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A COMBINED METHOD FOR INTERPOLATION OF SCATTERED DATA BASED ON TRIANGULATION AND LAGRANGE INTERPOLATION

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. We introduce a combined method for interpolation on scattered data sets in the plane, based on the triangulation method introduced by R. Franke and G. Nielson in [10].

1. **Introduction**

A certain method based on triangulation, introduced by R. Franke and G. Nielson in [10], and the Shepard method, introduced in [15], are superior to other methods used in interpolation of very large scattered data sets. Both methods have very comparable fitting capabilities, but there exist situations in which one or the other may be preferable [10].

We present first the original method based on triangulation, introduced in [10].

Let f be a real-valued function defined on $X \subset \mathbb{R}^2$, and $V_i(x_i, y_i) \in X,\ i = 1, ..., N$ some distinct points. Denote by $r_i\,(x, y)$, the distances between a given point $(x, y) \in X$ and the points $V_i\,(x_i, y_i)\,,\ i = 1, ..., N.$ The interpolant with regard to the data $V_i(x_i, y_i),\ i = 1, ..., N$, is of the form

$$(Pf)(x,y) = \sum_{i=1}^{N} W_i(x,y)P_i(x,y), \tag{1}$$

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where W_i, $i = 1, ..., N$ are the weight functions and P_i, $i = 1, ..., N$ are some local interpolation operators. We have

$$W_i(x_j, y_j) = \delta_{ij}, \quad i, j = 1, ..., N. \tag{2}$$

The method of defining the weight functions W_i, $i = 1, ..., N$ given in [10] requires the triangulation of the data $V_i(x_i, y_i)$, $i = 1, ..., N$. Each W_i will be a globally defined C^1 function with support $S_i = \cup_{jkl \in M_i} T_{jkl}$, where T_{jkl} denotes the triangle with vertices V_j, V_k, V_l and $M_i = \{jkl : T_{jkl}$ is a triangle with vertex $V_i\}$. For $(x, y) \in T_{jkl} \subset S_i$, the weight functions have the form [10]:

$$\begin{aligned} W_i(x,y) &= \\ &= b_i^2(3-2b_i) + 3\frac{b_i^2 b_j b_k}{b_i b_j + b_i b_k + b_j b_k}\left[b_j \frac{\|e_i\|^2 + \|e_k\|^2 - \|e_j\|^2}{\|e_k\|^2} + b_k \frac{\|e_i\|^2 + \|e_j\|^2 - \|e_k\|^2}{\|e_j\|^2}\right], \end{aligned} \tag{3}$$

where b_i, b_j, b_k are the barycentric coordinates of (x, y) with respect to the triangle T_{jkl} and $\|e_p\|$, $p = i, j, k$ represent the length of the edge opposite to V_p, $p = i, j, k$. The barycentric coordinates are given by the equations:

$$\begin{aligned} x &= b_i x_i + b_j x_j + b_k x_k, \\ y &= b_i y_i + b_j y_j + b_k y_k, \\ 1 &= b_i + b_j + b_k. \end{aligned}$$

For an arbitrary triangle T_{jkl} the only weights which are nonzero are W_i, W_j and W_k, and therefore the interpolant (1) becomes

$$(Pf)(x, y) = W_i(x, y)P_i(x, y) + W_j(x, y)P_j(x, y) + W_k(x, y)P_k(x, y). \tag{4}$$

We note that

$$W_i + W_j + W_k = 1. \tag{5}$$

As local interpolation operators P_i, $i = 1, ..., N$ R. Franke and G. Nielson considered the solution of the inverse distance weighted least squares problem at $(x, y) = (x_i, y_i)$, i.e.,

$$\begin{aligned} P_i(x,y) = & f_i + \bar{a}_{i2}(x-x_i) + \bar{a}_{i3}(y-y_i) + \bar{a}_{i4}(x-x_i)^2 \\ & + \bar{a}_{i5}(x-x_i)(y-y_i) + \bar{a}_{i6}(y-y_i)^2, \end{aligned}$$

where $f_i = f(x_i, y_i)$. The coefficients $\bar{a}_{ik}$, $k = 2, ..., 6$ are the solutions of the system [10]

$$\min_{a_{i2},\ldots,a_{i6}} \sum_{\substack{k-1 \\ k \neq i}}^{N} \left[\frac{f_i + a_{i2}(x_k - x_i) + ... + a_{i6}(y_k - y_i)^2 - f_k}{\rho_k(x_i, y_i)} \right]^2,$$

with ρ_i given by

$$\frac{1}{\rho_i} = \frac{(R_i - r_i)_+}{R_i r_i},$$

with

$$z_+ = \begin{cases} z, & z > 0 \\ 0, & z \leq 0, \end{cases}$$

and R_i is a radius of influence about the node (x_i, y_i) and it is varying with i. The proper choice of the radius is critical to the success of the method [10]. This is taken as the distance from node i to the jth closest node to (x_i, y_i) for $j > N_w$ (N_w is a fixed value) and j as small as possible within the constraint that the jth closest node is significantly more distant than the $(j-1)$st closest node (see, e.g. [14]).

2. **Main results**

In this section we consider a new combined interpolation operator. It is obtained using the Lagrange interpolation operator as a local interpolation operator.

Let Λ be the set of Lagrange type information,

$$\Lambda := \Lambda_L = \{\lambda_i : \lambda_i(f) = f(x_i, y_i),\, i = 1, \ldots, N\}. \tag{6}$$

Let L_i be the bivariate Lagrange operators of degree n (associated to the node (x_i, y_i)), $i = 1, \ldots, N$, (see, e.g., [6]), that interpolates the function f, respectively, at the sets of points

$$Z_{m,i} := \{z_i, z_{i+1}, \ldots, z_{i+m-1}\}, \quad i = 1, \ldots, N,\, m < N, \tag{7}$$

with $z_{N+i} = z_i$, $i = 1, \ldots, m-1$ and $m := (n+1)(n+2)/2$ is the number of the coefficients of a bivariate polynomial of the degree n, $\sum_{i+j\leq n} a_{ij}x^i y^j$.

Remark 1. [6] *For given N, it can be considered only operators L_i^n with n such that $(n+1)(n+2)/2 < N$, i.e., for $n \in \{1, \ldots, \nu\}$, where $\nu = \text{integer}[(\sqrt{8N+1} - 3)/2]$.*

The existence and the uniqueness of the operators L_i^n are assured by the following theorem.

Theorem 2. [1] *Let $z_i := (x_i, y_i)$, $i = 1, \ldots, (n+1)(n+2)/2$ be different points in plane that do not lie on the same algebraic curve of n-th degree. Then, for every function f defined at the points z_i, $i = 1, \ldots, (n+1)(n+2)/2$ there exists a unique polynomial of degree n that interpolates f at z_i, $i = 1, \ldots, (n+1)(n+2)/2$.*

Hence, if the points z_k, $k = i, \ldots, i+m-1$ of the set (7) do not lie on an algebraic curve of n-th degree for all $i = 1, \ldots, N$ then L_i^n exists and it is unique for all $i = 1, \ldots, N$.

In what follows we suppose that the existence and the uniqueness conditions of the operators L_i^n, $i = 1, \ldots, N$, are satisfied.

We have

$$(L_i^n f)(x,y) = \sum_{k=i}^{i+m-1} l_k(x,y) f(x_k, y_k), \quad i = 1, \ldots, N,$$

where l_k are the corresponding cardinal polynomials:

$$l_k(x_j, y_j) = \delta_{kj}, \; k, j = i, \ldots, i+m-1.$$

The operators L_i^n have the following interpolatory properties:

$$(L_i^n f)(x_k, y_k) = f(x_k, y_k), \quad k = i, \ldots, i+m-1 \tag{8}$$

and the degree of exactness is

$$\text{dex}(L_i^n) = n, \tag{9}$$

for all $i = 1, \ldots, N$.

Next we use these Lagrange polynomials as local interpolation operators P_i, $i = 1, ..., N$, in (4). In this way we obtain a new interpolant of the data $V_i(x_i, y_i)$, $i =$

$1, ..., N$, with respect to the Lagrange type information, namely:

$$(Pf)(x,y) = W_i(x,y)(L_i^n f)(x,y) + W_j(x,y)(L_j^n f)(x,y) + W_k(x,y)(L_k^n f)(x,y), \quad (10)$$

with W_i, W_j, W_k given by (3).

Remark 3. *As the Lagrange operator is linear then the combined operator P is also linear.*

Theorem 4. *The combined operator P has the following interpolation properties:*

$$(Pf)(x_p, y_p) = f(x_p, y_p), \; p = 1, \dots, N. \quad (11)$$

Proof. We have

$$(Pf)(x_p, y_p) = W_i(x_p, y_p)(L_i^n f)(x_p, y_p) + W_j(x_p, y_p)(L_j^n f)(x_p, y_p) + W_k(x_p, y_p)(L_k^n f)(x_p, y_p).$$

Taking into account (2) and the interpolation properties of the Lagrange operators the conclusion follows. □

Theorem 5. *The degree of exactness of the combined operator P is*

$$\mathrm{dex}(P) = n \quad (12)$$

Proof. We have

$$(L_h^n e_{pq})(x,y) = e_{pq}(x,y), \; p + q \leq n, \; h = i, j, k$$

where $e_{pq}(x,y) = x^p y^q$. We have

$$\begin{aligned}(Pe_{pq})(x,y) &= W_i(x,y)e_{pq}(x,y) + W_j(x,y)e_{pq}(x,y) + W_k(x,y)e_{pq}(x,y)\\ &= e_{pq}(x,y)(W_i(x,y) + W_j(x,y) + W_k(x,y))\end{aligned}$$

and taking into account (5) it follows that

$$(Pe_{pq})(x,y) = e_{pq}(x,y), \quad p + q \leq n,$$

But $\mathrm{dex}(L_h^n) = n$ so it means that there exists a $(p,q) \in \mathbb{N}^2$ with $p + q = n + 1$ such that $L_h^n e_{pq} \neq e_{pq}$, which implies that $Pe_{pq} \neq e_{pq}$. So the conclusion follows. □

The new obtained operator presents a higher degree of exactness compared to that of the Shepard operator (introduced in [15]), which is usually used in this kind of interpolation problems.

Remark 6. *For the particular case $n = 1$ we have*

$$(L_i^1 f)(x,y) = l_i(x,y)f(x_i,y_i) + l_{i+1}(x,y)f(x_{i+1},y_{i+1}) + l_{i+2}(x,y)f(x_{i+2},y_{i+2}) \quad (13)$$

for $i = 1, \ldots, N$. We have

$$\begin{aligned} l_i(x,y) &= \tfrac{(y_{i+1}-y_{i+2})x+(x_{i+2}-x_{i+1})y+x_{i+1}y_{i+2}-x_{i+2}y_{i+1}}{(x_i-x_{i+1})(y_{i+1}-y_{i+2})-(x_{i+1}-x_{i+2})(y_i-y_{i+1})} \\ l_{i+1}(x,y) &= \tfrac{(y_{i+2}-y_i)x+(x_i-x_{i+2})y+x_{i+2}y_i-x_iy_{i+2}}{(x_{i+1}-x_{i+2})(y_{i+2}-y_i)-(x_{i+2}-x_i)(y_{i+1}-y_{i+2})} \\ l_{i+2}(x,y) &= \tfrac{(y_i-y_{i+1})x+(x_{i+1}-x_i)y+x_iy_{i+1}-x_{i+1}y_i}{(x_{i+2}-x_i)(y_i-y_{i+1})-(x_i-x_{i+1})(y_{i+2}-y_i)}. \end{aligned}$$

In this case the new interpolant is of the form:

$$(Pf)(x,y) = W_i(x,y)(L_i^1 f)(x,y) + W_j(x,y)(L_j^1 f)(x,y) + W_k(x,y)(L_k^1 f)(x,y),$$

with W_i, W_j, W_k given by (3).

Remark 7. *The existence and uniqueness condition of L_i^1 is that the points (x_i,y_i), (x_{i+1},y_{i+1}), (x_{i+2},y_{i+2}) do not lie on a line $Ax + By + C = 0$.*

The steps of the algorithm of computation of the previously obtained interpolant (10) are:

1. Form a triangulation of the points $V_i(x_i,y_i)$, $i = 1,...,M$.
2. Given a point (x,y) determine the triangle V_{i_0,j_0,k_0}, containing (x,y).
3. Compute the Lagrange interpolation polynomials, L_i^n, $i = i_0, j_0, k_0$, given by

$$(L_i^n f)(x,y) = \sum_{k=i}^{i+m-1} l_k(x,y)f(x_k,y_k), \quad i = i_0, j_0, k_0.$$

4. Compute the weights functions.
5. Compute the interpolant given by (10).

3. Test results

We consider several of the generally used test functions, [13], [14], [9]:

Gentle $f_1(x,y) = \exp\big[-\frac{81}{16}((x-0.5)^2 + (y-0.5)^2)\big]/3,$

Saddle $f_2(x,y) = \big[1.25 + \cos(5.4y)\big]/\big[6 + 6(3x-1)^2\big],$

Steep $f_3(x,y) = \exp\big[-\frac{81}{4}((x-0.5)^2 + (y-0.5)^2)\big]/3,$

Cliff $f_4(x,y) = \big[\tanh(9y-9x) + 1\big]/9.$

The following table contains mean errors for interpolation by

$$(Pf_l)(x,y) = W_i(x,y)(L_i^1 f_l)(x,y) + W_j(x,y)(L_j^1 f_l)(x,y) + W_k(x,y)(L_k^1 f_l)(x,y),$$

for $l = 1,...,4$, with W_i, W_j, W_k given by (3), L_i^1, L_j^1, L_k^1 of the form (13) and considering the consecutive interpolation nodes as the vertices of the triangle. We took 100 random generated nodes in the square $[-1,1]\times[-1,1]$. In Figures 1–4 we plot the graphics of f_i and Pf_i, $i = 1,\ldots 4$.

Function	Interpolation error
f_1	0.0210
f_2	0.0340
f_3	0.0073
f_4	0.0498

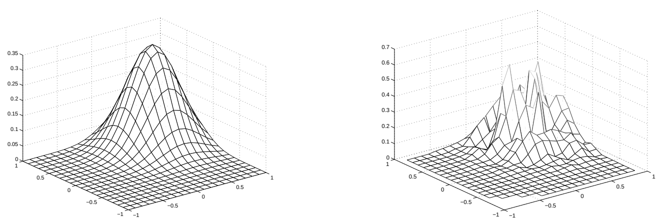

FIGURE 1. Function f_1 and interpolant Pf_1.

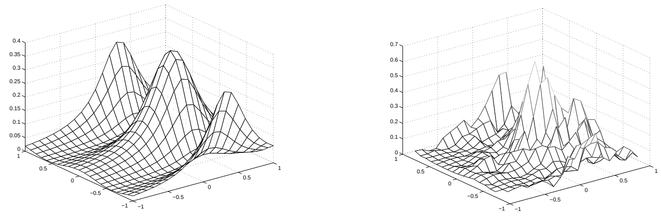

FIGURE 2. Function f_2 and interpolant Pf_2.

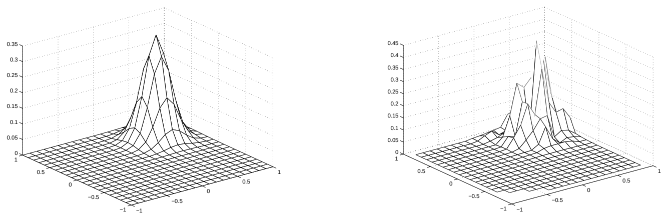

FIGURE 3. Function f_3 and interpolant Pf_3.

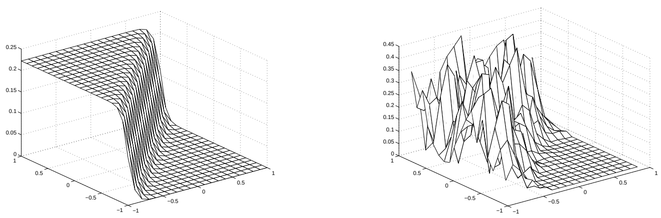

FIGURE 4. Function f_4 and interpolant Pf_4.

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