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SOME NOTES ON (σ,τ)-AMENABILITY OF BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a Banach algebra and σ,τ be continuous homomorphisms on $\mathcal{A}$. Suppose that $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A linear mapping $d:\mathcal{A}\longrightarrow\mathcal{X}$ is a (σ,τ)-derivation if

$$d(ab)=d(a)\sigma(b)+\tau(a)d(b) \qquad (a,b\in\mathcal{A}),$$

and is a (σ,τ)-inner derivation if there exists $x\in\mathcal{X}$ such that

$$d(a)=x\sigma(a)-\tau(a)x \qquad (a\in\mathcal{A}).$$

The Banach algebra $\mathcal{A}$ is called (σ,τ)-amenable if every (σ,τ)-derivation is (σ,τ)-inner. In this paper, we investigate the relation between amenability and (σ,τ)-amenability of Banach algebras and also hereditary properties of (σ,τ)-amenability. We give the notion σ-virtual diagonal and σ-approximate diagonal and apply them in study of σ-amenability.

1. **Introduction and preliminaries**

The notion of amenable Banach algebra was introduced by B.E. Johnson in his monograph [4]. This class of Banach algebras arises naturally out of the cohomology theory for Banach algebras, the algebraic version of which was developed by Hochschild [3]. For a comprehensive account on amenability the reader is referred to the books [2, 10, 11].

Throughout the paper, $\mathcal{A}$ is a Banach algebra and $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. We denote by $\mathcal{A}\mathcal{X}$ and $\mathcal{X}\mathcal{A}$ the closed linear span of $\{ax : a\in\mathcal{A}, x\in\mathcal{X}\}$ and $\{xa : a\in\mathcal{A}, x\in\mathcal{X}\}$, respectively. A Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is pseudo-unital if $\mathcal{A}\mathcal{X}\mathcal{A}=\mathcal{X}$,

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where $\mathcal{AXA}$ is the closed linear span of $\{axb : a, b \in \mathcal{A}, x \in \mathcal{X}\}$. The space $\mathcal{X}^*$ is a Banach $\mathcal{A}$-bimodule via the following module actions:

$$\begin{aligned}(a \cdot f)(x) &= f(xa),\\ (f \cdot a)(x) &= f(ax),\end{aligned}$$

$a \in \mathcal{A}$, $x \in \mathcal{X}$, $f \in \mathcal{X}^*$.

We also denote *weak**-limits with $w^* - \lim$. For a closed subspace $\mathcal{M}$ of $\mathcal{X}$, we denote by $\mathcal{M}^\perp$ the set $\{f \in \mathcal{X}^* : f|_\mathcal{M} = 0\}$.

Let σ, τ be continuous homomorphisms from $\mathcal{A}$ to $\mathcal{A}$. A linear mapping $d : \mathcal{A} \longrightarrow \mathcal{X}$ is a (σ, τ)-*derivation* if

$$d(ab) = d(a)\sigma(b) + \tau(a)d(b),$$

for all $a, b \in \mathcal{A}$. A linear map $d : \mathcal{A} \longrightarrow \mathcal{X}$ is a (σ, τ)-*inner derivation* if there exists $x \in \mathcal{X}$ such that $d(a) = x\sigma(a) - \tau(a)x$ for all $a \in \mathcal{A}$.

A wide range of examples are as follows (see [5, 6]):

(i) Every ordinary derivation of an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{X}$ is an $id_\mathcal{A}$-derivation, where $id_\mathcal{A}$ is the identity map on the algebra $\mathcal{A}$.

(ii) Every endomorphism α on $\mathcal{A}$ is an $\frac{\alpha}{2}$-derivation.

(iii) Given a character θ on $\mathcal{A}$, a θ-derivation is nothing than a point derivation $d : \mathcal{A} \to \mathbb{C}$ at the character θ.

We use notations $Z^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ for the space of all continuous (σ, τ)-derivations $d : \mathcal{A} \longrightarrow \mathcal{X}$, $B^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ for those which are inner (σ, τ)-derivations, and $H^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ for the quotient space $Z^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})/B^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ which we call the first (σ, τ)-cohomology group of $\mathcal{X}$.

A Banach algebra $\mathcal{A}$ is said to be (σ, τ)-*amenable*(resp. (σ, τ)-*contractible*) if $H^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}^*) = 0$ (resp. $H^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}) = 0$) for all $\mathcal{A}$-bimodules $\mathcal{X}$. See [8, 7].

If $\sigma = \tau$ we simply use the terminologies σ-derivation, σ-amenability, etc. For definitions and elementary properties of Banach algebras we refer the reader to [1, 2, 9].

Modifying some known definition and techniques in the theory of amenability of Banach algebras and using some ideas and terminology of [11], we investigate the relation between amenability and (σ,τ)-amenability of Banach algebras and also hereditary properties of (σ,τ)-amenability. We give σ-virtual diagonal and σ-approximate diagonal and apply them in study of σ-amenability.

2. General properties of (σ,τ)-amenability

In this section we study general properties of (σ,τ)-amenable Banach algebras. Our first result reads as follows.

Proposition 2.1. *Let σ,τ be two continuous homomorphisms on Banach algebra $\mathcal{A}$. If $\mathcal{A}$ is (σ,τ)-amenable then it is $(\lambda\circ\sigma,\mu\circ\tau)$-amenable too, for any continuous homomorphisms λ,μ on $\mathcal{A}$.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D:\mathcal{A}\longrightarrow\mathcal{X}^*$ be a continuous $(\lambda\circ\sigma,\mu\circ\tau)$-derivation. We define another $\mathcal{A}$-module product on $\mathcal{X}$ by

$$\begin{aligned} a\Box x &= \lambda(a)x \\ x\Box a &= x\mu(a) \end{aligned}$$

for all $a\in\mathcal{A}$ and $x\in\mathcal{X}$. Then $\mathcal{X}$ with this product is a Banach $\mathcal{A}$-bimodule, and $D(ab)=D(a)(\lambda\circ\sigma)(b)+(\mu\circ\tau)(a)D(b)=D(a)\Box\sigma(b)+\tau(a)\Box D(b)$. Therefore D is a (σ,τ)-derivation, and so, by (σ,τ)-amenability of $\mathcal{A}$, there exists $f\in\mathcal{X}^*$ such that $D(a)=f\Box\sigma(a)-\tau(a)\Box f=f(\lambda\circ\sigma)(a)-(\mu\circ\tau)(a)f$. □

Corollary 2.2. *If $\mathcal{A}$ is amenable, then $\mathcal{A}$ is (σ,τ)-amenable for every two homomorphisms σ and τ.*

The following proposition provides a converse for Proposition 2.1 in a special case.

Proposition 2.3. *Let $\sigma:\mathcal{A}\longrightarrow\mathcal{A}$ be an epimorphism. If $\mathcal{A}$ is σ-amenable, then $\mathcal{A}$ is amenable.*

Proof. Suppose $\mathcal{X}$ is an $\mathcal{A}$-bimodule and $d : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a derivation. Then $D = d\circ\sigma$ is a σ-derivation, since

$$\begin{aligned} D(ab) &= d\circ\sigma(ab) \\ &= d(\sigma(a)\sigma(b)) \\ &= d(\sigma(a))\sigma(b) + \sigma(a)d(\sigma(b)) \\ &= D(a)\sigma(b) + \sigma(a)D(b). \end{aligned}$$

By σ-amenability of $\mathcal{A}$, there exists $f \in \mathcal{X}^*$ such that $D(a) = f\cdot\sigma(a) - \sigma(a)\cdot f$ for each $a \in \mathcal{A}$. Let $b \in \mathcal{A}$. Then there exists $a \in \mathcal{A}$ such that $\sigma(a) = b$ and so $d(b) = d\circ\sigma(a) = D(a) = f\cdot\sigma(a) - \sigma(a)\cdot f = f\cdot b - b\cdot f$. Therefore d is inner. □

The proof of next result is clear and we omit it.

Proposition 2.4. *Let $\mathcal{A}$ be a Banach algebra with a left identity e. Let σ be a homomorphism on $\mathcal{A}$ and $\tau = 0$. Then $\mathcal{A}$ is (σ,τ)-contractible.*

3. Hereditary properties of (σ,τ)-amenability

This section is devoted to study hereditary properties of (σ,τ)-amenable Banach algebras. The results of this section are extensions of the known theorems in the classical setting; cf. [11, Subsection 2.3]. Suppose that $\tau,\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ are two endomorphisms, and $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ such that $\sigma(\mathcal{I}) \subseteq \mathcal{I}, \tau(\mathcal{I}) \subseteq \mathcal{I}$. Then one can define the map $\widehat{\tau},\widehat{\sigma} : \frac{\mathcal{A}}{\mathcal{I}} \longrightarrow \frac{\mathcal{A}}{\mathcal{I}}$ by $\widehat{\sigma}(a+\mathcal{I}) = \sigma(a)+\mathcal{I}, \widehat{\tau}(a+\mathcal{I}) = \tau(a)+\mathcal{I}$.

Let $\mathcal{M}$ be a closed subspace of $\mathcal{X}$. We identify $\mathcal{M}^\perp$ with $(\frac{\mathcal{X}}{\mathcal{M}})^*$ via $f \mapsto \widetilde{f}$, $\widetilde{f}(x+\mathcal{M}) = f(x)$.

Proposition 3.1. *Let $\mathcal{I},\sigma,\tau$ be as above. If $\mathcal{A}$ is (σ,τ)-amenable then $\frac{\mathcal{A}}{\mathcal{I}}$ is $(\widehat{\sigma},\widehat{\tau})$-amenable.*

Proof. Let $\mathcal{X}$ be a $\frac{\mathcal{A}}{\mathcal{I}}$-bimodule, and $d : \frac{\mathcal{A}}{\mathcal{I}} \longrightarrow \mathcal{X}^*$ is a $(\widehat{\sigma},\widehat{\tau})$-derivation. $\mathcal{X}$ is a $\mathcal{A}$-bimodule via, $a\cdot x = (a+\mathcal{I})x, x\cdot a = x(a+\mathcal{I})$. Define $D = d\circ\pi : \mathcal{A} \longrightarrow \mathcal{X}^*$, where $\pi : \mathcal{A} \longrightarrow \frac{\mathcal{A}}{\mathcal{I}}$ is the natural homomorphism. Clearly D is a (σ,τ)-derivation on $\mathcal{A}$. Since $\mathcal{A}$ is (σ,τ)-amenable, D is (σ,τ)-inner. Hence there exists $f \in \mathcal{X}^*$ such that $D = d_f$. So that, $D(a) = d \circ \pi(a) = \sigma(a) \cdot f - f \cdot \tau(a)$ for all $a \in \mathcal{A}$. Hence $d(a+\mathcal{I}) = (\sigma(a)+\mathcal{I})f - f(\tau(a)+\mathcal{I}) = \widehat{\sigma}(a+\mathcal{I})f - f\widehat{\tau}(a+\mathcal{I})$ for all $a \in \mathcal{A}$. □

Proposition 3.2. *Let $\mathcal{I}, \sigma, \tau$ be as above and let σ, τ be idempotent homomorphisms. If $\mathcal{I}$ is (σ,τ)-amenable and $\frac{\mathcal{A}}{\mathcal{I}}$ is $(\widehat{\sigma},\widehat{\tau})$-amenable, then $\mathcal{A}$ is (σ,τ)-amenable.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D' : \mathcal{A} \longrightarrow \mathcal{X}^*$ is an arbitrary (σ,τ)-derivation. Then $D'|_{\mathcal{I}} : \mathcal{I} \longrightarrow \mathcal{X}^*$is a (σ,τ)-derivation. Then there exists $f \in \mathcal{X}^*$ such that $D'|_{\mathcal{I}} = d_f$, and so $(D' - d_f)(\mathcal{I}) = 0$. Define $D := D' - d_f$. Then $D(\mathcal{I}) = 0$. We have $0 = D(ab)(x) = D(a)(\sigma(b)(x)) + \tau(a)D(b)(x) = D(a)(\sigma(b)(x))$ for all $a \in \mathcal{A}, b \in \mathcal{I}, x \in \mathcal{X}$. Also $0 = D(ba)(x) = D(b)(\sigma(a)x) + \tau(b)D(a)(x)$. Hence $\tau(b)D(a)(x) = 0$ and so $D(a)(x\tau(b)) = 0$ for all $a \in \mathcal{A}, b \in \mathcal{I}, x \in \mathcal{X}$. Let $\mathcal{X}_{\mathcal{I}}$ be the closed submodule generated by $\sigma(\mathcal{I})\mathcal{X} \cup \mathcal{X}\tau(\mathcal{I})$. Then $D(\mathcal{A}) \subseteq \mathcal{X}_I^{\perp} = (\frac{\mathcal{X}}{\mathcal{X}_{\mathcal{I}}})^*$. But $\frac{\mathcal{X}}{\mathcal{X}_{\mathcal{I}}}$ is a Banach $\frac{\mathcal{A}}{\mathcal{I}}$-bimodule via

$$
\begin{aligned}
(a+\mathcal{I})(x+\mathcal{X}_{\mathcal{I}}) &= \sigma(a)x + \mathcal{X}_{\mathcal{I}} \\
(x+\mathcal{X}_{\mathcal{I}})(a+\mathcal{I}) &= x\tau(a) + \mathcal{X}_{\mathcal{I}}
\end{aligned}
$$

for all $a \in \mathcal{A}, x \in \mathcal{X}$. Since $D(\mathcal{I}) = 0$, we can define $\widetilde{D} : \frac{\mathcal{A}}{\mathcal{I}} \longrightarrow \mathcal{X}_{\mathcal{I}}^{\perp}$ by $\widetilde{D}(a+\mathcal{I}) = \widetilde{D(a)}$, $\widetilde{D(a)}(x+\mathcal{X}_{\mathcal{I}}) = D(a)(x)$ $(a \in \mathcal{A}, x \in \mathcal{X})$ which is a $(\widehat{\sigma},\widehat{\tau})$-derivation, since

$$
\begin{aligned}
\widetilde{D}(ab+\mathcal{I})(x+\mathcal{X}_{\mathcal{I}}) &= \widetilde{D(ab)}(x+\mathcal{X}_{\mathcal{I}}) \\
&= D(ab)(x) \\
&= D(a)\sigma(b)(x) + \tau(a)D(b)(x) \\
&= D(a)(\sigma(b)x) + D(b)(x\tau(b)) \\
&= \widetilde{D}(a+\mathcal{I})(\sigma(b)x+\mathcal{X}_{\mathcal{I}}) + \widetilde{D}(b+\mathcal{I})(x\tau(a)+\mathcal{X}_{\mathcal{I}}) \\
&= \widetilde{D}(a+\mathcal{I})(\sigma(b)+\mathcal{I})(x+\mathcal{X}_{\mathcal{I}}) + \widetilde{D}(b+\mathcal{I})(x+\mathcal{X}_{\mathcal{I}})(\tau(a)+\mathcal{I}) \\
&= [\widetilde{D}(a+\mathcal{I})\widehat{\sigma}(b+\mathcal{I}) + \widehat{\tau}(a+\mathcal{I})\widetilde{D}(b+\mathcal{I})](x+\mathcal{X}_{\mathcal{I}}).
\end{aligned}
$$

Hence $\widetilde{D}$ is $(\widehat{\sigma},\widehat{\tau})$-inner, so there exists $\tilde{g} \in (\frac{\mathcal{X}}{\mathcal{X}_{\mathcal{I}}})^*$ such that $\widetilde{D} = d_{\tilde{g}}$. Therefore $D' = d_{f+g}$. □

Proposition 3.3. *Let $\mathcal{A}, \mathcal{B}$ be Banach algebras and σ, τ be continuous endomorphisms of $\mathcal{A}$ and $\mathcal{B}$, respectively. If there is a continuous homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\varphi(\mathcal{A})$ is a dense subalgebra of $\mathcal{B}$ and $\tau\varphi = \varphi\sigma$, then σ-amenability of $\mathcal{A}$ implies τ-amenability of $\mathcal{B}$.*

Proof. Let $\mathcal{X}$ be a Banach B-module. Then $\mathcal{X}$ can be considered as a Banach $\mathcal{A}$-module, via $a \circ x = \varphi(a)x, x \circ a = x\varphi(a)$. Let $d : \mathcal{B} \longrightarrow \mathcal{X}^*$ be a τ-derivation, then $D = d \circ \varphi : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a σ-derivation. It follows from σ-amenability of $\mathcal{A}$ that there exist $f \in \mathcal{X}^*$ such that $D = d_f$. Therefore

$$\begin{aligned} d(\varphi(a)) = D(a) &= \sigma(a) \circ f - f \circ \sigma(a) \\ &= \varphi(\sigma(a))f - f\varphi(\sigma(a)) \\ &= \tau(\varphi(a))f - f\tau(\varphi(a)) \end{aligned}$$

Hence $d(c) = \tau(c)f - f\tau(c)(c \in B)$. □

4. Approximate identity and σ-amenability

We start our work with following extension of [11, Proposition 2.2.1] to show the existence of a bounded approximate identity.

Proposition 4.1. *If σ, τ are two idempotent endomorphisms on $\mathcal{A}$ such that $\sigma(\mathcal{A})$ and $\tau(\mathcal{A})$ are dense subalgebras of $\mathcal{A}$ and $\mathcal{A}$ is (σ, τ)-amenable Banach algebra,then it has a bounded approximate identity.*

Proof. Suppose that $\mathcal{A}$ is (σ, τ)-amenable. Note that $\mathcal{X} = \mathcal{A}$ is a Banach $\mathcal{A}$-bimodule under the actions $a \bullet x = \tau(a)x$ and $x \circ a = 0$. Then $\mathcal{A}^{**}$, as a Banach $\mathcal{A}$-bimodule, has the property $\mathcal{A}^{**} \cdot \mathcal{A} = \{0\}$. The linear map $d : \mathcal{A} \longrightarrow \mathcal{A}^{**}$ defined by $d(a) = \widehat{\tau(a)}$ is a bounded (σ, τ)-derivation, where $\widehat{}$ denotes the Gelfand transform. In fact,

$$\begin{aligned} d(ab)(f) &= \widehat{\tau(ab)}(f) = f(\tau(ab)) \\ &= f(\tau(a)\tau(b)) = f(\tau^2(a)\tau(b)) \\ &= (\tau(a) \bullet f)(\tau(b)) = \widehat{\tau(b)}(\tau(a) \bullet f) \\ &= \big(\tau(a) \bullet \widehat{\tau(b)}\big)(f) = \big(\tau(a) \bullet d(b)\big)f). \end{aligned}$$

Therefore $d(ab) = \tau(a)\bullet d(b) = d(a)\circ\sigma(b) + \tau(a)\bullet d(b)$. Also $\|d(a)\| = \|\widehat{\tau(a)}\| = \|\tau(a)\| \leq \|\tau\|\|a\|$. By (σ,τ)-amenability of $\mathcal{A}$ there exists $E \in \mathcal{A}^{**}$ such that $d(a) = \widehat{\tau(a)} = -E\circ\sigma(a) + \tau(a)\bullet E$ for all $a \in \mathcal{A}$. Therefore $\widehat{\tau(a)} = \tau(a)\bullet E$. By the Goldstine theorem, let $\{e_\alpha\}$ be a bounded net in $\mathcal{A}$ such that $w^* - \lim \widehat{e_\alpha} = E$. Then $w^* - \lim \tau(a)\bullet\widehat{e_\alpha} = \tau(a)\bullet E = \widehat{\tau(a)}$. Hence

$$\begin{aligned} f(\tau(a)) &= \widehat{\tau(a)}(f) = \lim_\alpha(\tau(a)\bullet\widehat{e_\alpha})(f) \\ &= \lim_\alpha \widehat{e_\alpha}(\tau(a)\bullet f) = \lim_\alpha(\tau(a)\bullet f)(e_\alpha) \\ &= \lim_\alpha f(\tau(a)\bullet e_\alpha) = \lim_\alpha f(\tau(a)e_\alpha). \end{aligned}$$

It follows from $\overline{\tau(\mathcal{A})} = \mathcal{A}$, boundedness of f and boundedness of $\{e_\alpha\}$ that $f(ae_\alpha) \longrightarrow f(a)$ for all $a \in \mathcal{A}, f \in \mathcal{A}^*$. Hence $\{e_\alpha\}$ is a weakly right approximate identity for $\mathcal{A}$. It induces a right approximate identity for $\mathcal{A}$ say $\{d_\alpha\}$. With similar argument we can find a left approximate identity $\{c_\beta\}$. Thus by [1, Proposition 11.1.5] $d_\alpha\Box c_\beta = d_\alpha + c_\beta - d_\alpha c_\beta$ give us an approximate identity for $\mathcal{A}$. $\Box$

Let $\mathcal{A}$ be a Banach algebra. Recall that $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule under the module actions $a(b\otimes c) = ab\otimes c$ and $(b\otimes c)a = b\otimes ca$. Moreover, let $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow A$ be the canonical linear mapping defined by $\pi(a\otimes b) = ab$. We denote the first and the second conjugates of π by $\pi^* : \mathcal{A}^* \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ and $\pi^{**} : (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**} \longrightarrow \mathcal{A}^{**}$, respectively.

A net $\{m_\alpha\} \subseteq \mathcal{A}\widehat{\otimes}\mathcal{A}$ is said to be an *σ-approximate diagonal* for $\mathcal{A}$ if $\lim_\alpha m_\alpha\sigma(a) - \sigma(a)m_\alpha = 0$ and $\lim_\alpha \pi(m_\alpha).\sigma(a) = \sigma(a)$ for all $a \in \mathcal{A}$.

An element $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ is said to be a *σ-virtual diagonal* for $\mathcal{A}$, if $\sigma(a).M = M.\sigma(a)$, $\pi^{**}(M).\sigma(a) = \sigma(a)$ for all $a \in \mathcal{A}$.

The following theorem is an extension of [11, Theorem 2.2.4]

Theorem 4.2. *Let σ be a continuous idempotent homomorphism on a Banach algebra $\mathcal{A}$ with a bounded approximate identity $\{e_\alpha\}$. If $\mathcal{A}$ is σ-amenable then it has a σ-virtual diagonal.*

Proof. Let $\mathcal{A}$ be σ-amenable. The bounded net $\{e_\alpha\otimes e_\alpha\}$ has a w^*-cluster point in $(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$; say E. We can assume that $w^* - \lim e_\alpha\otimes e_\alpha = E$. Consider the inner

σ-derivation $d_E : \mathcal{A} \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ defined by $d_E(a) = E.\sigma(a) - \sigma(a).E$. We have

$$\begin{aligned} \pi^{**}(d_E(a)) &= \pi^{**}(E.\sigma(a) - \sigma(a).E) \\ &= w^* - \lim \pi(e_\alpha \otimes e_\alpha\sigma(a) - \sigma(a)e_\alpha \otimes e_\alpha) \\ &= \lim_\alpha (e_\alpha^2\sigma(a) - \sigma(a)e_\alpha^2) \\ &= \sigma(a) - \sigma(a) \\ &= 0. \end{aligned}$$

Therefore $d_E(\mathcal{A}) \subseteq ker(\pi^{**})$. It is Known that $ker(\pi^{**}) = (ker\pi)^{**}$ [11, Page 45]. Thus $d_E \in Z_{(\sigma,\sigma)}(\mathcal{A}, (ker\pi)^{**})$. By σ-amenability of $\mathcal{A}$, there exists $N \in (ker\pi)^{**}$ such that $d_E = d_N$. Put $M = E - N$. Then $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ and for all $a \in \mathcal{A}$ we have $\sigma(a) \cdot M - M \cdot \sigma(a) = d_M(a) = d_E(a) - d_N(a) = 0$ and

$$\begin{aligned} \pi^{**}(M) \cdot \sigma(a) &= \pi^{**}(E) \cdot \sigma(a) - \pi^{**}(N) \cdot \sigma(a) \\ &= \pi^{**}(E) \cdot \sigma(a) \\ &= w^* - \lim_\alpha (\pi(e_\alpha \otimes e_\alpha)\sigma(a)) \\ &= \lim_\alpha e_\alpha^2\sigma(a) \\ &= \sigma(a). \end{aligned}$$

□

The following result is a generalization of [1, Lemma 8].

Proposition 4.3. *Let σ be a continuous idempotent homomorphism on a Banach algebra $\mathcal{A}$ with a bounded approximate identity $\{e_\alpha\}$. If $\mathcal{A}$ has a σ-virtual diagonal then it has a σ-approximate diagonal.*

Proof. Let M be a σ-virtual diagonal for $\mathcal{A}$. Since $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ there exists a bounded net $\{p_\alpha\}$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $w^* - \lim_\alpha p_\alpha = M$. Then $w^* - \lim_\alpha(\sigma(a).p_\alpha - p_\alpha.\sigma(a)) = \sigma(a)M - M\sigma(a) = 0$ and so $w^* - \lim_\alpha(\sigma(a).p_\alpha - p_\alpha.\sigma(a)) = 0$. Moreover, $w^* - \lim_\alpha \pi(p_\alpha).\sigma(a) = \pi^{**}(M).\sigma(a)$ and so $w^* - \lim_\alpha \pi(p_\alpha)\sigma(a) = \sigma(a)$. By passing to a convex combination, we conclude the existence of a net $\{m_\alpha\}$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\sigma(a)\cdot m_\alpha - m_\alpha\cdot\sigma(a) \longrightarrow 0$ and $\sigma(a)\cdot\pi(m_\alpha) \longrightarrow \sigma(a)$. □

Proposition 4.4. *Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $\mathcal{A}$ has a bounded approximate identity. If $\mathcal{A}\mathcal{X} = \{0\}$ (or $\mathcal{X}\mathcal{A} = \{0\}$) then any (σ,τ)-derivation $d : \mathcal{A} \longrightarrow \mathcal{X}^*$ is inner.*

Proof. Let $\{e_\alpha\}$ be a bounded approximate identity for $\mathcal{A}$. $\mathcal{A}\mathcal{X} = \{0\}$ implies that $\mathcal{X}^*\mathcal{A} = \{0\}$. Without lose of generality we can assume that $w^* - \lim d(e_\alpha) = -f$ for some $f \in \mathcal{X}^*$. Then

$$\begin{aligned} d(a) &= w^* - \lim d(ae_\alpha) \\ &= w^* - \lim(d(a)\sigma(e_\alpha) + \tau(a)d(e_\alpha)) \\ &= -\tau(a)f \\ &= f\sigma(a) - \tau(a)f \\ &= d_f(a). \end{aligned}$$

□

The next proposition is an extension of [10, Proposition 0.3]

Proposition 4.5. *Let σ,τ be two homomorphisms on a Banach algebra $\mathcal{A}$ having a bounded approximate identity. Then $H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*) = H^1_{(\sigma,\tau)}(\mathcal{A},(\mathcal{A}\mathcal{X}\mathcal{A})^*)$ for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$.*

The following proposition is an extension of [11, Theorem 2.2.4].

Proposition 4.6. *Let σ be a continuous idempotent epimorphism on a Banach algebra $\mathcal{A}$ which has a σ-approximate diagonal. Then $\mathcal{A}$ is σ-amenable.*

Proof. Let $\{m_\alpha\}$ be a σ-approximate diagonal for $\mathcal{A}$. For each α there are two sequences $\{a^\alpha_n\}$, $\{b^\alpha_n\}$ such that $m_\alpha = \sum_{n=1}^\infty a^\alpha_n \otimes b^\alpha_n$ and $\sum_{n=1}^\infty ||a^\alpha_n||||b^\alpha_n|| < \infty$. Let $\mathcal{X}$ be a pseudo-unital $\mathcal{A}$-bimodule and $D \in Z_{(\sigma,\sigma)}(\mathcal{A},\mathcal{X}^*)$. The bounded net $\{\sum_{n=1}^\infty \sigma(a^\alpha_n)D(b^\alpha_n)\}$ in $\mathcal{X}^*$ has a w^*-cluster point, say $\phi \in \mathcal{X}^*$. Then by passing to a subnet, if necessary, we have

$$\begin{aligned}
\sigma(a)\phi &= \sigma(a)\lim_{\alpha}\sum_{n=1}^{\infty}\sigma(a_n^{\alpha})D(b_n^{\alpha}) \\
&= \lim_{\alpha}\sum_{n=1}^{\infty}\sigma(a)\sigma(a_n^{\alpha})D(b_n^{\alpha}) \\
&= \lim_{\alpha}F(\sigma(a)m_{\alpha}) \\
&= \lim_{\alpha}F(m_{\alpha}\sigma(a)) \\
&= \lim_{\alpha}\sum_{n=1}^{\infty}\sigma(a_n^{\alpha})D(b_n^{\alpha}\sigma(a)) \\
&= \lim_{\alpha}\sum_{n=1}^{\infty}\sigma(a_n^{\alpha})\Big(\sigma(b_n^{\alpha})D(\sigma(a)) + D(b_n^{\alpha})\sigma(a)\Big) \\
&= \lim_{\alpha}\Big(\sum_{n=1}^{\infty}\sigma(a_n^{\alpha})\sigma(b_n^{\alpha})D(\sigma(a)) + \sum_{n=1}^{\infty}\sigma(a_n^{\alpha})D(b_n^{\alpha})\sigma(a)\Big) \\
&= \lim_{\alpha}\sigma(\pi(m_{\alpha}))D(\sigma(a)) + \phi\sigma(a) \qquad (a \in \mathcal{A}),
\end{aligned}$$

where $F : \mathcal{A} \otimes \mathcal{A} \to \mathcal{X}^*$ is defined by $F(a \otimes b) = \sigma(a)D(b)$. Therefore for each $b \in \mathcal{A}$ is implied that $\sigma(b)(\sigma(a)\phi - \phi\sigma(a) - D(\sigma(a))) = 0$. Since σ is an epimorphism and $\lim_{\alpha}\pi(m_{\alpha})b = b$ for all $b \in \mathcal{A}$, we conclude that $D(a) = \sigma(a)\phi - \phi\sigma(a)$. □

5. **An example**

We use a Banach algebra introduced by Yong Zhang [12] to introduce a Banach algebra that is (σ,τ)-weak amenable for all homomorphisms σ,τ but for some homomorphisms σ and τ it is not (σ,τ)-amenable.

It is easy to see that ℓ^1 is a Banach algebra equipped with the following product

$$a \cdot b = a(1)b \qquad (a, b \in \ell^1).$$

and ℓ^1 has a left identity e_1 defined by

$$e_1(n) = \begin{cases} 1 & if \quad n = 1 \\ 0 & if \quad n \neq 1. \end{cases}$$

The dual space $(\ell^1)^* = \ell^\infty$ is a ℓ^1-bimodule via the ordinary actions as follows

$$a \cdot f = f(a)e_1, \quad f \cdot a = a(1)f \qquad (a \in \ell^1, f \in \ell^\infty)$$

where e_1 is regarded as an element of ℓ^∞.

Next let $\sigma : \ell^1 \longrightarrow \ell^1$ be a bounded homomorphism. We have $a(1)\sigma(b) = \sigma(a \cdot b) = \sigma(a) \cdot \sigma(b) = \sigma(a)(1)\sigma(b)$ and so $(a(1) - \sigma(a)(1))\sigma(b) = 0$ for all $a, b \in \mathbb{N}$. Since $\sigma \neq 0$, we have

$$\sigma(a)(1) = a(1) \quad (a \in \ell^1). \tag{5.1}$$

Now let σ, τ be homomorphisms and let $D : \ell^1 \longrightarrow \ell^\infty$ be a bounded (σ,τ)-derivation. Then for all $a, b \in \ell^1$ we have

$$D(a \cdot b) = D(a) \cdot \sigma(b) + \tau(a) \cdot D(b) \tag{5.2}$$

$$a(1)D(b) = \sigma(b)(1)D(a) + \tau(a) \cdot D(b). \tag{5.3}$$

By taking $b = a$ in (5.2) and using (5.1), we get $\tau(a) \cdot D(a) = 0$ for all $a \in \ell^1$. Therefore we have $\tau(a+b) \cdot D(a+b) = 0$ and so $\tau(a) \cdot D(b) = -\tau(b) \cdot D(a)$ for all $a, b \in \ell^1$.

Then

$$\begin{aligned} D(a) = D(e_1 \cdot a) &= D(e_1) \cdot \sigma(a) + \tau(e_1) \cdot D(a) \\ &= D(e_1) \cdot \sigma(a) - \tau(a) \cdot D(e_1) \end{aligned}$$

for all $a \in \ell^1$. Therefore D is (σ,τ)-inner. Thus ℓ^1 is (σ,τ)-weakly amenable for all homomorphisms σ and τ on ℓ^1.

Remark 5.1. The Banach algebra ℓ^1 is not amenable since it clearly has no bounded right approximate identity. Then, by Corollary 2.2, there exist homomorphisms σ_1 and τ_1 that ℓ^1 is not (σ_1,τ_1)-amenable. By Proposition 2.4, it is however $(\sigma,0)$-amenable for all homomorphisms σ on ℓ^1.

References

[1] Bonsall, F.F., Duncan, J., *Complete Normed Algebra*, Springer-Verlag 1973.

[2] Dales, H.G., *Banach algebra and Automatic continuity*, Oxford university Press, 2001.

[3] Hochschild, G., *On the cohmology groups of an associative algebra*, Ann. Of Math. (2) **46** (1945), 58-67.

[4] Johnson, B.E., *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., **127** (1972).

[5] Mirzavaziri, M., Moslehian, M.S., *Automatic continuity of σ-derivations on C^*-algebras*, Proc. Amer. Math. Soc. **134** (2006), no. 11, 3319-3327.

[6] Mirzavaziri, M., Moslehian, M.S., *σ-derivations in Banach algebras*, Bull. Iranian Math. Soc. **32** (2006), no. 1, 65-78.

[7] Mirzavaziri, M., Moslehian, M.S., *σ-amenability of Banach algebras*, Southeast Asian Bull. Math. (to appear).

[8] Moslehian, M.S., *Approximate $(\sigma - \tau)$-contractibility*, Nonlinear Funct. Anal. Appl., **11** (2006), no. 5, 805-813.

[9] Palmer, T.W., *Banach algebras and the general theory of $*$-algebras,* Vol. I, *Algebras and Banach algebras*, Encyclopedia of Mathematics and its Applications 49, Cambridge University Press, Cambridge, 1994.

[10] Pier, J., *Amenable Banach Algebra*, Pitman Research Notes in Mathematical Series 172, Longeman, Essex, 1988.

[11] Runde, V., *Lectures on Amenability*, Lecture Notes in Mathematics 1774, Springer, 2002.

[12] Zhang, Y., *Weak Amenability of a Class of Banach Algebras*, Canada, Math. Bull., **44**(4), 2001, 504-508.

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