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A CYCLIC ODD-EVEN REDUCTION TECHNIQUE APPLIED TO A PARALLEL EVALUATION OF AN EXPLICITE SCHEME IN MATHEMATICAL FINANCE

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. The purpose of this paper is to give a possibility of reducing the execution time involved in evaluating a financial option by means of an explicite scheme, using a cyclic odd-even reduction technique.

1. Introduction

The concept of arbitrage is largely used in the domain of mathematical finance. It allows us to establish precise relationships between prices and thence to determine them. Connected with it, the strategies of an option is very important. In the literature, the celebrated Black-Scholes differential equation for the price of the so-called European vanilla option is the best known and used.

Many papers study this equation and indicate different numerical methods in order to get the approximate solution. E.g., in [3], the finite difference method is presented. In [4], the method of radial basis is used, to avoid the mesh of discretized points.

In this paper, considering the idea given by the cyclic odd-even reduction (see [1]), we start from an explicit scheme obtained by means of finite differences, and give an alternative of evaluating of the approximate values, using a cyclic odd-even reduction type technique, which generates a logarithmic time of execution.

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2. Recalling the Black-Scholes equation

As in [3], denoting by:

- V, the value of an option
- S, the current value of the underlying asset
- t, the time
- σ, the volatility of the underlying asset
- T, the expiry
- r, the interest rate
- E, the exercise price,

We get the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \cdot s^2 \cdot \frac{\partial^2 V}{\partial s^2} + r \cdot s \cdot \frac{\partial V}{\partial s} - r \cdot V = 0 \tag{1}$$

with the boundary conditions:

$$C(0, t) = 0$$

$$C(S, T) = \max(S - E, 0).$$

3. The finite difference methods

Finite difference methods (see [2]) are a means of obtaining numerical solutions to partial differential equations (see [2], [3]). They constitute a very powerful and flexible technique and, if applied correctly, are capable of generating accurate numerical solutions to all of the mathematical finance models, also for the Black-Scholes equation (1).

So, considering a mesh of equal S-steps of size δS and equal time-steps of size δt, with $(N+1)^2$ points, central differences for S derivatives and backward differences for time derivatives, we get the explicit discretization of the Black-Scholes equation:

$$\begin{gathered} B_0 V_n^m + C_0 V_1^m = V_0^m \\ A_n V_{n-1}^m + B_n V_n^m + C_n V_{n+1}^m = V_n^{m+1}, \quad n = 1, 2, \ldots, N \end{gathered} \tag{2}$$

where m indicates the moment of time,

$$A_n = -\frac{1}{2}(\sigma^2 n^2 - (r - s_0)n)\delta t$$

$$B_n = 1 + (\sigma^2 n^2 + r)\delta t$$

$$C_n = -\frac{1}{2}(\sigma^2 n^2 + (r - s_0)n)\delta t$$

4. **A technique of the cyclic odd-even reduction type**

Relation (2) generates a system of equations of the following form:

$$\begin{bmatrix} B_0 & C_0 & 0 & \dots & 0 \\ A_1 & B_1 & C_1 & \dots & 0 \\ 0 & A_2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & C_N \\ 0 & 0 & \dots & A_N & B_N \end{bmatrix} \cdot \begin{bmatrix} V_0^m \\ V_1^m \\ \vdots \\ V_{N-1}^m \\ V_N^m \end{bmatrix} = \begin{bmatrix} V_0^{m+1} \\ V_1^{m+1} \\ \vdots \\ V_{N-1}^{m+1} \\ V_N^{m+1} \end{bmatrix} \tag{3}$$

which involves a time of execution of $O(N^2)$. Then the following theorem holds:

Theorem 1. *Using a cyclic odd-even reduction type technique, equation (2) can be computed in a time $O(N[\log_2 m])$ time.*

Proof. Rewriting (2),

$$A_n V_{n-1}^m + B_n V_n^m + C_n V_{n+1}^m = V_n^{m+1}$$

for one single value n, and replacing V_n^m using the same connection among values, we get:

$$A_n V_{n-1}^m + B_n(A_n V_{n-1}^{m-1} + B_n V_n^{m-1} + C_n V_{n+1}^{m-1}) + C_n V_{n+1}^m = V_n^{m+1}$$

or, making some computations:

$$A_n(V_{n-1}^m + B_n V_{n-1}^{m-1}) + B_n^2 V_n^{m-1} + C_n(B_n V_{n+1}^{m-1} + V_{n+1}^m) = V_n^{m+1}.$$

So, for n given, value V_n^{m+1} can be computed by means of values from two previous moments of time. Repeating the same substitution, we finally get:

$$A_n(a_m V_{n-1}^m + a_{m-1} V_{n-1}^{m-1} + \dots + a_0 V_{n-1}^0) + B_n^m V_n^0$$
$$+ C_n(a_m^1 V_{n-1}^m + a_{m-1}^1 V_{n-1}^{m-1} + \dots + a_0^1 V_{n-1}^0) = V_n^{m+1} \tag{4}$$

where we denoted by a_i and a_i^1, $i = \overline{0, m}$ the final coefficients.

Using the double recursive technique (see [1]), in $[\log_2 m]$ parallel steps, the values in parenthesis are computed.

Finally, for $n = 1, 2, \ldots, N$, the total execution time is $O(N[\log_2 m])$. $\square$

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