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NUMERICAL METHOD FOR FREE SURFACE VISCOUS FLOWS

TIBERIU IOANA AND TITUS PETRILA

Abstract. In this paper we present a new algorithm for studying the flow of viscous fluids with a free surface. This algorithm is based on an optimization solution strategy. Numerical results are presented in the case of a particular fluid flow problem.

1. Mathematical model and solution strategy

A viscous incompressible fluid of dynamic viscosity η, pressure p, density ρ and velocity $\mathbf{u}$ flows over a solid boundary Γ . The fluid is up bounded by a free surface S, the fluid domain being denoted by D. It is assumed that the flow is steady and the exterior force is represented by gravity.

For this problem we write the system of equations

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\varphi - Re^{-1}\nabla\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right) &= 0, \qquad \mathbf{x}\in D && (1)\\
\nabla\cdot\mathbf{u} &= 0, \qquad \mathbf{x}\in D && (2)\\
\varphi - Fr^{-2}y &= 0, \qquad \mathbf{x}\in S && (3)\\
Re^{-1}\mathbf{t}\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right)\mathbf{n} &= 0, \qquad \mathbf{x}\in S && (4)\\
\mathbf{u}\cdot\mathbf{n} &= 0, \qquad \mathbf{x}\in S && (5)\\
\mathbf{u} &= 0, \qquad \mathbf{x}\in \Gamma && (6)\\
\mathbf{u} &= \mathbf{u}_d, \qquad x\to\infty, && (7)
\end{aligned}
$$

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where φ is the hydrodynamic component of the fluid pressure, $\varphi(x, y) = p(x, y) + Fr^{-2}y$. Here we denote by Re and Fr the Reynolds and Froude numbers respectively, while $\mathbf{t}$ and $\mathbf{n}$ are the unit tanget and outward normal vectors respectively.

For solving this problem we use a solution strategy based on an optimization approach. Let's denote by S_0 the initial position of the free boundary. We assume that the new position of the free boundary S_ϵ, is related with its original position by

$$(x_0, y_0) \longrightarrow (x_\epsilon, y_\epsilon) = (x_0, y_0) - \epsilon\mathbf{n}.$$

(such a mapping is used for instance in [7])

Now an algorithm is constructed to obtain the shape of the free surface and the velocity field. Precisely, we have to chose first an initial position of the unknown free boundary and this position will be updated with $-\gamma \mathrm{grad}_{\mathbf{n}} J^*$, where

$$\begin{aligned} J^*(S) &= \int_S p^2 dS + \int_D \mathbf{v} \cdot \left((\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla\varphi - Re^{-1}\nabla \cdot \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T \right) \right) dD \\ &+ \int_D w \, (\nabla \cdot \mathbf{u}) \, dD. \end{aligned} \tag{8}$$

Then the procedure restart, with that new free boundary and so on. This procedure will stop when $\mathrm{grad}_{\mathbf{n}} J^* < \epsilon$. Folowing [2], [3] we get

$$\begin{aligned} \mathrm{grad}_{\mathbf{n}} J^* &= \int_S \{ (1 - \varphi + Fr^{-2}y)\mathbf{n} \cdot \nabla\varphi - Fr^{-2}\mathbf{n} \cdot \mathbf{j} \\ &\quad - \mathbf{v} \cdot Re^{-1} \left((\nabla (\mathbf{n} \cdot \nabla\mathbf{u})) + (\nabla (\mathbf{n} \cdot \nabla\mathbf{u}))^T \right) \mathbf{n} \} dS, \end{aligned} \tag{9}$$

where w and $\mathbf{v}$ are Lagrange multipilers, $\mathbf{v}$ is got from the boundary value problem (10) - (15), i.e.,

$$\begin{aligned}
\mathbf{u}\cdot\nabla\mathbf{v}+(\nabla\mathbf{u})\,\mathbf{v}+\nabla w+\nabla\cdot Re^{-1}\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right) &= 0, \quad \mathbf{x}\in D && (10)\\
\nabla\cdot\mathbf{v} &= 0, \quad \mathbf{x}\in D && (11)\\
\mathbf{v}\cdot\mathbf{n} &= -p, \ \mathbf{x}\in S && (12)\\
Re^{-1}\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right)\mathbf{n} &= 0, \quad \mathbf{x}\in S && (13)\\
\mathbf{v} &= 0, \quad \mathbf{x}\in\Gamma && (14)\\
\mathbf{v} &= 0, \quad |x|\to\infty. && (15)
\end{aligned}$$

Obviously, this is a linear problem in $\mathbf{v}$ and w, while $\mathbf{u}$ is the solution of the boundary value problem (1), (2), (3), (5), (6), (7).

2. **Numerical results**

Let's apply the optimization algorithm for a fluid configuration D considered in the sequel. Precisely let's consider the fluid flow of parameters $\rho = 1kg/m^3$, $\eta = 10Pa\cdot s$, $Fr = 0.7$, $u1_0 = 7m/s$, $u2_0 = 0m/s$ where $\mathbf{u_0} = (u1_0, u2_0)$ is the inflow velocity. For the output flow we have used some appropriate Neumann boundary conditions. Let's define the initial shape of the free boundary by a straight line (1).

To solve the respective boundary value problem and to update successively the free surface we have used the software packages Comsol and Matlab.

Using a step size $\gamma = 10^-2$ we get the new shape of the free surface (2), the fluid flow domain and the velocity field (3), the velocity surface (4), the stream lines (5), the velocity vector field (6).

We remark that the optimization algorithm proposed in this paper could be extended for three dimensional flows and this will be the target of a next paper.

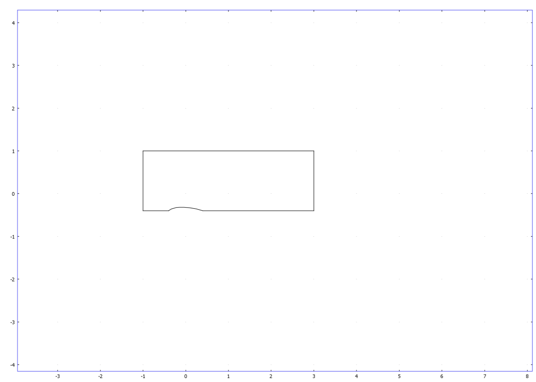

Figure 1. Initial fluid flow domain

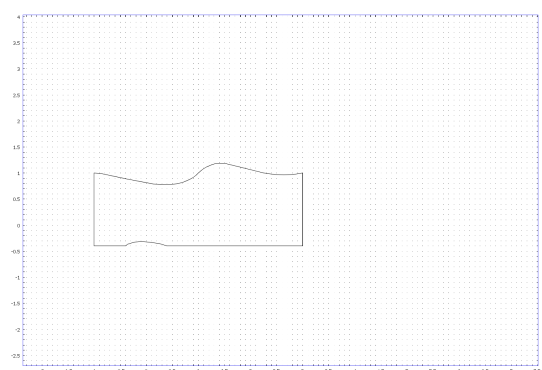

Figure 2. Fluid flow domain

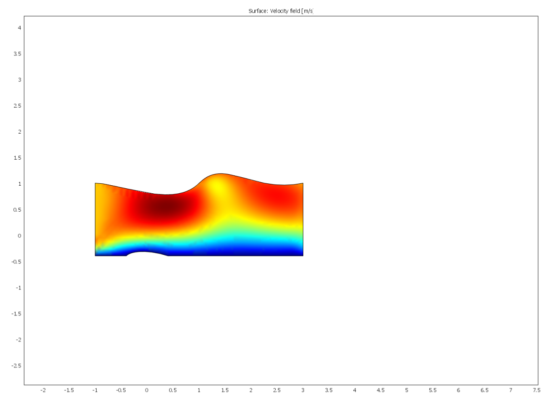

FIGURE 3. Fluid flow domain and the velocity filed

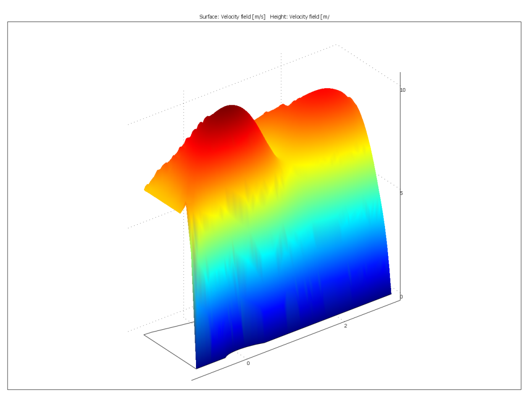

FIGURE 4. Velocity surface

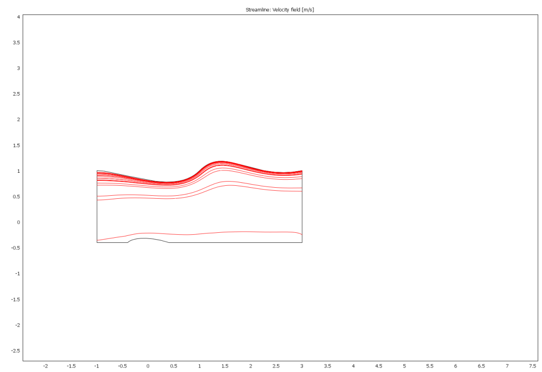

FIGURE 5. Stream lines

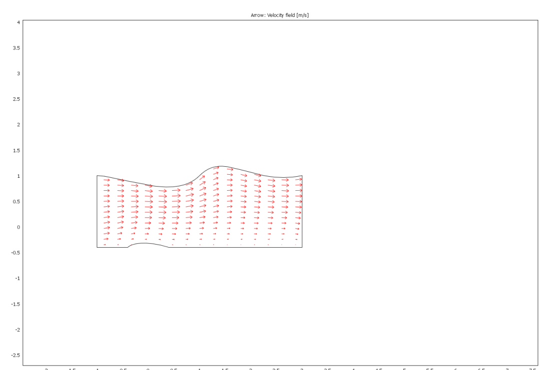

FIGURE 6. Velocity vector filed

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Babeş-Bolyai University, 1, M. Kogălniceanu Str.,
400084 Cluj-Napoca, Romania
E-mail address: tioana@math.ubbcluj.ro

Babeş-Bolyai University, 1, M. Kogălniceanu Str.,
400084 Cluj-Napoca, Romania
E-mail address: tpetrila@cs.ubbcluj.ro