$, with respect to the problem (1.1), [5]. Moreover, each eigenfunction u_n has n distinct roots x_n^k, $1 \le k \le n$, in the interval $(0, \pi)$, [4], i.e. $0 < x_n^1 < x_n^2 < x_n^3 < \cdots < x_n^n < \pi$; in this paper, we shall put $x_n^0 = 0$, $x_n^{n+1} = \pi$.

Introduce the natural numbers m_0, m_n as

$$m_0 = \begin{cases} 1, & \text{if} \quad a \in \mathbb{R} \\ 0, & \text{if} \quad |a| = \infty \end{cases} ; \quad m_n = \begin{cases} n, & \text{if} \quad A \in \mathbb{R} \\ n+1, & \text{if} \quad |A| = \infty \end{cases}$$

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and define, analogously to Lagrange interpolation, the *"fundamental interpolating" functions* $s_n^k \in C[0,\pi]$, $m_0 \le k \le m_n$, by

$$s_n^k(x) = \begin{cases} \dfrac{u_n(x)}{(x-x_n^k)u_n'(x_n^k)}, & \text{if} \quad x \neq x_n^k \\ 1, & \text{if} \quad x = x_n^k \end{cases}$$

The linear operators

$$\begin{cases} S_n : C[0,\pi] \to C[0,\pi];\ f \mapsto S_n f;\ n \ge 1 \\ (S_n f)(x) = \displaystyle\sum_{k=m_0}^{m_n} f(x_n^k) s_n^k(x) \end{cases} \tag{1.2}$$

are said to be the *"interpolating" operators* associated to the *Sturm-Liouville node matrix*

$$\mathcal{M}_{SL} = \{x_n^k :\ n \ge 1;\ m_0 \le k \le m_n\}.$$

Further, if x_0 is a given point of $[0,\pi]$, the linear functionals

$$S_n^0 : C[0,\pi] \to \mathbb{R}, \quad S_n^0 f = (S_n f)(x_0), \quad n \ge 1 \tag{1.3}$$

are named the *"interpolatory" functionals* associated to x_0 or *"point-interpolatory" functionals* at x_0.

Now, denote by $\omega(f;\cdot)$ the modulus of continuity of a function $f \in C[0,\pi]$ and let $\|\cdot\|$ be the uniform norm of f. G.I. Natanson, [3] and L.I. Tuchinskiĭ, [4] established the following estimation concerning the "interpolating" operators S_n:

The relation

$$|f(x) - (S_n f)(x)| = \left[\omega\left(f;\frac{1}{n}\right) + \frac{1}{n}\|f\|\right] O(\ln n), \quad f \in C[0,\pi] \tag{1.4}$$

holds uniformly on each interval $[a,\pi-a]$*, for any given* $a \in \left(0,\dfrac{\pi}{2}\right)$.

Based on this result, they proved the following convergence theorem:

If $f \in C[0,\pi]$ *satisfies a Dini-Lipschitz condition*

$$\lim_{\delta \searrow 0} \omega(f;\delta)\ln\delta = 0, \tag{1.5}$$

then the sequence $(S_n f)_{n\ge1}$ *is uniformly convergent to* f *on each segment* $[a,\pi-a]$*,* $0 < a < \dfrac{\pi}{2}$.

The main aim of this paper is to establish the superdense unbounded divergence of the family of interpolating operators $\{S_n:\ n \geq 1\}$. (Recall that a subset of a topological space T is said to be *superdense* in T if it is residual, i.e. its complement is of first Baire category, uncountable and dense in T). To this end, we shall define, in the next section, the functions and the constants of Lebesgue with respect to the node matrix $\mathcal{M}_{SL}$.

2. The functions and the constants of Lebesgue associated to the Sturm-Liouville node matrix

The functions $L_n : [0,\pi] \to \mathbb{R}$, $L_n(x) = \sum_{l=m_0}^{m_n} |s_n^k(x)|$, $0 \leq x \leq \pi$ and the positive numbers $\Lambda_n = \|L_n\|$, $n \geq 1$, are said to be the *Lebesgue functions*, respectively the *Lebesgue constants* associated to the node matrix $\mathcal{M}_{SL}$.

Standard arguments show that S_n, $n \geq 1$, are linear continuous operators and

$$\|S_n\| = \Lambda_n, \quad n \geq 1. \tag{2.1}$$

Indeed, using (1.2) we obtain:

$$|(S_n f)(x)| \leq L_n(x)\|f\| \leq \Lambda_n \|f\|, \text{ for all } x \in [0,\pi],$$

so

$$\|S_n f\| \leq \Lambda_n \|f\|, \text{ i.e. } \|S_n\| \leq \Lambda_n.$$

Conversely, for an arbitrary $t \in [0,\pi]$ let us define the function f_t by:

$$f_t(x) = \begin{cases} \operatorname{sign} s_n^k(t), & \text{if } x \in \{x_n^k:\ m_0 \leq k \leq m_n\} \\ \text{linear}, & \text{otherwise;} \end{cases}$$

we have $f_t \in C[0,\pi]$, $\|f_t\| = 1$ and

$$\begin{aligned} \|S_n\| &= \sup\{\|S_n f\|:\ f \in C[0,\pi],\ \|f\| \leq 1\} \geq \|S_n f_t\| \\ &\geq |(S_n f_t)(t)| = \left|\sum_{k=m_0}^{m_n} f_t(x_n^k) s_n^k(t)\right| = L_n(t),\ \forall\, t \in [0,\pi], \end{aligned}$$

which leads to $\|S_n\| \geq \Lambda_n$.

Similarly, concerning the functionals S_n^0 of (1.3), we get:

$$\|S_n^0\| = L_n(x_0) \tag{2.2}$$

for every $x_0 \in [0, \pi]$.

In what follows, we shall use the following estimation regarding the Lebesgue functions L_n, [4]:

$$L_n(x_0) = 1 + \frac{1}{\sqrt{2\pi}} |u_n(x_0)| [\ln n + \ln(n \sin x_0 + 1) + O(1)], \tag{2.3}$$

where $x_0 \in [0, \pi]$ and

$$u_n(x_0) = \sqrt{\frac{2}{\pi}} \cos(\alpha_n x_0 + \varepsilon\pi) + O\left(\frac{1}{n}\right) \tag{2.4}$$

where

$$\varepsilon = \varepsilon(a) = \begin{cases} 0, & \text{if} \quad a \in \mathbb{R} \\ 1/2, & \text{if} \quad |a| = \infty \end{cases} \tag{2.5}$$

$$\alpha_n = \alpha_n(a, A) = \begin{cases} n, & \text{if} \quad a \in \mathbb{R},\ A \in \mathbb{R} \\ n + 1/2, & \text{if} \quad a \in \mathbb{R},\ |A| = \infty \text{ or } |a| = \infty,\ A \in \mathbb{R} \\ n + 1, & \text{if} \quad |a| = |A| = \infty \end{cases} \tag{2.6}$$

for every $x_0 \in [0, \pi]$.

3. Superdense unbounded divergence of the family of "interpolating" operators

The main result of this paper is the following

Theorem 3.1. *The set of unbounded divergence of the family of "interpolating" operators $\{S_n : \ n \geq 1\}$, i.e.*

$$\left\{ f \in C[0, \pi] : \ \limsup_{n \to \infty} \|S_n f\| = \infty \right\},$$

is superdense in the Banach space $(C[0, \pi], \| \cdot \|)$.

Proof. In what follows M_k, $k \geq 1$, will be positive constants which do not depend on n. We deduce from (2.3) and (2.4):

$$|u_n(x_0)| \leq M_1; \quad |L_n(x_0)| \leq M_2 \ln n,\ \forall\ x_0 \in [0, \pi],$$

so that, according to (2.1), we get:

$$S_n = \Lambda_n \le M_2 \ln n, \text{ for sufficiently large } n. \tag{3.1}$$

Let us establish the converse of (3.1). According to (2.5) and (2.6), there are four possibilities.

1°. If $a \in \mathbb{R}$ and $A \in \mathbb{R}$, then $\alpha_n = n$ and $\varepsilon = 0$, so we deduce from (2.3) and (2.4):

$$L_n(0) = 1 + \left(\frac{1}{\pi} + O\left(\frac{1}{n}\right)\right)(\ln n + O(1))$$

and $\|S_n\| = \Lambda_n \ge L_n(0) \ge M_3 \ln n$.

2°. If $a \in \mathbb{R}$ and $|A| = \infty$, then $\varepsilon = 0$ and $\alpha_n = n + 1/2$, so (2.3), (2.4) give:

$$L_n\left(\frac{\pi}{2}\right) = 1 + \left(\frac{1}{\pi\sqrt{2}} + O\left(\frac{1}{n}\right)\right)(\ln(n^2 + n) + O(1)),$$

therefore

$$\|S_n\| = \Lambda_n \ge L_n\left(\frac{\pi}{2}\right) \ge M_4 \ln n.$$

3°. If $|a| = \infty$ and $A \in \mathbb{R}$, then $\varepsilon = 1/2$ and $\alpha_n = n + \dfrac{1}{2}$, so:

$$\|S_n\| = \Lambda_n \ge L_n(\pi) \ge M_5 \ln n.$$

4°. If $|a| = |A| = \infty$, then $\varepsilon = 1/2$ and $\alpha_n = n + 1$, so we get from (2.3) and (2.4):

$$L_{2n}\left(\frac{\pi}{2}\right) = 1 + \frac{1}{\sqrt{2\pi}}\left|\sqrt{\frac{2}{\pi}}(-1)^{n+1} + O\left(\frac{1}{n}\right)\right|(\ln(4n^2 + 2n) + O(1)),$$

so:

$$\|S_{2n}\| \ge L_{2n}(\pi) \ge M_6 \ln n \tag{3.2}$$

$$L_{2n+1}\left(\frac{\pi}{4}\right) = 1 + \frac{1}{\sqrt{2\pi}}\left|\cos\left(\frac{n\pi}{2} + \frac{3\pi}{4}\right) + O\left(\frac{1}{n}\right)\right|(2\ln n + O(1)),$$

so:

$$\|S_{2n+1}\| \ge L_{2n+1}\left(\frac{\pi}{4}\right) \ge M_7 \ln n. \tag{3.3}$$

Now, (3.2) and (3.3) give:

$$\|S_n\| = \Lambda_n \ge M_8 \ln n.$$

It follows from 1°, 2°, 3° and 4°:

$$\|S_n\| = \Lambda_n \geq M_9 \ln n, \text{ for sufficiently large } n. \tag{3.4}$$

The relations (3.1) and (3.4) lead to the estimation:

$$\|S_n\| = \Lambda_n \sim \ln n, \tag{3.5}$$

i.e.

$$M_9 \ln n \leq \|S_n\| = \Lambda_n \leq M_2 \ln n, \text{ for sufficiently large } n.$$

To prove the conclusion of this theorem, we shall apply the following principle of condensation of the singularities, [1]:

If X is a Banach space, Y is a normed space and $(A_n)_{n\geq 1}$ is a sequence of continuous linear operators from X into Y so that the set of norms $\{\|A_n\| : n \geq 1\}$ is unbounded, then the set of singularities of the family $\{A_n : n \geq 1\}$, i.e.

$$\left\{x \in X : \limsup_{n\to\infty} \|A_n x\| = \infty\right\},$$

is superdense in X.

Now, choose $X = Y = (C[0,\pi], \|\cdot\|)$ and take into account the estimation (3.5), which completes the proof.

4. **On the convergence of the point-interpolatory functionals**

Let consider the point-interpolatory functionals S_n^0, $n \geq 1$, given by (1.3) and suppose $x_0 \in (0,\pi)$.

According to (2.2), we have $\|S_n^0\| = L_n(x_0)$; moreover, if $\dfrac{x_0}{\pi} \in \mathbb{Q}$, then the set of values of Lebesgue functions at x_0 is unbounded, [2], so that the following result, similar to that of Theorem 3.1, holds:

If $\dfrac{x_0}{\pi} \in \mathbb{Q} \cap (0,1)$, then the set of unbounded divergence of the family of point-interpolatory functionals $\{S_n^0 : n \geq 1\}$, i.e.

$$\left\{f \in C[0,\pi] : \limsup_{n\to\infty} |S_n^0 f| = \infty\right\},$$

is superdense in the Banach space $(C[0,\pi], \|\cdot\|)$.

On the other hand, let $a \in \left(0, \frac{\pi}{2}\right)$ and $x_0 \in [a, \pi - a]$. It follows from (1.4):

$$|S_n^0 f - f(x_0)| \leq M_{10} \left[\omega\left(f; \frac{1}{n}\right) + \frac{1}{n}\|f\|\right] \ln n, \ \forall\ f \in C[0, \pi],$$

which leads to the following statement:

Theorem 4.1. *If $x_0 \in (0, \pi)$, then the family of point-interpolatory functionals $\{S_n^0 :\ n \geq 1\}$ is convergent on the set $DLC[0, \pi]$ of all functions $f \in C[0, \pi]$ satisfying the Dini-Lipschitz condition (1.5), namely*

$$\lim_{n\to\infty} S_n^0 f = f(x_0), \ \forall\ f \in DLC[0, \pi].$$

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