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INDEX OF THE ELASTICITY OPERATOR WITH CONTACT WITHOUT FRICTION BOUNDARY CONDITIONS

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Abstract. In this paper, one considers a contact without friction problem for the elasticity system, using the results given by P. Grisvard and B. Benabderrahmane respectively in ([1]: Far East J.Appl. Maths., Vol.24, No.3, p.373-380, (2006) and [2]: C.R. Acad. Sci. Paris, Ser.I Math. 304(3) (1987), 71-73), one proves that the Laplace operator is injective and with closed image of codimension N in $H^s(\Omega)^2$, and consequently Δ have an index which is equal to $-N$, where N denotes the number of the singular solutions of the considered problem. Using the above results one proves that the elasticity operator, denoted by $\mathcal{L}$ has an index which is equal to $-2N$, by basing on the *Fredholm* alternative. This enables us to deduce the explicitly singular solutions and to describe the singular behavior of the solutions in the polygon.

1. Problem statement

The aim of this statement is to deduce the index results for the contact without friction problem which is governed by the *Lamé* system in a polygon. Consequently, it can be given the explicitly singular solutions and to describe the singular behavior of the solutions in the polygon. Let $f \in L^2(\Omega)$, consider the following problem

$$(P):\left\{\begin{array}{c} \mathcal{L}u = f \quad \text{in } \Omega \\ \left\{\begin{array}{c} u.\eta^j = 0 \\ \left(\Sigma(u).\eta^j\right).\tau^j = 0 \end{array}\right. \quad \text{on } \Gamma_j, j = 1, ..., J, \end{array}\right.$$

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where Ω is homogeneous, elastic and isotropic medium occupying a bounded domain in $\mathbb{R}^2$, limited by straight polygonal boundary $\Gamma = \overset{J}{\underset{j=1}{\cup}} \overline{\Gamma}_j$, $\Gamma_i \cap \Gamma_j = \varnothing, \forall i \neq j$, $\Gamma_j =]S_j, S_{j+1}[$, where S_j are the different corners of Ω. $\eta^j = \left(\eta_1^j, \eta_2^j\right)$, $\tau^j = \left(\tau_1^j, \tau_2^j\right)$ designate the outward unit normal vector, and the tangential unit vector in Γ_j respectively. $\omega_j, (0 < \omega \leq 2\pi)$ represents the opening of the angle that makes Γ_j and Γ_{j+1} toward the interior of Ω.

$\mathcal{L}$ is the *Lamé* operator defined by:

$$\mathcal{L} : \lambda\Delta + (\lambda + \mu)\nabla div;$$

where Δ, ∇ and div represent respectively the *Laplace*, *Gradient* and *Divergence* . u, f is the displacement vector, and external forces density respectively. $\Sigma(u) = (\sigma_{ij}(u))_{ij}$ is the stress tensor given by *Hook*'s law using *Lamé* coefficients λ and μ which are strictly positive and such that $(\lambda + \mu) > 0$,

$$\sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda tr(\varepsilon(u))\delta_{ij},$$

where δ_{ij} is a *Kronecker* symbol and $\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the linearized tensor of linear elasticity.

The particular case $\lambda + \mu \to 0$, reduces the problem to

$$(E) : \begin{cases} \Delta u = \frac{f}{\lambda}(= f) \text{ in } \Omega \\ \begin{cases} u.\eta^j = 0 \\ \left(\Sigma(u).\eta^j\right).\tau^j = 0 \end{cases} \text{on } \Gamma_j, j = 1, ..., J, \end{cases}$$

with $\sigma_{ij}(u) = \mu(\partial_i u_j + \partial_j u_i) - (\partial_1 u_2 + \partial_2 u_1)\delta_{ij}$, where $\partial_j u$ is used as the partial derivative of u with respect to x_j.

Generally, the problem (E) hasn't sufficiently regular solution, hence we try to impose conditions on f in order to obtain desired solutions, i.e., we search for necessary and sufficient conditions on f allowing variational solution included in the space V such as

$$V = \left\{u \in H^1(\Omega)^2;\ u.\eta = 0 \text{ on } \Gamma\right\}$$

is in $H^{s+2}(\Omega)^2 \cap V$ $(s \geq 0)$, where H^{s+2} denotes $(s+2)$ order *Sobolev* space.

The resolution of this problem is based on the following inequality

$$\|u\|_{s+2} \le C_s \|u\|_s\,,\ u \in H^{s+2}(\Omega)^2 \cap V. \tag{1.1}$$

This inequality is not always true, for example the case when Ω is a polygon. However, we may prove this a priori inequality is verified all the same, when imposing a supplementary condition: $D_x u + D_y u \in H_0^s(\Omega)$ i.e.

$$u \in W_s(\Omega) = \left\{u \in H^{s+2}(\Omega)^2 \cap V;\ \ (\Sigma(u).\eta)\,.\tau = 0 \text{ on } \Gamma,\ D_x u + D_y u \in H_0^s(\Omega)\right\}.$$

By explicit calculations (see [1]), studying the boundary conditions considered, we prove the following Lemma.

Lemma 1. *The problem (P) amount to the two problems of oblique derivatives boundary conditions without coupling:*

$$(E_k) : \begin{cases} \Delta u_k = f_k \ \ on \ \ \Omega \\ \alpha_j D_x u_k + \beta_j D_y u_k = 0 \ \ in\ \Gamma_j,\ \ j = 1, ..., J \\ \alpha_j^2 + \beta_j^2 \neq 0. \end{cases} ,\ \ k = 1, 2$$

2. A priori inequality

This section is dedicated to demonstration of a priory inequality (1.1). To simplify the study, in all of this section we will write u instead u_k and (E) instead (E_k), because the $(E_k), k = 1, 2$ are two similar one-dimensional problems.

The inequality (1.1) follows from the following simple inequality

$$\|u\|_2 \le C_0 \left(\|\Delta u\|_0 + \|u\|_1\right),\ \forall u \in K_{\alpha,\beta}(\Omega), \tag{1.2}$$

with

$$K_{\alpha,\beta}(\Omega) = \left\{\ u \in H^2(\Omega); \alpha_j D_x u + \beta_j D_y u = 0 \text{ on } \Gamma_j,\ \alpha_j^2 + \beta_j^2 \neq 0,\ j = 1, ..., J\right\}.$$

Remark 1. *It is known (in [1]) that there is a constant C_0 such as the inequality (1.2) is verified for all $u \in K_{\alpha,\beta}(\Omega)$.*

Proposition 1. *There is a constant C such that the inequality* (1.1) *takes place for all* $u \in W_s(\Omega)$.

Proof. The essential idea of the demonstration is to search the boundary conditions verified by :

$$v = D_x^n D_y^m u, \text{ with } n + m \leq s.$$

For this, we parameterize the segments Γ_j using the following applications:

$$[0,1] \longrightarrow \mathbb{R}^2$$

$$\lambda \mapsto \left(t_{1j}\lambda + t'_{1j}, t_{2j}\lambda + t'_{2j}\right).$$

We have the condition $\alpha_j D_x u + \beta_j D_y u = 0$ on Γ_j, $j = 1, .., J$, therefore

$$\alpha_j D_x u\left(t_{1j}\lambda + t'_{1j}, t_{2j}\lambda + t'_{2j}\right) + \beta_j D_y u\left(t_{1j}\lambda + t'_{1j}, t_{2j}\lambda + t'_{2j}\right) = 0,$$

$\lambda \in [0,1]$, $j = 1, ..., J$, hence by derivation, we obtain

$$\alpha_j \sum_{k=0}^{s} C_s^k t_{1j}^k t_{2j}^{s-k} D_x^{k+1} D_y^{s-k} u + \beta_j \sum_{k=0}^{s} C_s^k t_{1j}^k t_{2j}^{s-k} D_x^k D_y^{s+1-k} u = 0, \tag{1.3}$$

on the other hand we have $D_x u + D_y u \in H_0^s(\Omega)^2$, therefore

$$D_x^p D_y^q (D_x u + D_y u) = 0, \text{ on } \Gamma_j, \ j = 1, ..., J, \text{ for } p + q = s (s \geq 1)$$

and we obtain consequently:

$$D_x^{p+1} D_y^q u = -D_x^p D_y^{q+1} u \text{ on } \Gamma_j \tag{1.4}$$

from which we deduce that:

$$\left\{ \begin{array}{l} D_x^k D_y^{s+1-k} u = (-1)^k D_y^{s+1} u, \ k = 0, 1, ... \\ D_x^{k+1} D_y^{s-k} u = (-1)^k D_x D_y^s u, \ k = 0, 1, ... \end{array} \right. \text{on } \Gamma_j, \ j = 1, ..., J$$

from which, we can rewrite the equation (1.3) as follow:

$$\alpha_j \sum_{k=0}^{s} C_s^k t_{1j}^k t_{2j}^{s-k} (-1)^k D_x D_y^s u + \beta_j \sum_{k=0}^{s} C_s^k t_{1j}^k t_{2j}^{s-k} (-1)^k D_y D_y^s u = 0. \tag{1.5}$$

Thus we obtain the condition of oblique derivative verified by $v = D_y^s u$. This condition is

$$\alpha_j^{'} D_x v + \beta_j^{'} D_y v = 0 \text{ on } \Gamma_j, \quad j = 1, ..., J \tag{1.6}$$

with

$$\begin{cases} \alpha_j^{'} = \sum\limits_{k=0}^{s} (-1)^{\alpha} \alpha_j C_s^k t_{1j}^k t_{2j}^{s-k} \\ \beta_j^{'} = \sum\limits_{k=0}^{s} (-1)^{\alpha} \beta_j C_s^k t_{1j}^k t_{2j}^{s-k} \end{cases} \tag{1.7}$$

According to (1.4) the condition is also verified by $D_x^1 D_y^{s-1} u, D_x^2 D_y^{s-2} u, ...$, we can apply the previous remark, if $\left(\alpha_j^{'}\right)^2 + \left(\beta_j^{'}\right)^2 \neq 0$ is verified. For this we note that

$$\begin{aligned} \left(\alpha_j^{'}\right)^2 + \left(\beta_j^{'}\right)^2 &= \left[\sum_{k=0}^{s} (-1)^{\alpha} C_s^k t_{1j}^k t_{2j}^{s-k}\right]^2 \left(\ \alpha_j^2 + \beta_j^2\right) \\ &= \left[(t_{1j} + t_{2j})^s\right]^2 \left(\ \alpha_j^2 + \beta_j^2\right) \neq 0. \end{aligned}$$

Since the two numbers t_{1j}, t_{2j} can't vanish simultaneously (because they are the coefficients of parameterization of Γ_j). Then we have proved that for $n + m \leq s$ $(s \geq 1)$, there exist α and β such as $D_x^n D_y^m u \in K_{\alpha,\beta}$ for all $u \in W_s(\Omega)$. In the case $s = 0$, the condition of oblique derivative (1.2) is verified by u. Then

$$\sum_{n+m=0}^{s} \left\| D_x^n D_y^m u \right\|_2 \leq C_0 \left(\sum_{n+m=0}^{s} \left\| D_x^n D_y^m \Delta u \right\|_0 + \sum_{n+m=0}^{s} \left\| D_x^n D_y^m u \right\|_1 \right)$$

from which we deduce the inequality $\|u\|_{s+2} \leq C_S \left(\|\Delta u\|_s + \|u\|_1\right)$ and we obtain the inequality (1.1), using the well know inequality $\|u\|_1 \leq C \|\Delta u\|_0$ in V. □

3. **Fredholm alternative**

Let $R_s(\Omega)$ be the subspace of $H^s(\Omega)^2$ defined by

$$R_s(\Omega) = \left\{f = \Delta u; u \in W_s(\Omega)\right\}.$$

Remark 2. *Using the inequality* (1.1), *there can be seen that* $R_s(\Omega)$ *is a closed subspace of* $H^s(\Omega)^2$. *Let* $N_s(\Omega)$ *be the orthogonal of* $R_s(\Omega)$ *in* $H^{-s}(\Omega)^2$, *i.e.*

$$N_s(\Omega) = \left\{v \in H^{-s}(\Omega)^2,\ (v, f) = 0 \text{ for all } f \in R_s(\Omega)\right\},$$

where (v, f) *represents the duality pairing between* $H^s(\Omega)^2$ *and* $H^{-s}(\Omega)^2$.

Thanks to a generalization of *Green* formula, it can be proved the following Lemma:

Lemma 2. *The orthogonal of* $R_s(\Omega)$ *in* $H^{-s}(\Omega)$ *is the vector subspace,* $N_s(\Omega)$, *of* $H^{-s}(\Omega)$ *defined by :*

$$N_s(\Omega) = \left\{v \in H^{-s}(\Omega); \Delta v = 0 \ in \ \Omega \ ; \gamma_s\left(v.\eta^j, \left(\Sigma(v).\eta^j\right).\tau^j\right) = \mathbf{0}\right\},$$

where γ_s *is a generalized operator trace defined by duality.*

According to a generalization of *Green* formula, we will see that the orthogonal of R_s in H^{-s} is

$$N_s(\Omega) = \left\{v \in H^{-s}; \Delta v = 0, \text{ in } \Omega;\ \gamma_s\left(u.\eta, \left(\Sigma(u).\eta\right).\tau\right) = 0\right\}.$$

We will have the necessary and sufficiencies conditions on $f \in H^s(\Omega)^2$, in order to allow for a variational solution to be in $H^{s+2} \cap V$. This condition is expressed as follows:

$$(f, v) = 0, \text{ for all } v \in N_s(\Omega).$$

3.1. **Laplace operator Index.** In the case, when Ω doesn't have any angle ω of the form

$$\frac{\ell\pi}{k+2};\ \ell, k \in \mathbb{N},\ \ell \neq (k+2),\ k = 1, ..., s$$

the dimension of $N_s(\Omega)$ is exactly equal to N, where

$$N = \left\{k \in \mathbb{N}; 1 \leq k \leq \left(\frac{\omega}{\pi}\right) s\right\}.$$

The ω are well specified at the end of the last section.

Using the techniques of Grisvard [2], it is shown the following result:

Lemma 3. *Suppose* Ω *is a simply connected,* $\Delta : W_s(\Omega) \to H^s(\Omega)^2$ *is an operator with index. More precisely, thanks to inequality* (1.1), *the Laplace operator* Δ *is injective, has a closed image of codimension equal to* $N < +\infty$ *in* $H^s(\Omega)^2$. *Consequently*

$$Ind(\Delta) = \dim Ker(\Delta) - co\dim(\Delta) = 0 - N = -N.$$

3.2. **Calculation of the operator $\mathcal{L}$ index.** Now come back to the problem (P), as defined above. In the following, and for the problem (P), essentially we are interested by the demonstration of the following inequality (1.8) :

$$\|u\|_{H^2(\Omega)^2} \leq C \|u\|_{L^2(\Omega)^2}, \tag{1.8}$$

where C is an independent constant of Lamé coefficients.

Lemma 4. *We have*

$$\left(D_x^2 u, D_y^2 u\right) = \|D_x D_y u\|^2_{L^2(\Omega)^2}, \forall u \in H^2(\Omega)^2 \cap V. \tag{1.9}$$

The proof of this Lemma is made by a party integration, using the density of $H^3(\Omega)^2 \cap V$ in $H^2(\Omega)^2 \cap V$.

Thus, we recover the restriction on the coefficients of Lamé $|\lambda| < \sqrt{3}|\mu|$ (see [2]), which is necessary so that the inequality (1.8) is verified. Thanks to inequality (1.8), the operator of Lamé is injective, has a closed image of $H^2(\Omega)^2 \cap V$ in $L^2(\Omega)^2$. Therefore, $\mathcal{L}$ is semi-*Fredholm* operator. As the operator $\mathcal{L}$ depends continuously of λ, its index (see [4]) is independent of λ. In the particular case where $\lambda = -\mu$ and according to the Lemma 1, the problem (P) amounts to problems $(E_k), k = 1, 2$, where $Ind(E_k) = N, k = 1, 2$, and consequently the index of the operator $\mathcal{L}$ is equal to $-2N$.

4. Singular solutions

Thanks to the index of the operator $\mathcal{L}$ that there exist $2N$ linearly independent functions S_j and $S'_j \in V$, such as

$$S_j, S'_j \notin H^2(\Omega)^2 \text{ and } \mathcal{L}S_j, \mathcal{L}S'_j \in L^2(\Omega)^2$$

and as $\mathcal{L}$ is an isomorphism of

$$Sp\left(H^2(\Omega)^2, S_j, S'_j\right) \cap V \text{ on } L^2(\Omega)^2, j = 1, ..., J,$$

where the symbol Sp designates the vector space generated by the elements continued in the bracket that follow.

We can calculate these functions explicitly, by searching S such as

$$S(r,\theta) = r^{\alpha}\Psi_{\alpha}(\theta),$$

solution of $\mathcal{L}S = 0$ in the sector

$$\Sigma = \{\theta; 0 < \theta < \omega\},$$

where

$$\Psi_{\alpha}(\theta) = \left(W_1(\theta)\cos\theta - W_2(\theta)\sin\theta, W_1(\theta)\sin\theta + W_2(\theta)\cos\theta\right)^t,$$

with

$$\begin{cases} W_1'(0) = W_2(0) = 0 \\ W_1'(\omega) = W_2(\omega) = 0. \end{cases}$$

Then, we find that the number α must be a solution of the following transcendent equation

$$\sin^2 \alpha\omega = \sin\omega \tag{1.10}$$

and such that

$$\Psi_{\alpha}(\theta) = \begin{cases} ((\rho_0+\rho_1)\cos(\alpha-2)\theta - (\rho_1-\rho_0)\cos\alpha\theta)\sin(\alpha+1)\omega + \\ \quad +2\rho_1\sin(\alpha-1)\omega\cos\alpha\theta \\ (-(\rho_0+\rho_1)\sin(\alpha-2)\theta - (\rho_1-\rho_0)\sin\alpha\theta)\sin(\alpha+1)\omega + \\ \quad +2\rho_1\sin(\alpha-1)\omega\sin\alpha\theta \end{cases}$$

with $\rho_0 = \nu_0(\alpha-1)-2, \rho_1 = \nu_0(\alpha+1)+2$, $\nu_0 = \frac{1}{1-2\nu}$, where ν is the *Poisson* coefficient.

It is well clear that the solutions of the transcendent equation (1.10) are real and explicitly given by

$$\alpha_\ell = \frac{\ell\pi}{\omega} \pm 1, \ell \in \mathbb{Z}^*.$$

Besides, if $\omega \neq \frac{k\pi}{\omega}, k \in \mathbb{Z}^*$ then the solutions are simple, else they are double.

In conclusion the transcendent equation (1.10) possesses one simple solution α in $]0,1[$, when $\omega \in \left]\frac{\pi}{2},\pi\right[\cup \left]\frac{3\pi}{2},2\pi\right[$ and it has only one solution double $\alpha' = \frac{1}{3}$, when $\omega = \frac{3\pi}{2}$. This will permit the demonstration of the following theorem which is described the singular behavior of the solution of the problem (P).

Theorem 1. *For $f \in L^2(\Omega)^2$, if $u \in V$ is a variational solution of the problem (P), then there are constants C_α and C'_α such as*

$$u - C_\alpha r^\alpha \Psi_\alpha(\theta) - C'_\alpha \left(\log r \Psi_\alpha(\theta) + \frac{\partial \Psi_\alpha(\theta)}{\partial \alpha} \right)_{\alpha = \frac{1}{3}} \in H^2(\Omega)^2 .$$

The first sum in this expression is extended to all real numbers α simple solution of the transcendent equation (1.10), whereas the second sum is extended to all real numbers α double solution of the equation (1.10).

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